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A BIRKHOFF TYPE THEOREM FOR STRONG VARIETIES

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Abstract

Algebraic systems with partial operations have different ways to interpret equality between two terms of the language. A strong identity is a formula which says that two terms are equal in the algebra if the existence of one of them implies the existence of the other one and in the case of existence their values are equal. A class of partial algebras defined by a set of strong identities is called a strong variety. In the characterization of strong varieties in the case of partial algebras by means of a Birkhoff-type theorem there appeared a new concept, regularity of partial homomorphisms and partial subalgebras. Here we define and study these operators from two different perspectives. Firstly, in their relation with other well known concecpts of partial homomorphisms and partial subalgebras, as well as with the po-monoid of Pigozzi for the H, S and P operators. Secondly, in regard to the preservation of the different types of formulae that represent equality in the case of partial algebras for these operators. Finally, we give a characterization of the strong varieties as classes closed under regular homomorphisms, regular subalgebras, direct products and that satisfy a closure condition.

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1. Introduction

The study of partial algebras has its origin in the works of Grätzer and Tarski, five decades ago. In [4] , there is a chapter devoted to these systems and most of the concepts and notations that will be used here, are introduced there. At that time, partial algebras appeared as subsets of total algebras that are not closed under the operations of the algebra. At present, partial algebras appear in different fields of mathematics and in many cases, it can be proved that it is impossible to build a total algebra that will work as a completion, in the sense that it contains a copy of the original one, and preserves its structure.

One of the most important theorems about total algebras is due to Birkhoff and says that the classes of (total) algebras defined by identities are exactly those which are closed under homomorphisms, subalgebras and direct products. When dealing with partial algebras, the notions of identity, homomorphism and subalgebra, split into several different notions and it becomes relevant to study the relations between them. We shall define regular homomorphisms and regular subalgebras which are the appropriate notions in order to work with classes of partial algebras which satisfy a set of formulae known as strong identities [see 2.5]. The concepts of regularity allow us to obtain a characterization which generalizes Birkhoff's theorem.

We consider partial algebras as structures with a non-empty set as its universe and with operations that each are defined on a subset of a power of the universe. We use a first order language with finitary operation symbols. The axiomatic system is a generalization of the usual system to which we add some specific axioms in order to assure that the variables will always be assigned to elements of the algebra, to identify all the terms that are not defined and to secure that all the subterms of a defined term are also defined. This axiomatic system was developed by Irene Mikenberg in [5] . Some characterization results for certain equational classes of partial algebras may be found in the works of Mikenberg [5] , who worked with strong identities and built a closure for a particular class of partial algebras closed by normal subalgebras which preserves the strong identities; Peter Burmeister ([2]) who worked with infinitary languages and gave a Birkhoff type theorem for varieties defined by existentially identities; Ferdinand Börner ([1]) who characterized the strong varieties as classes closed by closed homomorphism, closed subalgebras and reduced products; and Bozena and Bogdan Staruch ([7]) who characterized the strong varieties in terms of five operators including closed homomorphisms and closed subalgebras. This work contains another independent characterization which uses neither closed homomorphism nor closed subalgebras but instead a weak notion, the regularity.

2. Preliminaries

In order to obtain a Birkhoff type theorem for a class of partial algebras it is necessary to make precise the concepts of identity, homomorphism and subalgebra that we will use. Definitions of regular homomorphism and subalgebra appear from the need of finding algebraic operators which preserve some kind of identities called strong identities, and they are the key to the characterization theorem for strong varieties. We use the following axiomatic system developed by Mikenberg in [5] .
Let $F_1 = F_1(x_1, \ldots, x_n)$, $F_2 = F_2(x_1, \ldots, x_n)$, $F_3 = F_3(x_1, \ldots, x_n)$ first order formulae and let $\sigma = \sigma(x_1, \ldots, x_n)$, $\tau = \tau(x_1, \ldots, x_n)$ be terms of the language:

(A1) $(F_1 \Rightarrow F_2) \Rightarrow ((F_2 \Rightarrow F_3) \Rightarrow (F_1 \Rightarrow F_3))$

(A2) $(\neg F_1 \Rightarrow F_1) \Rightarrow F_1$

(A3) $F_1 \Rightarrow (\neg F_1 \Rightarrow F_2)$

(A4) $\forall x(F_1 \Rightarrow F_2) \Rightarrow (\forall x\, F_1 \Rightarrow \forall x\, F_2)$

(A5) $\forall x\, F_1 \Rightarrow F_1$

(A6) $F_1 \Rightarrow \forall x\, F_1$, where x does not appear in F_1.

(A7) $\exists x\, (x \approx y)$, where $x \neq y$ and y is a variable symbol.

(A8) $(\forall x\, \neg(x \approx \sigma) \wedge \forall x\, \neg(x \approx \tau)) \Rightarrow \sigma \approx \tau$

(A9) $\sigma \approx \tau \Rightarrow \sigma_1 \approx \tau_1$, where τ_1 is obtained from σ_1 substituting one or more occurrences of τ by σ.

(A10) $\sigma \approx \tau \Rightarrow (\sigma_1 \approx \tau_1 \Rightarrow \sigma_2 \approx \tau_2)$, where the atomic formula $\sigma_2 \approx \tau_2$ is obtained from $\sigma_1 \approx \tau_1$ substituting one or more occurrences of τ by σ.

