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ON THE LOCAL CONVERGENCE OF A NEWTON–TYPE METHOD IN BANACH SPACES UNDER A GAMMA–TYPE CONDITION

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Abstract

We provide a local convergence analysis for a Newton–type method to approximate a locally unique solution of an operator equation in Banach spaces. The local convergence of this method was studied in the elegant work by Werner in [11], using information on the domain of the operator. Here, we use information only at a point and a gamma–type condition [4], [10]. It turns out that our radius of convergence is larger, and more general than the corresponding one in [10]. Moreover the same can hold true when our radius is compared with the ones given in [9] and [11]. A numerical example is also provided.

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1. Introduction

In this paper we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice–Fréchet–differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

We revisit the Newton–type method given by x_0, $y_0 \in D$ by

$$\begin{aligned} x_{n+1} &= x_n - F'(z_n)^{-1}\, F(x_n), \qquad z_n = \frac{x_n + y_n}{2},\ (n \geq 0), \\ y_{n+1} &= x_n - F'(z_n)^{-1}\, F(x_{n+1}), \end{aligned} \tag{1.2}$$

to generate a sequence $\{x_n\}$, $(n \geq 0)$ approximating $x^\star$ [4], [11].

Let us illustrate how this method is conceived:

We start with the identity

$$F(x) - F(y) = \int_0^1 F'(y + t(x-y))\, dt\ (x - y) \quad \text{for all} \quad x, y \in D. \tag{1.3}$$

If $x^\star$ is a solution of equation (1.1), then identity (1.3) gives

$$F(x) = \int_0^1 F'(x + t(x^\star - x))\, dt\ (x^\star - x) \quad \text{for all} \quad x \in D. \tag{1.4}$$

The linear operator in (1.4) can be approximated in different ways [1], [3], [4], [12].

If for example

$$\int_0^1 F'(x + t(x^\star - x))\, dt \simeq F'(x) \quad \text{for all} \quad x \in D, \tag{1.5}$$

then (1.4) suggests the famous Newton's method [1]–[12]:

$$x_{n+1} = x_n - F'(x_n)^{-1}\, F(x_n)\ (n \geq 0). \tag{1.6}$$

Another choice is given by

$$\int_0^1 F'(x + t(x^\star - x))\,dt \simeq F'\left(\frac{x^\star + x}{2}\right) \quad \text{for all} \quad x \in D, \tag{1.7}$$

which leads to the implicit iteration:

$$x_{n+1} = x_n - F'\left(\frac{x_n + x_{n+1}}{2}\right)^{-1} F(x_n), \quad (n \geq 0). \tag{1.8}$$

Unfortunately iterates in (1.8) can only be computed in very restrictive cases, and numerically, the method (1.8) is not a pratical procedure.

That is why we consider y_n given in (1.2) as a suitable replacement for x_{n+1} $(n \geq 0)$. Hence, we arrive at method (1.2), which requires the computation of two iterates x_n and y_n. The computation of the additional iterate y_n can be seen as a step to calculate the iterate x_{n+1} using Newton's method (1.8).

This shows that iterate x_{n+1}, thus defined is corrected by computing the iterate y_{n+1} using (1.8). Another advantage of method (1.2) is that the particular case $x_0 = y_0$ corresponds to the classical Newton's method (1.8). Procedure (1.2) has a geometrical interpretation similar to the tangent–Secant method in the scalar case, and was introduced by King [8] (see procedure (I, II), p. 299), and extended into Banach space by Werner in [11], where the R–order $1 + \sqrt{2}$ local convergence was established.

Here, we provide a local convergence analysis of the Newton–type method (1.2) using a γ–type condition (see (2.3) and (2.4)). Our radius of convergence r_A (see Theorem 2.2) is larger than the corresponding one denoted by r_W (see (2.28)) given in the elegant work by Wang and Zhao [10]. Note also that a special choice of γ denoted by $\gamma^\star$ (see (2.29)) used in [10].
As it turns out the radius of convergence can be larger than the radii given in [9], [11] where information on a domain is used (see (2.30) and (2.32)) instead of only information at a point used by us.
A numerical example is also provided.

