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CHARACTERIZATION OF LALLEMENT ORDER ON A REGULAR SEMIGROUP

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Abstract

In this paper a study of properties of the Mitsch order relation 'μ' on a regular semigroup and Nambooripads order 'ν' on any arbitrary regular semigroup is made. Mainly a characterization of Lallement order on a regular semigroup is obtained. The necessary and sufficient condition for the restriction of Lallement order 'λ' to B(S) to be usual order on an orthodox semigroup is also obtained.

Key Words : *Nambooripads Order ν, Mitsch Order 'μ', Lallement order λ.*

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1. Introduction

In this paper a study of properties of the Mitsch Order relation 'μ' on a regular semigroup S, Lallement order 'λ' on a regular semi group and Nambooripads order ν on a regular semigroup of 'S is made. The common property enjoyed by all the three partial orders λ, 'μ' and ν is that the set E (S) is an initial segment (See def. 1.5) of 'S' under each of these partial orders. It is interesting to note that λ is a compatible partial order on a regular semigroup such that $\lambda \cap [E(S)XE(S)]$ is contained in the usual order on E(S), where as Nambooripads partial order 'ν' is not in general compatible but $\nu \cap [E(S)xE(S)]$ is the usual order on E(S). It is obtained in theorem (2.15) that the restriction of Lallement order λ to $E(S)$ is the usual order on $E(S)$. Nambooripad himself proved that on a regular semi group 'ν' is compatible iff 'S' is a locally inverse semigroup (See def. 1.4). A necessary and sufficient condition for restriction of Lallement order λ to $E(S)$ is the usual order on $E(S)$ is also the same i.e., 'S' is a locally inverse semi group is also obtained and in this case both λ and ν are same. It is also observed that in order to show that ν is compatible on 'S' it is enough to show that $(xey, xfy) \in \nu$ whenever $\forall(e, f) \in \nu\forall$ and $\forall x, y \in S^1$. The necessary and sufficient condition for the restriction of lallement order 'λ' to $B(S)$ to be the usual order on $B(S)$ is that $B(S)$ is a normal band. It is also obtained as a corollary that a band B is narmal band if and only if 'λ' is equal to usual partial order on $B(S)$.

First we start with the following preliminaries

Definition 1.0 : Suppose 'S' is a semigroup. An element $a \in S$ is said to be regular if there exists $x \in S$ such that $axa = a$. If every element of 'S' is regular then 'S' is called a regular semigroup.

Definition 1.1 : On a semigroup 'S' define the relation '$\leq$' on 'S' by $a \leq b$ if there exists two idempotents e, f in S^1 (S^1 is the monoid obtained from 'S' by adjoining an identity 1) such that $a = eb = bf$.

Definition 1.2 : Suppose (S,.) is a regular semigroup. Then the Lallement order λ on 'S' is defined by the rule that $a\lambda b$ iff for all x, y in 'S' $x\mathbf{R}xa \Rightarrow xa = xb$ and $y\mathcal{L}ay \Rightarrow ay = by$.

Definition 1.3 : Suppose (S,.) is a regular semigroup and $E(S)$ is the set of all Idempotents of 'S'. For any two elements a, b of S, define a relation ν on 'S' by $a\nu b$ if $R_a \leq R_b$ and (there exist $e \in E \cap Ra)a = eb$. 'ν' is called Nambooripads order on 'S', Ra is the principal right ideal containing a.

Definition 1.4 : A regular semigroup S with set E of Idempotents will be called locally inverse if eSe is an inverse semigroup for every e in E.

Definition 1.5 : Suppose $(X, \leq)$ is a partially order set. Then a subset A of X is called an initial segment of $X if x \in A$ whenever $x \leq a \in A$.

Definition 1.6 : Suppose (S,.) is a semigroup. Then the Mistch order relation "μ" on 'S' is defined for any a, b $\in S, a'\mu$' b, iff there exists $s, t \in S^1$ such that $sa = sb = a = at = bt$. Now we start with the following lemma

Lemma 2.0 : Suppose "μ" is a Mitsch order relation on 'S'.Then the restriction of "μ" to the set of Idempotents of 'S' coincides with the usual order on E(S).

