Proyecciones
Vol. 27, N$^o$ 2, pp. 171–183, August 2008.
Universidad Católica del Norte
Antofagasta - Chile
DOI:10.4067/S0716-09172008000200004

# EXISTENCE OF SOLUTIONS OF SEMILINEAR SYSTEMS IN $\ell^2$ *

*RUBÉN HIDALGO*
*UNIVERSIDAD TÉCNICA FEDERICO SANTA MARÍA, CHILE*
*and*
*MAURICIO GODOY*
*UNIVERSITY OF BERGEN, NORWAY*

*Received : March 2008. Accepted : July 2008*

**Abstract**

*Let $Q : \ell^2 \to \ell^2$ be a symmetric and positive semi-definite linear operator and $f_j : \mathbf{R} \to \mathbf{R}$ $(j = 1, 2, ...)$ be real functions so that, $f_j(0) = 0$ and, for every $x = (x_1, x_2, ....) \in \ell^2$, it holds that $f(x) := (f_1(x_1), f_2(x_2), ...) \in \ell^2$. Sufficient conditions for the existence of non-trivial solutions to the semilinear problem $Qx = f(x)$ are provided. Moreover, if $G$ is a group of orthogonal linear automorphisms of $\ell^2$ which commute with $Q$, then such sufficient conditions ensure the existence of non-trivial solutions which are invariant under $G$. As a consequence, sufficient conditions to ensure solutions of nonlinear partial difference equations on finite degree graphs with vertex set being either finite or infinitely countable are obtained. We consider adaptations to graphs of both Matukuma type equations and Helmholtz equations and study the existence of their solutions.*

**Keywords :** *Graphs, Partial difference equations, Nonlinear elliptic equations, Laplacian*

**Subjclass :** *[2000] 05C12, 39A12, 35J05.*

---

*Partially supported by Projects Fondecyt 1070271 and UTFSM 12.08.01

## 1. A semilinear problem in $\ell^2$

Let us consider a Hilbert space $\mathcal{H}$, a positive semi-definite linear operator $Q : \mathcal{H} \to \mathcal{H}$ and a function $f : \mathcal{H} \to \mathcal{H}$ so that $f(0) = 0$. Associated to the above is the semilinear problem

$$Qx = f(x). \tag{1.1}$$

In many problems of different areas, such as biology, computer science, economy, ecology and difference equations, one searches for the existence of non-trivial solutions of a problem of type (1.1).

In [6,10] sufficient conditions for the existence of non-trivial solutions of (1.1) are provided if $\mathcal{H} = \mathbf{R}^n$ and $f(x_1, ..., x_n) = (f_1(x_1), f_2(x_2), ..., f_n(x_n))$, where $f_j : \mathbf{R} \to \mathbf{R}$ are continuous functions so that $f_j(0) = 0$. In particular, the results in there were used to study the existence of solutions of partial difference equations on finite graphs.

The aim of this note is to see that such conditions can be naturally extended to the infinite countable case

$$\mathcal{H} = \ell^2 = \{x = (x_1, x_2, ...) : x_j \in \mathbf{R},\ \sum_{j=1}^{\infty} x_j^2 < \infty\},\ < x, y >= \sum_{j=1}^{\infty} x_j y_j,$$

and $f(x) := (f_1(x_1), f_2(x_2), ...)$ where $f_j : \mathbf{R} \to \mathbf{R}$ satisfy $f_j(0) = 0$ and, for every $x = (x_1, x_2, ....) \in \ell^2$, it holds that $f(x) := (f_1(x_1), f_2(x_2), ...) \in \ell^2$. We apply this existence results to study the existence of solutions of partial difference equations on graphs with an infinite countable set of vertices.

A function $f : \mathbf{R} \to \mathbf{R}$, with $f(0) = 0$, is said locally increasing at 0 (respectively, locally decreasing at 0) if there is some $\epsilon > 0$ so that $f$ restricted to $(-\epsilon, \epsilon)$ is increasing (respectively, decreasing).