(A11) $\exists x\, (x \approx \varphi(\sigma_1, \ldots, \sigma_n)) \Rightarrow \bigwedge_{i=1}^{n} \exists x (x \approx \sigma_i)$, where $\sigma_1, \ldots, \sigma_n$ are terms and φ is an n-ary operation symbol of the language.

(A12) $\exists x \exists x_1 \ldots \exists x_n (x \approx \varphi(x_1, \ldots, x_n))$ for all n-ary operation symbol φ of the language.

From now on, $\mathbf{L}$ is a fixed first order language with similarity type Ω of finitary operations symbols, denoted by small Greeks letters $\varphi, \psi, \ldots$. The class of partial algebras which satisfies a set Σ of $\mathbf{L}$-formulas is denoted by $Mod_p(\Sigma)$.

Definition 2.1. *Let $A = \langle A, \varphi^A \rangle_{\varphi \in \Omega}$, $B = \langle B, \varphi^B \rangle_{\varphi \in \Omega}$ be partial algebras of $\mathbf{L}$ and let $h : A \to B$ be a mapping. We say that $h : A \to B$ is a* **weak (partial) homomorphism** *if and only if for any n-ary function symbol φ of $\mathbf{L}$, for all $a_1, \ldots, a_n \in A$, if $(a_1, \ldots, a_n) \in Dom\, \varphi^A$, then $(h(a_1), \ldots, h(a_n)) \in Dom\, \varphi^B$ and in this case, $h(\varphi^A(a_1, \ldots, a_n)) = \varphi^B(h(a_1), \ldots, h(a_n))$. In particular, if c is a constant symbol, $h(c^A) = c^B$.*

Definition 2.2. *Let h be a weak homomorphism from A to B. We say that h is a* **full homomorphism** *if and only if $\varphi^B(h(a_1), \ldots, h(a_n)) = h(a_0)$ implies that there exist elements $a'_0, a'_1, \ldots, a'_n \in A$ such that $h(a_i) = h(a'_i)$ and $\varphi^A(a'_1, \ldots, a'_n) = a'_0$.*

Furthermore, a weak homomorphism h is a **closed homomorphism** *if and only if $\varphi^B(h(a_1), \ldots, h(a_n)) = h(a_0)$ implies that $(a_1, \ldots, a_n) \in Dom\, \varphi^A$.*

If K is a class of partial algebras of the same type, we denote by $H_w(K)$, $H_f(K)$ and $H_c(K)$ respectively the classes of weak, full and closed surjective homomorphic images of K.

If $h : A \to B$ is a closed and bijective homomorphism we say that h is an **isomorphism** from A onto B or that A is isomorphic to B. $I(K)$ denotes the class of algebras which are isomorphic to some partial algebra $A \in K$.

Definition 2.3. *A partial algebra B is a* **weak (partial) subalgebra** *of the partial algebra A if $id_B : B \to A$ is a weak homomorphism, that is to say, for any $b, b_1, \ldots, b_n \in B$ we have that $\varphi^B(b_1, \ldots, b_n) = b$ implies that $\varphi^A(b_1, \ldots, b_n) = b$.*

We say that a partial algebra B is a **relative (partial) subalgebra** *of the partial algebra A if $id_B : B \to A$ is a full homomorphism, that is to say, for any $b, b_1, \ldots, b_n \in B$ we have that $\varphi^A(b_1, \ldots, b_n) = b$ implies that $\varphi^B(b_1, \ldots, b_n) = b$.*

The relative subalgebra B is a **normal subalgebra** *of A if $\varphi^A(a_1,\ldots,a_n) = b \in B$ implies $a_1,\ldots,a_n \in B$.*

The relative subalgebra B is a **closed subalgebra** *of A if $id_B : B \to A$ is a closed homomorphism, that is to say for any $b, b_1,\ldots,b_n \in B$ we have that $(b_1,\ldots,b_n) \in Dom\,\varphi^A$ if and only if $(b_1,\ldots,b_n) \in Dom\,\varphi^B$ and $\varphi^A(b_1,\ldots,b_n) = \varphi^B(b_1,\ldots,b_n)$.*

We use $S_w(K), S_r(K), S_n(K)$, $S_c(K)$ respectively to denote the class of partial algebras which are isomorphic to a weak, relative, normal or closed subalgebra of A.

Definition 2.4. *Let $\{A_i\}_{i\in I}$ be a family of partial algebras of type Ω and let D be a proper filter over I. The* **direct product** *of $\{A_i\}_{i\in I}$ is defined as the partial algebra $A = \langle \prod\limits_{i\in I} A_i, \varphi^A\rangle_{\varphi\in\Omega}$ where for every n-ary operation symbol φ and every $f_1\ldots,f_n \in \prod_{i\in I} A_i$ we have $(f_1\ldots,f_n) \in Dom\,\varphi^A \Leftrightarrow \forall\, i \in I\,((f_1(i),\ldots,f_n(i)) \in Dom\,\varphi_i^A)$.*

The **reduced product** *of $\{A_i\}_{i\in I}$ is the partial algebra $A_D = \langle \prod\limits_D A_i, \varphi_D^A\rangle_{\varphi\in\Omega}$, where $\prod_D A_i = \{[f]_D : f \in \prod_{i\in I} A_i\}$ with $[f]_D = \{g \in \prod\limits_{i\in I} A_i : \{i \in I : f(i) = g(i)\} \in D\}$ and for every n-ary operation symbol φ and every $f_1\ldots,f_n \in \prod_{i\in I} A_i$ we have $([f_1]_D\ldots,[f_n])_D \in Dom\,\varphi_D^A \Leftrightarrow \{i \in I((f_1(i),\ldots,f_n(i)) \in Dom\,\varphi_i^A)\} \in D$. In this case, $\varphi_D^A([f_1]_D\ldots,[f_n])_D = [f]_D$ where*

$$f(i) = \begin{cases} \varphi_i^A(f_1(i),\ldots,f_n(i)) & \text{if } (f_1(i),\ldots,f_n(i)) \in Dom\,\varphi_i^A \\ f_1(i) & \text{if not.} \end{cases}$$

As usual, if D is an ultrafilter we call the algebra A_D, an ultraproduct.