2. Local convergence analysis of the midpoint method (1.2)

Let us define scalar function f on $[0, \dfrac{1}{\gamma})$ by

$$f(t) = b - t + \frac{\gamma\, t^2}{1 - \gamma\, t}, \tag{2.1}$$

where $b \geq 0$, and $\gamma > 0$ are given.

It is known [9] that if

$$\alpha = b\,\gamma \leq 3 - 2\sqrt{2}, \tag{2.2}$$

then function f has two roots

$$t^\star = \frac{1+\alpha-\sqrt{(1+\alpha)^2 - 8\,\alpha}}{2\,\gamma}, \quad t^{\star\star} = \frac{1+\alpha+\sqrt{(1+\alpha)^2 - 8\,\alpha}}{2\,\gamma}$$

satisfying

$$b \leq t^\star \leq (1+\frac{1}{\sqrt{2}})\; b \leq (1-\frac{1}{\sqrt{2}})\;\frac{1}{\gamma} \leq t^{\star\star} \leq \frac{1}{2\,\gamma}.$$

We use throughout this paper the concept of γ–conditions:

Definition 2.1. *An operator $F\,:\,D \subseteq X \longrightarrow Y$ satisfies γ–conditions if the following hold:*

(i) There exists a zero $x^\star \in D$ of operator F such that

$$F'(x^\star)^{-1} \in L(Y,X);$$

(ii) Operator F is thrice–Fréchet–differentiable on D, and for all $x \in D$

$$\parallel F'(x^\star)^{-1}\; F''(x^\star) \parallel \leq 2\;\gamma, \tag{2.3}$$

and

$$\parallel F'(x^\star)^{-1}\; F'''(x^\star) \parallel \leq \frac{6\;\gamma^2}{(1-\gamma\;\parallel x - x^\star \parallel)^4} = f'''(\parallel x - x^\star \parallel). \tag{2.4}$$

In view of (2.1), we have

$$f'(t) = \frac{1-2\,(1-\gamma\;t)^2}{(1-\gamma\;t)^2},$$

$$f''(t) = \frac{2\,\gamma}{(1-\gamma\;t)^3},$$

and

$$f'''(t) = \frac{6\,\gamma^2}{(1-\gamma\ t)^4}.$$

We need the following Lemma:

Lemma 2.2. *Under the γ–conditions given by Definition 2.1, and for all $x \in U(x^\star, (1-\frac{1}{\sqrt{2}})\ \frac{1}{\gamma} = r_0) = \{x \in X\ :\ \| x - x^\star \| < r_0\} \subseteq D$, the following estimates hold:*

$$\| F'(x^\star)^{-1}\ F''(x) \| \leq f''(\| x - x^\star \|), \tag{2.5}$$

$$F'(x)^{-1} \in L(Y, X),$$

and

$$\| F'(x)^{-1}\ F'(x^\star) \| \leq -\frac{1}{f'(\| x - x^\star \|)}. \tag{2.6}$$

Proof. Using the γ–conditions, and the properties of function f, we obtain in turn:

$$\begin{aligned}
\| F'(x^\star)^{-1}\ F''(x) \| &\leq \| F'(x^\star)^{-1}\ F''(x^\star) \| + \| F'(x^\star)^{-1}\ (F''(x) - F''(x^\star)) \| \\
&= \| F'(x^\star)^{-1}\ F''(x^\star) \| + \\
&\quad \| \int_0^1 F'(x^\star)^{-1}\ F''(x^\star + t\,(x - x^\star))\,(x - x^\star)\,dt \| \\
&\leq 2\ \gamma + \int_0^1 f''(t\ \| x - x^\star \|)\ \| x - x^\star \|\ \ dt \\
&= 2\ \gamma + f''(\| x - x^\star \|) - f''(0) = f''(\| x - x^\star \|).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\| F'(x^\star)^{-1}\ (F'(x) - F'(x^\star)) \| &= \| F'(x^\star)^{-1}\ \int_0^1 F''(x^\star + t\,(x - x^\star))\,(x - x^\star)\,dt \| \\
&\leq \textstyle\int_0^1 f''(t\ \| x - x^\star \|)\ \| x - x^\star \|\ \ dt \\
&= f'(\| x - x^\star \|) - f'(0) = f'(\| x - x^\star \|) + 1 < 1.
\end{aligned}$$