**Theorem 1.** *Let $Q : \ell^2 \to \ell^2$ be a symmetric and positive semi-definite linear operator and $\{0\} \neq W < \ell^2$ be an invariant subspace under $Q$. Assume there are continuous maps $f_j : \mathbf{R} \to \mathbf{R}$, for $j = 1, 2, ...$, so that they all satisfy the following conditions*

*(1) $f_j(0) = 0$;*

*(2) either,*

*(2.1) for all $j$, $f_j$ is locally decreasing at $0$, or*

*(2.2) for all $j$, $f_j$ is locally increasing at $0$ and, for small $|t|$, it holds $\int_0^t f_j(s)\, ds < t^2$;*

*(3)* $\lim_{t\to+\infty} \frac{f_j(t)}{t} = +\infty$, *and*

*(4)* $\left|\int_0^{x_j} f_j(t)\,dt\right| \leq x_j^2$, *for every* $x = (x_1, ....) \in \ell^2$,

*then the semilinear problem (1.1) has non-trivial solutions in* $W$. *Moreover, if* $G$ *is a group of orthogonal linear automorphisms of* $\ell^2$ *so that each transformation in* $G$ *commutes with* $Q$, *then the semilinear problem (1.1) has non-trivial solutions invariant under the action of* $G$.

**Remark 2.** *Condition (2.2) in Theorem 1 trivially holds if, for every* $j$, $f_j$ *is locally increasing at* 0 *and, for small* $|t| \neq 0$, *it holds that* $f(t)/t < 2$.

In the finite dimensional situation this is not required. As we may identify $\mathbf{R}^n$ with the subspace of $\ell^2$ defined by the condition $x_j = 0$, for $j > n$, then we have the following consequence that extends a result in [6].

**Corollary 3.** *Let* $Q : \mathbf{R}^n \to \mathbf{R}^n$ *be a symmetric and positive semi-definite linear operator and* $\{0\} \neq W < \mathbf{R}^n$ *be an invariant subspace under* $Q$. *Assume there are continuous maps* $f_j : \mathbf{R} \to \mathbf{R}$, *for* $j = 1, 2, ...n$, *so that they all satisfy conditions (1), (2) and (3) of Theorem 1, then the semilinear problem (1.1) has non-trivial solutions in* $W$. *Moreover, if* $G$ *is a group of orthogonal linear automorphisms of* $\mathbf{R}^n$ *so that each transformation in* $G$ *commutes with* $Q$, *then the semilinear problem (1.1) has non-trivial solutions which are invariant under the action of* $G$.

The proof of Theorem 1 is provided in Section 3. In Section 2 we use the above existence results to the case of partial difference equations in metric graphs which are either finite or they have an infinite countable vertex set. In Sections 4 and 5 we use the above to study two examples.

## 2. Application to partial difference equations on graphs

In this section we consider graphs $\mathcal{G} = (V, E)$ which are simple and of finite degree (that is, each vertex has finite degree). In general one only need to take care on connected graphs, but we do not impose such a restriction at this point. To each vertex $v \in V$ there is associated the set $N(v) \subset V$, called the neighborhood of $v$, so that $w \in N(v)$ if $\{v, w\} \in E$; $w$ is called a neighbor of $v$. Now on, we only assume the vertex set $V$ is finite or infinite countable. Associated to the graph $\mathcal{G}$ are the real vector space $C^0(\mathcal{G})$ consisting of the real functions defined on $V$ and its subvector space $L^2(\mathcal{G})$

consisting of those $\mu \in C^0(\mathcal{G})$ so that $\sum_{j=1}^{\infty} \mu(v_j)^2 < \infty$. A derivation on the graph $\mathcal{G}$ is a linear operator $D : L^2(\mathcal{G}) \to L^2(\mathcal{G})$. The set of derivations is a real algebra $\Xi_{\mathbf{R}}(\mathcal{G})$. If $G : V \times \mathbf{R}^{k+2} \to \mathbf{R}$ is any function, $D_1, ..., D_k \in \Xi_{\mathbf{R}}(\mathcal{G})$ are any derivations, then we have associated a partial difference equation on the graph given by

$$G(v, \mu, D_1\mu, ..., D_k\mu) = 0. \tag{2.1}$$

A solution of (2.1) is a function $\mu \in L^2(\mathcal{G})$ satisfying $G(v, \mu(v), D_1\mu(v), ..., D_k\mu(v)) = 0$, for every vertex $v \in V$. We only consider partial difference equations of the form

$$D\mu = F(v, \mu) \tag{2.2}$$

where $F : V \times \mathbf{R} \to \mathbf{R}$ is some function.