We denote by $P(K)$, $P_r(K)$ and $P_U(K)$ respectively the class of direct product, reduced product and ultraproduct of partial algebras in K.

Like the operators H, S and P on classes of total algebras the set

$$\{H_w, H_f, H_{rg}, H_c, S_w, S_r, S_{rg}, S_n, S_c, P, P_r, P_U\}$$

can be considered as a set of operators on classes of similar partial algebras with the composition. We can compare operators O_1, O_2 by $O_1 \le O_2$ if and only if $O_1(K) \subseteq O_2(K)$ for any class of partial algebras K of the language. Note that $O_1 \le O_2$ implies $QO_1 \le QO_2$ and $O_1Q \le O_2Q$ for any operator Q.

Definition 2.5. *Let σ and τ be terms of the language.*

1. *An e-**identity**, briefly, $\sigma \approx_e \tau$, is the formula $(\exists x\ x \approx \sigma \wedge \exists x\ x \approx \tau \wedge \sigma \approx \tau)$.*

2. *A s**(strong)-identity**, $\sigma \approx_s \tau$, is the formula $(\exists x\ x \approx \sigma \vee \exists x\ x \approx \tau) \to (\sigma \approx \tau)$.*

3. *Let $\{\sigma_1, \ldots, \sigma_n\}$, $\{\tau_1, \ldots, \tau_m\}$ be all the proper subterms of σ and τ respectively. An E-**identity**, $\sigma \approx_E \tau$, is the formula:*

$$(\exists x\ x \approx \sigma \wedge \bigwedge_{j=1}^{m} \exists x\, x \approx \tau_j) \vee (\exists x\ x \approx \tau \wedge \bigwedge_{i=1}^{n} \exists x\, x \approx \sigma_i) \to (\sigma \approx \tau)$$

4. *A w-**identity**, $\sigma \approx_w \tau$, is the formula $(\exists x\ x \approx \sigma \wedge \exists x\ x \approx \tau) \to (\sigma \approx \tau)$.*

A class K of partial algebras is an e, s, E, w-variety if there exists a set Σ of e, s, E, w-identities respectively such that K is precisely the class of partial algebras which satisfies Σ, that is to say $K = Mod_p(\Sigma)$.

3. Properties of the operations on classes of partial algebras

First we give the definition of regular homomorphism and regular subalgebra. Let A, B and C be partial algebras of the same type of similarity.

1. A weak homomorphism $h : A \to B$ is a **regular homomorphism** if and only if for every term τ of the language $\mathbf{L}$, if $\tau^B(h(a_1), \ldots, h(a_n)) = h(a_0)$, there are elements $a'_0, a'_1, \ldots, a'_n \in A$ such that $h(a_i) = h(a'_i)$ and $\tau^A(a'_1, \ldots, a'_n) = a'_0$. We use $H_{rg}(K)$ to denote the class of partial algebras which are regular homomorphic images of some partial algebra $A \in K$.

2. We say that C is a **regular subalgebra** of A if $id_C : C \to A$ is a regular homomorphism, that is to say, if for every term τ of the language and $c_1, \ldots, c_n \in C$, $\tau^A(c_1, \ldots, c_n) = c \in C$, then $\tau^C(c_1, \ldots, c_n)$ is defined in C and it is equal to c. Similarly $S_{rg}(K)$ denotes the class of all partial algebras which are isomorphic to a regular subalgebra of some algebra $A \in K$.

We will start the study of the algebraic operators with the following crucial theorem.

Theorem 3.1. Homomorphism theorem for partial algebras. *Let A, B partial algebras of the same type and $h : A \to B$ a weak homomorphism. Then:*

1. *For all n-ary terms τ of the language, $\tau^A[a_1, \ldots, a_n] = a_0 \in A$ implies that $\tau^B[h(a_1), \ldots, h(a_n)) = h(a_0) \in B$. In particular, if $A \models \exists x\, x \approx \tau[a_1, \ldots, a_n]$ then $B \models \exists x\, x \approx \tau[h(a_1), \ldots, h(a_n)]$. Moreover, if h is a closed and surjective homomorphism, $\tau^B(h(a_1), \ldots, h(a_n)) = h(a_0) \in B$ implies that $\tau^A(a_1, \ldots, a_n) \in A$.*

2. *For all formula $F(x_1, \ldots, x_n)$ of the language such that for all subterm σ of F we have $\sigma^A[a_1, \ldots, a_n]$ is defined in A, then $A \models F[a_1, \ldots, a_n]$ implies that $B \models F[h(a_1), \ldots, h(a_n)]$. The condition of the existence of the subterms of F can be removed when h is a closed homomorphism.*

Proof: By induction of the length of the terms and formulas taking into account the special axioms (A7), (A8), (A11) and (A12).