It follows by the Banach Lemma on invertible operators [4], [12] that $F'(x)^{-1} \in L(Y, X)$, and

$$\begin{aligned}
\| F'(x)^{-1}\ F'(x^\star) \| &\leq \frac{1}{1 - \| F'(x^\star)^{-1}\ (F'(x) - F'(x^\star)) \|} \\
&\leq -\frac{1}{f'(\| x - x^\star \|)}.
\end{aligned}$$

That complete the proof of the Lemma. ♢

It is convenient for us to define sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ by

$$a_n = \frac{\gamma}{1-\gamma \parallel x_n - x^\star \parallel}, \quad b_n = \frac{\gamma^2}{4\,(1-\gamma \parallel x_n - x^\star \parallel)},$$

$$c_n = \frac{(1-\gamma \parallel z_n - x^\star \parallel)^2}{2\,(1-\gamma \parallel z_n - x^\star \parallel)^2 - 1},$$

$$d_n = \frac{\gamma}{4\,(1-\dfrac{\gamma}{2} \parallel x_n - x^\star \parallel)\,(1-\dfrac{\gamma}{2}\,(\parallel x_n - x^\star \parallel + \parallel y_n - x^\star \parallel))};$$

and functions a, b, c, d on $[0, r_0)$ by

$$a(r) = \frac{r}{1-r}, \quad b(r) = \frac{r^2}{4\,(1-r)},$$

$$c(r) = \frac{(1-r)^2}{2\,(1-r)^2 - 1},$$

$$d(r) = \frac{r}{4\,(1-\dfrac{r}{2})\,(1-r)}.$$

It is simple algebra to see that system of inequalities

$$c(r)\ [b(r) + 3\,d(r)] \le 1 \tag{2.7}$$

is satisfied for all

$$r \in [0, \frac{5-\sqrt{13}}{6}). \tag{2.8}$$

We shall also use the identities [4]:

$$F(x^\star) - F(x) - F'(x)\,(x^\star - x) = \int_0^1 F''(x + t(x^\star - x))\,(1-t)\,(x^\star - x)^2\,dt, \tag{2.9}$$

$$F(x) - F(y) - F'(z)\,(x-y) = \frac{1}{4}\int_0^1 \left[F''(z+\frac{t}{2}(x-y)) - F''(z+\frac{t}{2}(y-x))\right] (1-t)\,(x-y)^2\,dt$$
$$= \frac{1}{4}\int_0^1 F'''(z+\frac{t}{2}(y-x)+s\,t\,(x-y))\; s\,(1-t)\;(x-y)^3\,ds\,dt,$$

(2.10)

$$F'(z) - F'(\frac{x+x^\star}{2}) = \int_0^1 F''\left(\frac{x+x^\star}{2} + t\,(\frac{y-x^\star}{2})\right)\left(\frac{y-x^\star}{2}\right)dt,$$

(2.11)

and

$$F'(\frac{x^\star+w}{2}) - F'(z) = \int_0^1 F''\left(z+\frac{t}{2}\,(x^\star+w-x-y)\right)\left(\frac{x^\star+w-x-y}{2}\right)dt,$$

(2.12)

for $z = \dfrac{x+y}{2}$, and all x, y, $w \in D$.

We can show the local convergence theorem for the Newton–type method (1.2):

Theorem 2.3. *Under the γ–conditions given by Definition 2.1 for $x \in \overline{U}(x^\star, r^\star = \dfrac{5-\sqrt{13}}{6\,\gamma}) \subseteq D$, sequences $\{x_n\}$, $\{y_n\}$ generated by the Newton–type method (1.2) are well defined, remain in $U(x^\star, r^\star)$ for all $n \geq 0$, and converge to the unique zero of equation $F(x) = 0$ in $\overline{U}(x^\star, r^\star)$ provided that x_0, $y_0 \in U(x^\star, r^\star)$.*

Moreover the following estimates hold for all $n \geq 0$:

$$\| x_{n+1} - x^\star \| \leq c_n \left[b_n \ \| x_n - x^\star \|^2 + d_n \ \| y_n - x^\star \| \right] \ \| x_n - x^\star \|, \tag{2.13}$$

and

$$\| y_{n+1} - x^\star \| \leq$$

$$c_n \left[b_{n+1} \| x_{n+1} - x^\star \|^2 + d_n (\| x_{n+1} - x^\star \| + \| y_n - x^\star \| + \| x_n - x^\star \|) \right] \| x_{n+1} - x^\star \| . \tag{2.14}$$