Partial difference equations appear naturally by numerical discretization of partial differential equations on grids (see, for instance, [5] for linear elliptic difference equations on the plane). Due to applications in fields as biology, computer science, economy, ecology, etc., nonlinear partial difference equations has recently attracted a lot of attention.

If we set $V = \{v_1, v_2, ....\}$, then there is a natural isomorphism of real vector spaces $\phi : C^0(\mathcal{G}) \to \mathbf{R}^{\mathbf{N}}$ defined by $\phi(\mu) = {}^t[\mu(v_1)\ \mu(v_2)\ \cdots]$, with restriction $\phi : L^2(\mathcal{G}) \to \ell^2$. In this way, the partial diference equation (2.2) is equivalent to a semilinear problem (1.1). As a direct consequence of Theorem 1 is the following existence result.

**Corollary 4.** *Let $\mathcal{G} = (V, E)$ be an finite degree simple graph with vertex set $V$ either finite or countable infinite, say $V = \{v_1, ....\}$. Let $D \in \Xi_{\mathbf{R}}(\mathcal{G})$ be a symmetric positive semi-definite derivation, $F : V \times \mathbf{R} \to \mathbf{R}$ be continuous in the real variable and set $f_j(t) = F(v_j, t)$, for each $v_j \in V$. If, for each $j$, $f_j : \mathbf{R} \to \mathbf{R}$ satisfies the conditions (1), (2), (3) and, for $V$ infinite, also condition (4) of Theorem 1, then the partial difference equation (2.2) has non-trivial solutions. Moreover, if $G$ is a group of symmetries of the graph $\mathcal{G}$, then there are non-trivial solutions which are symmetric respect to $G$.*

A discussion of the above result for the particular case of finite graphs and the discrete Laplacian operator can be found in [10] and, for a more general class of derivations, in [6].

**Corollary 5 (Dirichlet's problem).** *Let $\mathcal{G} = (V, E)$ be an finite degree simple graph with vertex set $V$ either finite or countable infinite and $W < V$ so that $W \neq \emptyset$ and $V - W \neq \emptyset$. Let $F : (V - W) \times \mathbf{R} \to \mathbf{R}$ be some continuous function in the real variable.*

*(1) If $W$ is finite, then write $V = \{v_{-r+1}, v_{-r+2}, ....\}$, $W = \{v_{-r+1}, ..., v_0\} \subset V$. Consider real values $a_0, a_{-1}..., a_{1-r} \in \mathbf{R}$ and the column vector $a = {}^t[a_{1-r} \cdots a_0]$.*

*(2) If $W$ is infinite, then write $V = \{..., v_{-2}, v_{-1}, v_0, v_1, ....\}$, $W = \{..., v_{-2}, v_{-1}, v_0\} \subset V$. Consider real values $a_j \in \mathbf{R}$, where $j \leq 0$, and the infinite column vector $a = {}^t[\cdots a_{-1}\ a_0]$.*

*Assume, with this enumeration of the vertex set $V$, that the (infinite size) matrix representation of $D$ is*

$$J = \begin{bmatrix} R & {}^tU \\ U & S \end{bmatrix}$$

*where $R$ is the matrix corresponding to $W$ (of size $r \times r$ in (1) and of infinite size in (2)). Set $b = Ua$ and, $f_j(t) = F(v_j, t)$. If each $f_j : \mathbf{R} \to \mathbf{R}$ satisfies the conditions $f_j(0) = b_j$, for all $j$, and conditions (2), (3) and, if $V$ is infinite, also condition (4) of Theorem 1, then the Dirichlet problem*

$$\begin{cases} D\mu = F(v, \mu), & v \in V - W \\ \mu(v_j) = a_j, & v_j \in W \end{cases}$$

*has solutions.*

**Proof.** In this case, we need to solve $Sy = g(y)$, where $g(y) = f(y) - Ua$, $f(y)$ is the column vector whose $j$-th coordinate is $f_j(y_j)$ and $S$ is symmetric and positive semi-definite. The existence is now consequence of Theorem 1 using $g$ instead of $f$.
□