Corollary 3.2. *If B is a relative subalgebra of A, then for all n-ary terms τ of the language and all $b_1, \ldots, b_n \in B$, if $\tau^B(b_1, \ldots, b_n) = b \in B$ then $\tau^A(b_1, \ldots, b_n) = b \in A$.*

The basic properties of the regular homomorphism and regular subalgebras with respect to the other operators are established in the following

Theorem 3.3. *If H_p, $H_q \in \{H_c, H_{rg}, H_f, H_\omega\}$ and S_p, $S_q \in \{S_c, S_n, S_{rg}, S_r, S_w\}$ and $P_\alpha \in \{P, P_r, P_U\}$ then:*

1. $H_c < H_{rg} < H_f < H_w$
2. $S_c < S_{rg} < S_r < S_w$
3. $S_n < S_{rg}$
4. S_n *and* S_c *are not comparable.*
5. $S_pS_p = S_p$ *and* $H_pH_q = \max\{H_p, H_q\}$.
6. $S_qH_p \leq H_pS_q$
7. $P_\alpha H_p < H_pP_\alpha$, $P_\alpha S_q < S_qP_\alpha$
8. $S_nS_c = S_cS_n = S_{rg}$

Proof:

We just check those assertions concerning to regularity since the others are well known, see [2] .

1. Let $h : A \to B$ be a closed homomorphism and let τ be a term of the language such that $\tau^B(h(a_1), \ldots, h(a_n)) = h(a_0)$ for some elements $a_0, a_1, \ldots, a_n \in A$. By induction over the length of the term τ, we have that $\tau^A(a_1, \ldots, a_n) = a_0$, which implies that h is regular. The other inequality is obvious.

   To see that $H_c \neq H_{rg}$, let $A = \langle A, \varphi^A \rangle$, $B = \langle B, \varphi^B \rangle$ be the following mono-unary partial algebras: $A = \{a, b, c, d\}$, $\varphi(a) = \varphi(b) = \varphi(c) = c$ and not defined in d and $B = \{0, 1\}$, $\varphi(0) = \varphi(1) = 1$. Note that all the terms of the language are $\varphi^n(x)$, where $\varphi^0(x) = x$ and $\varphi^{n+1}(x) = \varphi(\varphi^n(x))$. Let $h : A \to B$ be the mapping defined by $h(a) = h(d) = 0$, $h(b) = h(c) = 1$. The next table shows that $h : A \to B$ is a regular but not closed homomorphism from A onto B. We omit the superscripts for simplicity.

x	a	b	c	d
$h(\varphi(x))$	1	1	1	-
$\varphi(h(x))$	1	1	1	1
$\varphi^n(h(x))$	1	1	1	1
$h(\varphi^n(x))$	1	1	1	-

   Finally, to see that $H_{rg} \neq H_f$, consider the partial algebras A, B and the mapping h as above, except that now $\varphi(c)$ is not defined. Then h is a full but not regular homomorphism. In fact, for every $x \in A$ we have $\varphi(h(x)) = 1 = h(c) = h(\varphi(a))$ and $\varphi^2(h(a)) = 1 = h(c) \in B$, but $\varphi^2(x)$ is not defined in A.

2. The proof of $S_c \leq S_{rg} \leq S_r$ is straightforward. To prove that equality does not hold, let $A = \langle A, \mathbf{s} \rangle$ where $A = \{0, 1, 2, 3, 4\}$ and $\mathbf{s}$ represents the successor function (hence $\mathbf{s}(4)$ is not defined in A). Let B the relative subalgebra of A with universe $B = \{1, 2\}$. All terms of the language are of the form $\mathbf{s}^n$, hence $B \in S_{rg}(A)$, but $B \notin S_c(A)$ because $\mathbf{s}(2)$ is not defined in B. On the other hand, the relative subalgebra C of A with universe $C = \{0, 3\}$ is not regular, because $s^3(0) = 3$ is defined in A but it is not defined in C.

3. The assertion $S_n \leq S_{rg}$ is immediate from the definition. The regular subalgebra B of the former example is not a normal subalgebra of A because $s(0) \in B$, which shows that we have strict inequality.

4. Let $A = \langle A, s \rangle$ the partial algebra of item 2. and let B' be the relative subalgebra of A with universe $\{0, 1, 2\}$. It is clear that B' is a normal but not closed subalgebra of A. On the other hand, the relative subalgebra C' of A with universe $\{2, 3, 4\}$ is a closed but not normal subalgebra of A.