Proof. By hypotheses x_0, $y_0 \in U(x^\star, r^\star)$, and for $x = \dfrac{x_0 + y_0}{2}$ in (2.6) we get $F'(z_0)^{-1}$ exists, and

$$\| F'(z_0)^{-1} \ F'(x^\star) \| \leq - \frac{1}{g'(\| z_0 - x^\star \|)}. \tag{2.15}$$

Let us assume that x_k, $y_k \in U(x^\star, r^\star)$ for $k = 0, 1, \cdots, n$. Then by (2.6) $F'(z_k)^{-1}$ exists, and

$$\| F'(z_k)^{-1} \ F'(x^\star) \| \leq - \frac{1}{g'(\| z_k - x^\star \|)}. \tag{2.16}$$

We shall show that x_{k+1}, $x_{k+1} \in U(x^\star, r^\star)$, and estimates (2.13), (2.14) hold true.

Using (1.2) we obtain the identity

$$\begin{aligned} x_{k+1} - x^\star &= x_k - F'(z_k)^{-1} F(x_k) - x^\star \\ &= F'(z_k)^{-1} \ [F'(z_k) (x_k - x^\star) - F(x_k) + F(x^\star)] \\ &= F'(z_k)^{-1} \ [F'(\frac{x_k + x^\star}{2}) (x_k - x^\star) - F(x_k) + F(x^\star)] + \\ & \quad F'(z_k)^{-1} \ [F'(z_k) - F'(\frac{x_k + x^\star}{2})] (x_k - x^\star). \end{aligned} \tag{2.17}$$

In view of (2.4), (2.5), (2.7), (2.10), (2.11), (2.16) and (2.17) we obtain

$$
\begin{aligned}
\| x_{k+1} - x^\star \| &\leq c_k \, [b_k \| x_k - x^\star \|^2 + d_k \; \| y_k - x^\star \|] \; \| x_k - x^\star \| \\
&\leq c(r) \, [b(r) + d(r)] \; \| x_k - x^\star \| \\
&< \| x_k - x^\star \| < r^\star,
\end{aligned}
\tag{2.18}
$$

which shows (2.13) for $n = k$ and $x_{k+1} \in U(x^\star, r^\star)$.
By (1.2) we obtain the identity

$$
\begin{aligned}
y_{k+1} - x^\star \; &= \; F'(z_k)^{-1} \Bigg[[F(x^\star) - F(x_{k+1}) - F'(\frac{x^\star + x_{k+1}}{2}) \; (x^\star - x_{k+1})] + \\
& [F'(\frac{x^\star + x_{k+1}}{2}) - F'(\frac{x_k + y_k}{2})] \; (x^* - x_{k+1}) \Bigg].
\end{aligned}
\tag{2.19}
$$

Using (2.4), (2.5), (2.7), (2.11), (2.12), (2.16) and (2.19) we get

$$
\begin{aligned}
\| y_{k+1} - x^\star \| \; &\leq \; c_k \Bigg[b_{k+1} \| x_{k+1} - x^\star \|^2 + d_k \, (\| x_{k+1} - x^\star \| + \| y_k - x^\star \| + \\
& \| x_k - x^\star \|) \Bigg] \; \| x_{k+1} - x^\star \| \\
&\leq \; c(r) \, [b(r) + 3 \, d(r)] \; \| x_{k+1} - x^\star \| \\
&< \; \| x_{k+1} - x^\star \| < r^\star,
\end{aligned}
\tag{2.20}
$$

which shows (2.14) for $n = k$ and $y_{k+1} \in U(x^\star, r^\star)$. Moreover by letting $k \longrightarrow \infty$ in (2.17), and (2.19) we get $\lim\limits_{k \longrightarrow \infty} x_k = \lim\limits_{k \longrightarrow \infty} y_k = x^\star$.

Finally, to show uniqueness let $y^\star \in \overline{U}(x^\star, r^\star)$ be a solution of equation (1.1).