Let us consider a finite degree, simple, connected graph $\mathcal{G} = (V, E)$, where we assume that the set $V$ of vertices is either finite or countable infinite. Let us fix some metric on the graph, that is, a positive function $d : E \to (0, +\infty)$. A classical derivation on $\mathcal{G}$, $\Delta_2 : L^2(\mathcal{G}) \to L^2(\mathcal{G})$, called the *Discrete Laplace operator*, is given by

$$\Delta_2\mu(v) = \sum_{w \in N(v)} \frac{\mu(v) - \mu(w)}{d(\{v, w\})^2}.$$

As we are assuming $N(v)$ to be finite for each $v \in V$, the sum in the above definition is a finite sum. It is well known that $\Delta_2$ is a symmetric and positive semi-definite operator (see for instance [2, 3, 4, 9]).

## 3. The proof of Theorem 1.

The proof of Theorem 1 is a simple application of the Mountain Pass Theorem, due to A. Ambrosetti and P. Rabinowitz, which we recall below as matter of completeness.

**Mountain Pass Theorem 6 ([6]).** Let $U$ a Hilbert space and $H : U \to \mathbf{R}$ be a $C^1(U; \mathbf{R})$ functional which is Lipschitz continuous on bounded sets of $U$ (that is, $|H(x) - H(y)| \leq K_\Omega \|x - y\|$ for $x, y \in \Omega \subset U$ bounded). Let us assume

(i) $H(0) = 0$;

(ii) there exists $a, r > 0$ such that $H(x) \geq a$, for $\|x\| = r$;

(iii) there exists $y \in U$, $\|y\| > r$ with $H(y) < 0$.

Let

$$A = \{\alpha : [0,1] \to U : \alpha(0) = 0,\ \alpha(1) = y,\ \alpha \in C^0([0,1], U)\}$$

and

$$a = \mathrm{Inf}_{\alpha \in A}\{\mathrm{Max}_{t \in [0,1]} H(\alpha(t))\} \in \mathbf{R}.$$

Then "$a$" is a critical value of $H$.

### 3.1. Proof of theorem 1.

Let us first consider the case $W = \ell^2$. We consider the real map

$$H_Q : \ell^2 \to \mathbf{R} : x \mapsto \frac{1}{2}{}^t x Q x - \sum_{j=1}^{\infty} \int_0^{x_j} f_j(t)\, dt. \tag{3.1}$$

The function $H_Q$ is a $C^1(\ell^2; \mathbf{R})$ whose gradient is

$$\nabla H_Q x = Qx - f(x). \tag{3.2}$$

It follows that the (non-trivial) critical points of $H_Q$ are exactly the solutions we are searching for. Condition (1) ensures $H_Q(0) = 0$. As

$f_j(0) = 0$ and $Q$ is positive semi-definite, condition (2) ensures the existence of positive values $a, r > 0$ so that $H_Q(x) \geq a > 0$ for $\|x\| = r$. Condition (3) asserts the existence of some $y \in \ell^2$ with $\|y\| > r$ and $H_Q(y) < 0$. Let $R > 0$ be fixed and let $x, y \in B(0; R) \subset \ell^2$, that is, $\|x\|^2 = \sum_{j=1}^{\infty} x_j^2 < R^2, \|y\|^2 = \sum_{j=1}^{\infty} y_j^2 < R^2$. Then, by condition (4),

$$|H_Q(x) - H_Q(y)| \leq \frac{1}{2} \left| {}^t xQx - {}^t yQy \right| + \left| \sum_{j=1}^{\infty} \int_{x_j}^{y_j} f_j(t)\, dt \right| \leq K(R)\|x - y\|,$$

for some constant $K(R) > 0$. All the above and Mountain Pass Theorem (see below) ensure the existence of non-trivial critical points of $H_Q$.

Let now consider a general $W < \ell^2$ which is invariant under $Q$. Conjugating by a suitable orthogonal linear automorphism of $\ell^2$, we may assume $W$ is given by the conditions $x_r = 0$, for some indices $r$. As $Q$ still being symmetric and positive semi-definite under such a rotation and its restriction to the subspace $W$ still having the same properties, the existence is provided by the above case.

If $G$ is a group of orthogonal transformations of $\ell^2$ so that each of its elements commutes with $Q$ and set $W = Fix(G) = \{x \in \ell^2 : g(x) = x, \forall g \in G\}$, then one obtains the existence of non-trivial solutions in $Fix(G)$.