5. Straightforward.

6. Let $B \in S_q H_{rg}(A)$. Then, there exists a partial algebra C and a mapping $h : A \to C$ such that $B \in S_q(C)$ and h is a regular homomorphism from A onto C. Consider the relative subalgebra $h^{-1}(B) \in S_r(A)$ with universe $h^{-1}[B]$. We will prove that $h_{h^{-1}(B)}$ is a regular homomorphism onto B. Let τ be a term of the language and let $a_0, a_1, \ldots, a_n \in h^{-1}[B]$ such that $\tau^B(h(a_1), \ldots, h(a_n)) = h(a_0)$. Then there exist $a'_0, \ldots, a'_n \in A$ such that $h(a_i) = h(a'_i), i = 1, \ldots, n$ and $\tau^A(a'_1, \ldots, a'_n) = a'_0$. But $h(a'_i) \in B$, therefore $a'_i \in h^{-1}[B]$. Hence $B \in H_{rg}S_r(A) \subseteq H_{rg}S_w(A)$. Now, if $B \in S_{rg}(C)$ and $\tau^A(b_1, \ldots, b_n) \in h^{-1}[B]$ with $b_1, \ldots, b_n \in h^{-1}[B]$ then $h(b_1), \ldots, h(b_n) \in B$ and $\tau^C(h(b_1), \ldots, h(b_n)) \in B$ which implies $\tau^B(h(b_1), \ldots, h(b_n))$ is defined in B. Hence $\tau^{h^{-1}(B)}(b_1, \ldots, b_n)$ is defined in $h^{-1}(B)$ which implies that $h^{-1}(B) \in S_{rg}(A)$. If $B \in S_n(C)$ and $\varphi^A(a_1, \ldots, a_n) = b \in h^{-1}[B]$,then $\varphi^C(h(a_1), \ldots, h(a_n)) = h(b) \in B$ and therefore $h(a_1), \ldots, h(a_n) \in B$ which implies $a_1, \ldots, a_n \in h^{-1}[B]$. Similarly we prove that $B \in S_c(C)$ implies $h^{-1}(B) \in S_c(A)$. Hence $B \in H_{rg}S_q(A)$. In the same way we prove that $S_{rg}H_p \leq H_p S_{rg}$.

7. Similar to the above proof.

8. Let $B \in S_n S_c(A)$. Then, there exists a partial algebra C such that $C \in S_c(A)$ and $B \in S_n(C)$. We will construct recursively a partial algebra D such that B is a closed subalgebra of D. Let $D = A \setminus \bigcup X_n$, where:

   $X_1 = \{a \in C \setminus B : \exists \varphi \in \Omega \, \exists b_1, \ldots, b_n \in B \, (\varphi^B(b_1, \ldots, b_n) = a\}$,

   $X_{n+1} = \{a \in C \setminus B : \exists \varphi \in \Omega \, \exists b_1, \ldots, b_n \in X_n \, (\varphi^C(b_1, \ldots, b_n) = a\}$

   Let D be the relative subalgebra of A with universe D. Then $B \in S_c(D)$ and $D \in S_n(A)$ by construction.

Now, let $B \in S_cS_n(A)$. Then, there exists a partial algebra C such that $C \in S_n(A)$ and $B \in S_c(C)$. We will construct recursively a partial algebra D such that B is a normal subalgebra of D. Let $D = A \setminus \bigcup X_n$, where

$X_1 = \{a \in C \setminus B : \exists \varphi \in \Omega\, \exists b_1, \dots, b_n \in C\, \varphi^C(b_1, \dots, b_n) \in B, \text{ and } a = b_i, \text{ for some } i \in \{1, \dots, n\}\}$,

$X_{n+1} = \{a \in C \setminus B : \exists \varphi \in \Omega\, \exists b_1, \dots, b_n \in C\, (\varphi^C(b_1, \dots, b_n) \in X_n, \text{ and } a = b_i, \text{ for some } i \in \{1, \dots, n\}\}$

Let D be the relative subalgebra of A with universe D. Then $B \in S_n(D)$ and $D \in S_c(A)$ by construction.

Finally, it is clear that if $B \in S_nS_c(A)$ then $B \in S_{rg}(A)$. Suppose that $B \in S_{rg}(A)$. Let $X = \{a \in A \setminus B$: there exists a term τ and there exist elements $b_1, \dots, b_n \in B$ such that $\tau^A(b_1, \dots, b_n) = a\}$ and let C be the relative subalgebra of A with universe $A \setminus X$. Then $B \in S_c(C)$ and $C \in S_n(A)$.

Proposition 3.4. *Let K be a class of total algebras of the same type Ω such that $K = SI(K)$ (S means total subalgebras). Then $S_n(K) = S_{rg}(K)$.*

Proof: $S_n(K) = S_nS(K) = S_nS_c(K) = S_{rg}(K)$.

Observation 3.5. *The operators S_n and S_{rg} are in general different for arbitrary classes of partial algebras and they are different for arbitrary classes of total algebras. This means that a regular subalgebra of a total algebra is not necessarily a normal subalgebra.*

In [6] the lattice of the algebraic operators on classes of total algebras H, S and P is characterized as the partially ordered monoid (po-monoid) with three generators and defining relations:

$$HH = H,\ SS = S,\ PP = P$$
$$1 \leq H,\ 1 \leq S,\ 1 \leq P$$
$$SH \leq HS,\ PH \leq HP,\ PS \leq SP$$

It is shown in that paper that this lattice, called S, has exactly 18 elements and any po-monoid with three generators and the above defining relations is an homomorphic image of S with at least 16 elements. The only relationships which are not directly obtained from the above conditions are $SHPS \neq HSP$ ($SHPS \leq HSP$ is always true) and $HPSPHS$. Since operators H_p, S_q and P, verify the defining relations we have the following

Corollary 3.6. *Let $H_p \in \{H_c, H_{rg}, H_f, H_\omega\}$ and $S_q \in \{S_c, S_n, S_{rg}, S_r, S_w\}$ and $P_\alpha \in \{P, P_r, P_U\}$. Then the operators H_p, S_q and P_α generate a po-monoid with at least 16 elements which is a homomorphic image of the lattice S.*

We don't know yet if $S_{rg}H_{rg}PS_{rg} \neq H_{rg}S_{rg}P$ and $H_{rg}PS_{rg}PH_{rg}S_{rg}$.