Using the identity

$$
F(x^\star) - F(y^\star) = \mathcal{L} \, (x^\star - y^\star), \tag{2.21}
$$

where,

$$
\mathcal{L} = \int_0^1 F'(y^\star + t \, (x^\star - y^\star)) \, dt, \tag{2.22}
$$

and Lemma 2.2 for x replaced by $y^\star + t \, (x^\star - y^\star)$ that $\mathcal{L}^{-1}$ exists. Hence, by (2.20), we deduce $x^\star = y^\star$. That completes the proof of the theorem. $\diamondsuit$

In order for us to determine the R–order of the Newton–type method (1.2) we need the Lemma :

Lemma 2.4. *[4], [11]*
Let $0 < \delta_0, \delta_1 < 1$, $p > 1$, $q \geq 0$, $c \geq 0$. If scalar sequence $\{\delta_n\}$ $(n \geq 0)$ satisfies

$$0 < \delta_{n+1} \leq c\, \delta_n^p\, \delta_{n-1}^q \qquad (n \geq 1) \tag{2.23}$$

then it converges to zero with R–order of convergence given by

$$R(p,q) = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q}. \tag{2.24}$$

Let us define functions g_1, g_2 and g_3 on $[0, \frac{1}{\gamma})$ by

$$g_1(r) = \frac{(1-r)^2\, r^2}{4\,(2\,(1-r)^2 - 1)\,(1-r)},$$

$$g_2(r) = \frac{r\,(1-r)^2}{4\,(2\,(1-r)^2 - 1)\,(1 - \frac{r}{2})\,(1-r)},$$

and

$$g_3(r) = g_1(r)\, r + \frac{3}{4}\, \frac{r\,(1-r)}{(1-\frac{r}{2})\,(2\,(1-r)^2 - 1)}.$$

Set

$$\lambda_1 = g_1(r^\star),\ \lambda_2 = g_2(r^\star),\ \text{and } \lambda_3 = g_3(r^\star). \tag{2.25}$$

In view of (2.7), (2.13), (2.14) and (2.23) we get

$$\| x_{n+1} - x^\star \| \leq \lambda_1 \ \| x_n - x^\star \|^3 + \lambda_2\ \| y_n - x^\star \|\ \| x_n - x^\star \|, \tag{2.26}$$

and

$$\| y_{n+1} - x^\star \| \leq \lambda_3\ \| x_{n+1} - x^\star \|\ \| x_n - x^\star \|\ . \tag{2.27}$$

It then follows from (2.24) and (2.25) that there exists $c > 0$ such that (2.21) holds true for $\delta_n = \| x_n - x^\star \|$, $p = 1$ and $q = 1$. Hence, we arrived at:

Corollary 2.5. *Under the hypotheses of Theorem 2.3, the Newton–type method (1.2) is of R–order of convergence* $1+\sqrt{2}$.

Remark 2.6. *As noted in [1], [3], [4], [5], [7], [12] the local results obtained here can be used for projection method such us Arnold's, the generalized minimum residual method (GMRES), the generalized conjugate residual method (GCR), for combined Newton/finite projection methods, and in connection with the mesh independence principle to develop the cheapest and most efficient mesh refinement strategies.*

Remark 2.7. *The local results obtained can also be used to solve equation of the form* $F(x)=0$*, where* F' *satisfies the autonomous differential equation [4]:*

$$F'(x)=P(F(x)), \tag{2.28}$$

where $P : Y \longrightarrow X$ *is a known continuous operator. Since* $F'(x^\star) = P(F(x^\star)) = P(0)$*, we can apply our results without actually knowing the solution of* $x^\star$ *of equation (1.1).*

Example 2.8. *Let* $X=Y=I\!R$*,* $D=U(0,1)$*, and define function* F *on* D *by*

$$F(x)=e^x-1. \tag{2.29}$$

Then, note that we can set $P(x)=x+1$ *in (2.26).*
We must have that conditions (2.3) and (2.4) hold for some $\gamma \geq 0$*. It can easily be seen that we can set* $\gamma = \frac{1}{2}$*. Hence the radius of convergence is* $r^\star = r_A = 2(\frac{5-\sqrt{13}}{6}) = .464816242$*. The radius of convergence* r_W *in [10] is given by*

$$r_W = \frac{1}{2\,\gamma^\star}\,(3-2\sqrt{2}) \tag{2.30}$$

with

$$\gamma^\star = \sup_{k\geq 2} \parallel F'(x^\star)^{-1} F^{(k)}(x^\star) \parallel^{\frac{1}{k-1}} \leq \frac{1}{2}. \tag{2.31}$$