## 4. Example I: Matukuma type equations

In this section we consider, as a first example, Matukuma type equations and adapt them to metric graphs. Then we use the results obtained previously in order to assert the existence of non-trivial solutions.

### 4.1. Classical Matukuma equations

In order to have a model to describe the dynamics of globular cluster of star, T. Matukuma [8] proposed the following one (*Matukuma equation*)

$$\Delta\mu + \frac{1}{1 + |x|^2}\mu^p = 0, \text{ in } \mathbf{R}^3, \tag{4.1}$$

where $p > 1$, $\mu^p = |\mu|^{p-1}\mu$ and $\mu > 0$ is the gravitational potential so that

$$\int_{\mathbf{R}^3} \frac{\mu^p}{4\pi(1 + |x|^2)} dx$$

represents the total mass. Many results have been obtained for radial solutions $\mu_a(r)$, where $r = |x|$ and $a > 0$ is so that $\mu_a(0) = a$. Even, in [8] obtained that $\mu_{\sqrt{3}}(r) = \sqrt{3/(1+r^2)}$. A more general kind of equations that generalize Matukuma equation are the *Matukuma type equations*

$$\Delta\mu + f(x)\mu^p = 0, \quad \text{in } \mathbf{R}^3, \tag{4.2}$$

where $f(x) > 0$ for every $x \in \mathbf{R}^3$. A discussion about positive solutions of Makutuma type equation can be found in [11].

### 4.2. Matukuma type equations on graphs

Matukuma type equations can be adapted to connected, finite degree metric graphs as follows. Let us consider a finite degree, simple, connected graph $\mathcal{G} = (V, E)$, where we assume that the set $V$ of vertices is either finite or countable infinite. Let us fix some metric on the graph, that is, a positive function $d : E \to (0, +\infty)$. Equation (4.2) is adapted to this metric graph as the partial difference equation

$$-\Delta_2\mu + f(v)\mu^p = 0, \; v \in V, \tag{4.3}$$

where $f : V \to (0, +\infty)$. More generally, if $W \subset V$ is so that $V - W \neq \emptyset$, then we may consider the Dirichlet problem

$$\left\{ \begin{array}{c} -\Delta_2\mu + f(v)\mu^p = 0, \; v \in V - W, \\ \mu(w) = 0, \; w \in W, \end{array} \right. \tag{4.4}$$

where $f : V \to (0, +\infty)$. If $W \neq \emptyset$, then Corollary 5 ensures the existence of non-trivial solutions of equation 9. In the case that $W = \emptyset$, Corollary 1 ensures the existence of non-trivial solutions of equation (4.3); moreover, it asserts the existence of non-trivial solutions which are invariant under the group of symmetries of the metric graph $(\mathcal{G}, d)$. Next theorem asserts that the non-trivial solutions of the equation (4.3) must be sign changing.

**Theorem 6.** *Equation (4.3) has non-trivial solutions and all of them must be a sign changing solution, in particular, there is no non-negative nor non-positive non-trivial solution.*

**Proof.** Set $V = \{v_1, v_2, ....\}$,

$$w_{ij} = \left\{ \begin{array}{cc} d(\{v_i, v_j\})^{-2}, & \{v_1, v_2\} \in E \\ 0, & \{v_1, v_2\} \notin E \end{array} \right.$$

and

$$b_{ij} = \begin{cases} 0, & i \neq j \\ \sum_{r=1}^{+\infty} w_{ir}, & i = j \end{cases}$$

Note that the sums in the definition of $b_{ij}$ are finite sums as we are assuming the graph $\mathcal{G}$ to be of finite degree. Also set $A = [w_{ij}]$, $B = [b_{ij}]$ and $J_2 = B - A$. The discrete Laplace operator $\Delta_2$ is then defined in $\mathbf{R}^{\mathbf{N}}$ as the linear operator $x \mapsto J_2 x$, for $x \in \mathbf{R}^{\mathbf{N}}$. As the sum of each column of $J_2$ (a finite sum) is equal to 0. It follows that for every $\mu \in C^0(\mathcal{G})$ it holds that

$$\sum_{v \in V} \Delta_2 \mu(v) = 0,$$

in particular, if $\mu$ is a solution of equation (4.3), then

$$\sum_{v \in V} f(v)\mu(v)^p = \sum_{v \in V} f(v)|\mu(v)|^{p-1}\mu(v) = 0.$$

□

## 5. Example II: Helmholtz equations

Many problems related to acoustic mechanics, electromagnetism, etc., reduce to study the *Helmholtz equation* [7]. We proceed to adapt such equation to finite degree graphs (with either finite or an infinite countable number of vertices) and we find conditions to ensure the existence and uniqueness of solutions.