Definition 3.7. *Let O be an operator and F a formula of the language. We say that O preserves the formula F if for any class K of the language, $K \models F \Rightarrow O(K) \models F$.*

In [2] it is proved that e-identities are preserved under H_w, S_c and P. We will prove similar results for the others identities. It is clear that if $O_1 \leq O_2$ and F is a formula, then O_2 preserves F implies that O_1 preserves F.

Proposition 3.8. *Let σ, τ be a terms of the language. Then*

1. *$\sigma \approx_s \tau$ is preserved by H_{rg} and S_{rg} but not by H_f and S_r.*
2. *$\sigma \approx_E \tau$ preserved by H_c and S_r but not by H_{rg} and S_w.*
3. *$\sigma \approx_w \tau$ preserved by H_c and S_w but not by H_{rg}.*

Proof:

Preservation is straightforward. For instance consider 1. and let $h : A \to B$ be a regular homomorphism from A onto B and let $\sigma \approx_s \tau(x_1, \ldots, x_n)$ be a s-identity such that $A \models \sigma \approx_s \tau$. Let $b_1, \ldots, b_n \in B$ such that $B \models \exists\, x\, x \approx \sigma[b_1, \ldots, b_n]$. This means that there exists an element $b_0 \in B$ such that $\sigma^B(b_1, \ldots, b_n) = b_0$. Because h is a surjective regular homomorphism, there exist $a_0, a_1, \ldots, a_n \in A$ such that for $i = 0, 1, \ldots, n$, $h(a_i) = b_i$ and $\sigma^A(a_1, \ldots, a_n) = a_0$. Hence $A \models \exists\, x\, x \approx \sigma[a_1, \ldots, a_n]$, so it follows that $A \models \exists\, x\, x \approx \tau[a_1, \ldots, a_n]$. Moreover, $\tau^A(a_1, \ldots, a_n) = a_0$.

Now, we give counterexamples for non preservation.

1. Let $A = \langle A, \varphi^A, \psi^A \rangle$ be the partial algebra of type $(1, 2)$ with universe $A = \{\, 0, 1, 2, 3, 4, 5, 6, 7\}$ such that $Dom\, \varphi = \{\, 0, 2, 5, 6\}$, $Dom\, \psi = \{\, (0, 2), (5, 5), (5, 6)\}$ and defined by $\varphi(0) = 1$, $\varphi(2) = 4$, $\varphi(5) = 6$, $\varphi(6) = 7$, $\psi(0, 2) = 3$, $\psi(5, 5) = 7$, $\psi(5, 6) = 7$.

   Let $B = \langle B, \varphi^B, \psi^B \rangle$ be the partial algebra of type $(1, 2)$ with universe $B = \{\, a, b, c, d, e, f, g\}$ and $Dom\, \varphi^B = \{\, a, b, e, f\}$, $Dom\, \psi^B =$

$\{(a,b), (e,e), (e,f)\}$ and defined by $\varphi(a) = b$, $\varphi(b) = d$, $\varphi(e) = f$, $\varphi(f) = g$, $\psi(a,b) = c$, $\psi(e,e) = g$, $\psi(e,f) = g$. Let $h : A \to B$ be the following mapping:

x	0	1	2	3	4	5	6	7
$h(x)$	a	b	b	c	d	e	f	g

It is easy to check that h is a full homomorphism from A onto B.

Then $A \models \psi(x,\varphi(x)) \approx_s \varphi(\varphi(x))$ because 5 is the only element of A which satisfies the formula $\exists x\, x \approx \psi(x,\varphi(x)) \vee \exists x\, x \approx \varphi(\varphi(x))$ but B does not preserve the s-identities of A because $\varphi(\varphi(a)) = c$ and $\psi(a,\varphi(a)) = d$. Moreover, the relative subalgebra C of A with universe $\{5,6,7\}$ shows that the s-identities are not preserved under S_r.

2. Consider the following partial algebras of type $(1,1)$ (The symbol ' - ' means that the operation is not defined):

x	a	b	c	d	e	f	g	h	i	j
$\varphi(x)$	-	c	-	e	-	-	h	-	j	-
$\rho(x)$	b	-	-	-	f	-	i	j	-	-

x	0	1	2	3	4	5	6	7	8
$\varphi(x)$	1	-	-	4	-	6	-	8	-
$\rho(x)$	3	2	-	-	-	7	8	-	-

Let $h : A \to B$ defined by :

x	a	b	c	d	e	f	g	h	i	j
$h(x)$	0	3	4	0	1	2	5	6	7	8

Then h is a partial homomorphism from A onto B:

$x \in Dom\,\varphi^A$	b	d	g	i		$x \in Dom\,\rho^A$	a	e	g	h
$h(\varphi(x))$	4	1	6	8		$h(\rho(x))$	3	2	7	8
$\varphi(h(x))$	4	1	6	8		$\rho(h(x))$	3	2	7	8

The next table shows the behaviour of all the terms defined in B.

$\varphi(\rho(0)) = 4$	$0 = h(a)$	$h(\varphi(\rho(a))) = h(c) = 4$
$\varphi(\rho(5)) = 8$	$5 = h(g)$	$h(\varphi(\rho(g))) = h(j) = 8$
$(\psi(\varphi(0)) = 2$	$0 = h(d)$	$h(\rho(\varphi(d))) = h(f) = 2$
$\rho(\varphi(5)) = 8$	$5 = h(g)$	$h(\rho(\varphi(g))) = h(j) = 8$

It follows that h is a regular homomorphism.