Therefore, (2.27) gives

$$r_W \leq \sqrt{3}-2\sqrt{2} = .171573.$$

Moreover, Rheinboldt radius [9] r_R *is given by*

$$r_R = \frac{2}{3\,l}, \tag{2.32}$$

where l *is the Lipschitz constant in condition:*

$$\| F'(x^\star)^{-1}\,(F'(x) - F'(y)) \| \leq l \ \| x - y \| \quad \text{for all} \quad x, y \in D. \tag{2.33}$$

Using (2.27) and (2.30) we get: $l = e$. *That is*

$$r_R = .245252961.$$

The radius r_{WW} *given by Werner in [11] is defined by*

$$r_{WW} = \frac{2}{\Gamma\, l_1}, \tag{2.34}$$

where

$$\| F'(x^\star)^{-1} \| \leq \Gamma \tag{2.35}$$

and

$$\| F'(x) - F'(y) \| \leq l_1 \ \| x - y \| \tag{2.36}$$

hold true for all $x, y \in D$.

Hence, since $\Gamma = l_1 = e$, *we get by (2.32)*

$$r_{WW} = .270670566. \tag{2.37}$$

Hence, we deduce

$$r_W < r_R < r_{WW} < r_A. \tag{2.38}$$

By comparing r_A *and* r_W *we see that it is always true that*

$$r_W < r_A. \tag{2.39}$$

Moreover note that under (2.2) the existence of $x^\star$ *in* $U(x_0, \frac{1}{\gamma}\,(1 - \frac{1}{\sqrt{2}}))$ *is guaranteed. However, in practice the existence of* $x^\star$ *may have been established by another way that avoids condition (2.2). Finally note that enlarging the radius of convergence is very important in computational* *mathematics since in this case we can obtain a wider range of initial guesses* x_0.

References

[1] E.L. Allgower, K. Böhmer, F.A. Potra, W.C. Rheinboldt, A mesh independence principle for operator equations and their discretizations, SIAM J. Numer. Anal., 23, pp. 160–169, (1986).

[2] S. Amat, S. Busquier, Convergence and numerical analysis of a family of two–step Steffensen's method, Comput. and Math. with Appl., 49, pp. 13–22, (2005).

[3] I.K. Argyros, A unifying local–semilocal convergence analysis and applications for two–point Newton–like methods in Banach space, J. Math. Anal. and Appl., 298, pp. 374–397, (2004).

[4] I.K. Argyros, Approximate solution of operator equations with applications, World Scientific Publ. Comp., New Jersey, U.S.A., (2005).

[5] P.N. Brown, A local convergence theory for combined inexact–Newton/finite difference projection methods, SIAM J. Numer. Anal., 24, pp. 407–434, (1987).

[6] J.M. Gutiérrez, M.A. Hernánadez, M.A. Salanova, Accessibility of solutions by Newton's method, Inter. J. Comput. Math., 57, pp. 239–241, (1995).

[7] L.V. Kantorovich, G.P. Akilov, Functional analysis in normed spaces, Pergamon Press, Oxford, (1982).

[8] R.F. King, Tangent methods for nonlinear equations, Numer. Math., 18, pp. 298–304, (1972).

[9] W.C. Rheinboldt, An adaptive continuation process for solving systems of nonlinear equations, Banach Center Publ., 3, pp. 129–142, (1975).

[10] D. Wang, F. Zhao, The theory of Smale's point estimation and its applications, J. Comput. Appl. Math., 60, pp. 253–269, (1995).

[11] W. Werner, Über ein verfahren ordnung $1+\sqrt{2}$ zür Nullstellenbestimmung, Num. Math., 32, pp. 333–342, (1979).

[12] T.J. Ypma, Local convergence of inexact Newton Methods, SIAM J. Numer. Anal., 21, pp. 583–590, (1984).

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