### 5.1. Classical Helmholtz equation

Let $\Omega \subset \mathbf{R}^n$ be some domain and let $h : \Omega \to \mathbf{R}$ be a function (in general with compact support). *Helmholtz equation* is given by

$$-\Delta\mu = k^2\mu + h \tag{5.1}$$

where $\Delta$ is the Laplace operator and $\mu : \Omega \to \mathbf{R}$ is some solution function (with the required regularity properties).

If $h = 0$, then the existence of non-trivial solutions of is equivalent to have $k^2$ as an eigenvalue of $-\Delta$ (on the domain $\Omega$). In this homogeneous case, the equation can be solved by separation of variables.

One may also consider the corresponding Newmann problem

$$(5.2) \qquad \begin{cases} -\Delta\mu &= k^2\mu + h, & \Omega \\ \partial_n\mu &= f, & \partial\Omega \end{cases}$$

where $\partial_n\mu$ denotes the normal derivative respect to the boundary $\partial\Omega \subset \mathbf{R}^n$.

### 5.2. Helmholtz equation on graphs

Let us assume that we are given a simple, finite degree and connected graph $\mathcal{G} = (V, E)$, with $V$ either finite or countable infinite, and a metric $d : E \to (0, +\infty)$. Let $W \subset V$, $W \neq \emptyset$, $V - W \neq \emptyset$, let $\sigma : W \to V - W$ be some function and let real values $b_w$ $(w \in W)$, $a_v$ $(v \in V - W)$ be no necessarily different ones. Newmann's problem (5.2) may be adapted to the above metric graph as

$$(5.3) \qquad \begin{cases} \Delta_2\mu(v) &= k^2\mu(v) + a_v, & v \in V - W \\ \mu(w) - \mu(\sigma(w)) &= b_w, & w \in W \end{cases}$$

#### 5.2.1. Finite graphs

Let us set $V = \{v_{-n+1}, ...., v_m\}$, $W = \{v_{-n+1}, ..., v_0\} \subset V$ and set $Ind(W) = \{-n+1, ..., -1, 0\}$ and $Ind(V) = \{1, ..., m\}$. Set $b_j = b_{v_j}$ $(j \leq 0)$, $a_r = a_{v_r}$ $(r = 1, ..., m)$, and consider the column vectors $b = {}^t\,[b_{-n+1} \cdots b_0]$ and $a = {}^t\,[a_1 \cdots a_m]$.

#### 5.2.2. Infinite graphs

(1) If $W$ is finite, then write $V = \{v_{-n+1}, v_{-n+2}, ....\}$, $W = \{v_{-n+1}, ..., v_0\} \subset V$, and set $Ind(W) = \{-n+1, ..., -1, 0\}$. Define $b_j = b_{v_j} \in \mathbf{R}$ $(j \leq 0)$, $a_r = a_{v_r}$ $(r \geq 1)$, and let us consider the column vectors $b = {}^t[b_{-n+1} \cdots b_0]$ and $a = {}^t[a_1 \cdots]$.

(2) If $W$ is infinite, then write $V = \{..., v_{-2}, v_{-1}, v_0, v_1, ....\}$, $W = \{..., v_{-2}, v_{-1}, v_0\} \subset V$, and set $Ind(W) = \{... -2, -1, 0\}$. Define $b_j = b_{v_j} \in \mathbf{R}$, where $j \leq 0$, $a_j = a_{v_j}$, for $j \geq 1$, and let us consider the column vectors $b = {}^t[\cdots b_{-1}\ b_0]$ and $a = {}^t[a_1 \cdots]$.

In the above two cases, we also set $Ind(V) = \{1, 2, ....\}$.