Now, consider the E-identity $\varphi(\rho(x)) \approx_E \rho(\varphi(x))$. It is easy to see that $A \models \varphi^A(\rho^A(x) \approx_E \rho^A(\varphi^A(x))$ but we have $\varphi^B(\psi^B(0)) = 4$ and $\rho^B(\varphi^B(0)) = 2$.

Finally, the weak subalgebra C of A does not satisfy the E-identity $\varphi(\rho(x)) \approx_E \rho(\varphi(x))$:

x	g	h	i	j
$\varphi(x)$	h	-	-	-
$\rho(x)$	i	j	-	-

3. The example just described in 2 shows that there is no preservation under regular homomorphisms using the w-identity $\varphi(\psi(x)) \approx_w \psi(\varphi(x))$.

4. An w-identity is a particular case of an e-quasi-identity, hence there is no preservation under H_{rg}. Now, let $A = \langle \{0, 1, 2, 3\}, \varphi\rangle$ with $\varphi(0) = 0$, $\varphi(1) = 2$, $\varphi(2) = \varphi(3) = 3$ and consider the normal (and regular) subalgebra B with universe $B = \{0, 1, 2\}$. Then, the following e-quasi-identity is true in A and false in B: $\varphi(y) \approx_e \varphi(y) \to \varphi(\varphi(y)) \approx_e \varphi(\varphi(\varphi(y)))$.

   To show that there is no preservation under closed homomorphism consider the following algebras A and B of type $(1, 1)$:

$x \in A$	0	1	2	3	4
$\varphi(x)$	2	-	-	4	4
$\rho(x)$	1	-	-	4	4

$x \in B$	a	b	c	d
$\varphi(x)$	b	-	d	d
$\rho(x)$	b	-	d	d

The mapping $h : A \to B$ is defined by:

$x \in A$	0	1	2	3	4
$h(x)$	a	b	b	c	d

Then h is a closed homomorphism from A onto B:

$x \in A$	0	1	2	3	4
$h(\varphi(x))$	b	-	-	d	d
$\varphi(h(x))$	b	-	-	d	d
$h(\rho(x))$	b	-	-	d	d
$\rho(h(x))$	b	-	-	d	d

4. Characterization of strong varieties

Definition 4.1. *Let* $\mathbf{A}$ *be a partial algebra and let* $\mathbf{B}$ *be a total algebra with the same similarity type. We say that* $\mathbf{B}$ *is a closure or* **completion** *of* $\mathbf{A}$ *if and only if* $\mathbf{A} \in S_r(\mathbf{B})$ *and* A *generates* B.

In this sense, the free completion $\mathbf{T}(\mathbf{A})$ of $\mathbf{A}$ generated by $\mathbf{A}$ like in a term algebra as usual is the **greatest completion** of the algebra $\mathbf{A}$.

Definition 4.2. *Let* $\mathbf{A} = \langle A, \varphi^{\mathbf{A}} \rangle_{\varphi \in \Omega}$ *be a partial algebra such that not all the operations are total in* $\mathbf{A}$. *We define the* **trivial completion** *for an external point of* $\mathbf{A}$ *as the algebra* $\mathbf{A}^\bullet = \langle A^\bullet, \varphi^{\mathbf{A}^\bullet} \rangle_{\varphi \in \Omega}$ *where* $p \notin A$, $A^\bullet = A \cup \{p\}$, *and for every* n*-ary operation symbol* φ, $a_1, \ldots, a_n \in A^\bullet$, $\varphi^{\mathbf{A}^\bullet}(a_1, \ldots, a_n) = \varphi^{\mathbf{A}}(a_1, \ldots, a_n)$ *if* $(a_1, \ldots, a_n) \in Dom\varphi^{\mathbf{A}}$ *and it is equal to* p *if not.*

If the algebra $\mathbf{A}$ *is a total algebra, then we define* $\mathbf{A}^\bullet = \mathbf{A}$.

Note first that the next proposition is obvious.

Proposition 4.3. *Let* $\mathbf{A}$ *be a partial algebra,* $\mathbf{A}^\bullet$ *its trivial completion. Then* $\mathbf{A} \in S_n(\mathbf{A}^\bullet)$ *and if* σ *and* τ *are* n*-ary terms of the language, then*

$$\mathbf{A} \models \sigma \approx_s \tau \text{ if and only if } \mathbf{A}^\bullet \models \sigma \approx \tau$$

Since for any partial algebra we can construct a total algebra which preserves its strong identities, we get a Birkhoff type theorem that characterizes strong varieties.

Theorem 4.4. *Let $\mathcal{K}$ be a nonempty class of partial algebras and let $\mathcal{K}^\bullet = \{\mathbf{A}^\bullet \,:\, \mathbf{A} \in \mathcal{K}\}$, $\overline{\mathcal{K}} = \{\mathbf{A} \in \mathcal{K} \,:\, \mathbf{A}$ is a total algebra$\}$. Then, the following are equivalent*

a) $\mathcal{K}$ is a strong variety.
b) $\mathcal{K} = H_{rg}S_{rg}P(\mathcal{K})$ and $\mathcal{K}^\bullet = \overline{\mathcal{K}}$.
c) $\mathcal{K} = S_n(\overline{\mathcal{K}})$ and $\overline{\mathcal{K}}$ is a variety.
d) $\mathcal{K} = S_{rg}(\overline{\mathcal{K}})$ and $\overline{\mathcal{K}}$ is a variety.
e) $\mathcal{K} = S_{rg}(\mathcal{K}^\bullet)$ and $\mathcal{K}^\bullet$ is closed with respect to H and P.
f) $\mathcal{K}^\bullet$ is a variety and $\mathcal{K} = \{\mathbf{A} : \mathbf{A}^\bullet \in \mathcal{K}^\bullet\}$.