### 5.2.3. Existence of solutions

Let us write the matrix $J_2$ (representing $\Delta_2$) as

$$J_2 = \begin{bmatrix} A & {}^tC \\ C & D \end{bmatrix}$$

where $A$ and $D$ are positive semi-definite symmetric matrices of sizes $|W|$ and $|V - W|$, respectively. The function $\sigma$ may be though as a function

$$\sigma : Ind(W) \to Ind(V).$$

Let us set $M_\sigma = [m_{ij}]$ (where $i \in Ind(W)$ and $j \in Ind(V)$) by

$$m_{ij} = \begin{cases} -1, & \sigma(i) = j \\ 0, & \sigma(i) \neq j \,. \end{cases}$$

Set

$$\mu(v_j) = \begin{cases} x_j, & j \in Ind(W) \\ y_j, & j \in Ind(V)\,. \end{cases}$$

With the above notation, equation (5.3) is equivalent to

$$\begin{bmatrix} I_n & M_\sigma \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ k^2 y + a \end{bmatrix} \tag{5.4}$$

then

$$\begin{cases} x = b - M_\sigma y \\ (D - CM_\sigma - k^2 I_n)y = a - Cb\,. \end{cases} \tag{5.5}$$

It nows easily follows the next result.

**Theorem 7.** *Newmann's problem (5.3) has a unique solution if and only if $k^2$ is not an eigenvalue of $D - CM_\sigma$. If $k^2$ is an eigenvalue of $D - CM$, then (5.3) may not have solution and, if it has at least one, then it must have infinitely many solutions.*

**Example 8.** *Let us consider the complete graph $K_2$. Set $V = \{v_1, v_2\}$, $w = 1/d(\{v_1, v_2\}) > 0$. Assume $W = \{v_1\}$, $b_{v_1} = b$ and $a_{v_2} = a$. In this case,*

$$J_2 = \begin{bmatrix} w & -w \\ -w & w \end{bmatrix}$$

$$M_\sigma = [-1]$$

$$D - CM_\sigma = 0\ .$$

*If $k \neq 0$, the equation (5.3) has a unique solution, this being given by*

$$\begin{cases} x = (bk^2 - a - wb)/k^2 \\ y = -(a + wb)/k^2 \end{cases}$$

*If $k = 0$ and $wb \neq -a$, then (5.3) hs no solution. If $k = 0$ and $wb = -a$, then (5.3) has infinitely many solutions*

## References


[1] Ambrosetti, A. and Rabinowitz, P. Dual Variational Methods in Critical Point Theory and Applications. *J. Functional Analysis* **14**, pp. 349-381, (1973).

[2] Bapat, R. B. The Laplacian Matrix of a Graph. Math. Student **65**, pp. 214-223, (1996).

[3] Colin de Verdiere, Y. Spectre d'operateurs différentiels sur les graphes. In Random walks and discrete potential theory, Cortona, June 22-28, pp. 1-26, (1997).

[4] Colin de Verdiere, Y. Spectres de graphes. Societé Mathématique de France (1998).

[5] FriedrichsK. and Lewy, H. Über die partiellen Differenzengleichungen der mathematischen Physik. (German) *Math. Ann.* **100** (1), pp. 32-74, (1928).

[6] Hidalgo, R. A. Zeros of semilinear systems with applications to nonlinear partial difference equations on graphs. To appear in *Journal of Difference Equations and Applications.*

[7] Howe, M. S. *Acoustics of fluid-structure interactions.* Cambridge, New York. Cambridge University Press (1998).

[8] Matukuma, T. *The Cosmos.* Iwanami Shoten, Tokio, (1938).

[9] Mohar, B. The laplacian spectrum of graphs. In Graph Theory, Combinatorics, and Applications **2**. Ed. Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk. Wiley, pp. 871-898, (1991).

[10] Neuberger, John M. Nonlinear Elliptic Partial Difference Equations on Graphs. *Experimental Mathematics* **15**, pp. 91-107, (2006).

[11] Yi, Li. On the positive solutions of the Matukuma equation. *Duke Math. J.* **70** (3), pp. 575-589, (1993).

**Rubén Hidalgo**
Departamento de Matemáticas
Universidad Técnica Federico Santa María,
Valparaíso,
Chile
e-mail : ruben.hidalgo@usm.cl

and

**Mauricio Godoy**
Department of Mathematics,
University of Bergen,
Bergen,
Norway
e-mail : mauricio.godoy@gmail.com