Proof:
a) $\Rightarrow$ *b)* It is easy to see that strong identities are preserved under regular homomorphisms, regular subalgebras and also under direct products. Then $\mathcal{K} = H_{rg}S_{rg}P(\mathcal{K})$. Furthermore, by definition, $\mathbf{A}^\bullet = \mathbf{A}$ if $\mathbf{A}$ is a total algebra, then $\overline{\mathcal{K}} \subseteq \mathcal{K}^\bullet$. But if $\mathbf{A} \in \mathbf{K}$ then the total algebra $\mathbf{A}^\bullet$ preserves the strong identities of the original partial algebra, hence $\mathbf{A}^\bullet \in \overline{\mathcal{K}}$. Therefore the other inclusion is true and we have $\mathcal{K}^\bullet = \overline{\mathcal{K}}$.

b) $\Rightarrow$ *c)* The completion $\mathbf{A}^\bullet$ contains the original partial algebra $\mathbf{A}$ as a normal subalgebra and, by hypothesis, it is in the class $\overline{\mathcal{K}}$. Furthermore, if $\mathbf{A}$ is a total algebra, then $HSP(\mathbf{A}) \subseteq \mathbf{H_{rg}S_{rg}P}(\mathbf{A})$, hence $HSP(\overline{\mathcal{K}}) \subseteq \overline{\mathcal{K}}$ and $\overline{\mathcal{K}}$ is a variety.

c) $\Rightarrow$ *d)* Given that $\overline{\mathcal{K}}$ is closed under subalgebras and every normal subalgebra is regular, we will prove that $S_n(\overline{\mathcal{K}}) = S_{rg}(\overline{\mathcal{K}})$. Although in general the operators S_n and S_{rg} are different.

Let $\mathbf{A} \in \overline{\mathbf{K}}$, $\mathbf{B} \in S_{rg}(\mathbf{A}) \setminus \mathbf{S_n}(\mathbf{A})$. We will construct recursively a subalgebra $\mathbf{C}$ of $\mathbf{A}$ such that $\mathbf{B}$ is a normal subalgebra of $\mathbf{C}$:

Let $C = A \setminus \bigcup C_m$ where

$$C_1 = \{a \in A \setminus B : \exists\, \varphi \in \Omega\ \exists\, (a_1, \dots, a_n) \in Dom\, \varphi^{\mathbf{A}}\ (a \in \{a_1, \dots, a_n\} \text{ and } \varphi^{\mathbf{A}}(a_1, \dots, a_n) \in B)\}$$

$$C_{m+1} = \{a \in A \setminus B : \exists\, \varphi \in \Omega\ \exists\, (a_1, \dots, a_n) \in Dom\, \varphi^{\mathbf{A}}\ (a \in \{a_1, \dots, a_n\} \text{ and } \varphi^{\mathbf{A}}(a_1, \dots, a_n) \in C_m)\}$$

We remove the elements of A that prevent $\mathbf{B}$ from being a normal subalgebra of $\mathbf{A}$.

It is important to note that if φ is an n-ary operation symbol and $b_1, \dots, b_n \in B$, then $\varphi^{\mathbf{A}}(b_1, \dots, b_n) \in C$, because $\mathbf{B}$ is a regular subalgebra of $\mathbf{A}$.

Let $\mathbf{C}$ be the relative subalgebra of $\mathbf{A}$ with universe C. Note that $\mathbf{C}$ is a total algebra and therefore $\mathbf{C} \in S(\mathbf{A})$.

Indeed, if φ is an n-ary operation symbol and $c_1, \dots, c_n \in C$, then the element $\varphi^{\mathbf{C}}(c_1, \dots, c_n) \in C$ by construction, otherwise each $c_i \in \{c_1, \dots, c_n\}$ should be in some C_{m_i}.

Finally, $\mathbf{B} \in S_n(\mathbf{C})$, by construction of C_1.

d) $\Rightarrow$ *e*) In order to prove e) we use the fact that $\overline{\mathcal{K}}$ is a variety, then, it satisfies a set of identities. The set of strong identities that arise from such set is preserved under regular subalgebras and therefore we have $\mathcal{K}^\bullet = \overline{\mathcal{K}}$.

e) $\Rightarrow$ *f*) It is not difficult to see that every (total) subalgebra of a total algebra $\mathbf{A}$ is a regular subalgebra of $\mathbf{A}$, too. Hence $\mathcal{K}^\bullet$ is closed under H, S and P that is to say $\mathcal{K}^\bullet$ is a variety. Furthermore, $\mathbf{A}^\bullet \in \mathcal{K}^\bullet$ implies $\mathbf{A} \in \mathbf{S_{rg}}(\mathcal{K}^\bullet)$, hence $K = \{\mathbf{A} : \mathbf{A}^\bullet \in \mathcal{K}^\bullet\}$.

f) $\Rightarrow$ *a*) If $\mathcal{K}^\bullet$ is a variety, there exists a set Σ of identities such that $\mathcal{K}^\bullet = Mod(\Sigma)$. We consider the set Σ_s of the strong identities that arises from the identities of Σ that axiomatize $\mathcal{K}^\bullet$. Then using the fact that $\mathbf{A} \in S_n(\mathbf{A}^\bullet)$, we obtain that $\mathcal{K}$ is a strong variety.

■

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