Proof : Let $(e, f) \in' \mu$', where $e, f \in E(S)$, so that $se = sf = e = et = ft$ for some $s, t \in S^1$ and hence $ef = sef = sf2 = sf = e$ so that fe = ef = e therefore $e \leq f$. Conversely if $e, f \in E(s)$ and $e.f = f.e = e$, then by choosing s = t = e, we have $se = sf = e = et = ft$ and hence $(e, f) \in' \mu$', so that the restriction of the order 'μ' to $E(S)$ coincides with the usual order on $E(S)$.

Lemma 2.1 : Suppose 'S' is a semigroup and a me where e is an Idempotent of 'S' then a is also an Idempotent of 'S'. In other words, the set of all Idempotents of 'S' is an Initial segment of 'S' under 'μ'.

Proof : Let $a\mu e$, where $e \in E(S)$, so that there exists $s, t \in S^1$ such that $sa = se = a = at = et$, and now $a^2 = seet = set = sa = a$ and hence a is an Idempotent of 'S'.

Theorem 2.2 : Suppose a is a regular element of a semigroup 'S' and if $(a, b) \in' \mu$' then $a \leq bi.e.a = eb = bf$ for $e, f \in E(S1)$.

Proof : Let $(a, b) \in' \mu$' so that there exists s, $t \in S^1$ such that $sa = sb = a = at = bt$. For any inverse a' of a, we have aa' and a'a are Idempotents of 'S'. Now, $a = aa'a = bta'a$ (as $a = bt$) $= b$ $(ta'a)$. Also $(ta'a)$ $(ta'a) = ta'ata'a = ta'aa'a = ta'a$ so that $ta'a \in E(S^1)$. Similarly $aa's \in E(S^1)$ for s $\in S^1$ and hence $a = aa'a = aa'sb = (aa's)b$. Thus $a = eb = bf$, Where $e = aa's$ and $f = ta'a$ and hence $a \leq b$.

Remark 2.3 : It can be easily observed that on any semigroup'S'; we have as a relation $\leq \subseteq' \mu$'. $\leq$ is not in general a transitive relation. However if a is a regular element of 'S' and $a \leq b$, $b \leq c$ then $a \leq c$.

Corollary 2.4 : Suppose 'S' is a semigroup and a is a regular element of 'S' such that $a = eb = bf$ and $b = gc = ch$ for $e, f, g, h \in E(S1)$ then $a \leq c$.

Proof : We have obviously $\leq\subseteq' \mu$'. Since $(a, b) \in\leq\subseteq' \mu$' and $(b, c) \in\leq\subseteq' \mu$' hence $(a, c) \in' \mu$' as 'μ' is transitive and hence $(a, c) \in\leq$.

Corollary 2.5 : The Mitsch order relation 'μ' on a semigroup 'S' is such that its restriction to the set $E(S)$ of Idempotents of 'S' is the usual order on $E(s)$.

Proof : Proof is obvious.

Remark 2.6 : If '$\leq$' is the binary relation defined on a semigroup 'S' by $a \leq b$ if and only if $a = b$ or $a = eb = bf$ for some Idempotents e, f of 'S' and if the Idempotents form a subsemigroup of 'S' then it can be easily verified that '$\leq$' is a partial order relation on 'S'.

Lemma 2.7 : Suppose 'S' is a semigroup and '$\leq$' on 'S' is defined by $a \leq b$ if either $a = b$ or $a = eb = bf$ for some Idempotents e, f of 'S'. If to each c$\in$ $Sande \in E(S)thereexists$g,h $\in E(S)$ such that $ce = gc$ and $fc = ch$, then '$\leq$' is compatible with multiplication.

Proof : If e, f are Idempotents of 'S', then from the given condition there exists an Idempotent g of 'S' such that $ef = ge$ so that $efe = ge2 = ge = ef$ and hence $ef.\ ef = ef2 = ef$. Thus the set of Idempotents of 'S' is a subsemigroup of 'S'. Hence by remark 2.6, '$\leq$' is a partial order relation on 'S'. For $a, b \in S$ with $a \leq b$, then $a = eb = bf$. Since $a = eb = bf$ so that $ac = ebc = bfc$. We have from the given condition $fc = ch$ for some Idempotent h and therefore, $ac = e(bc) = (bc)h$ so that $ac \leq bc$. Similarly it can be shown that $ca \leq cb$.

Remark 2.8 : The above condition is only sufficient but not necessary because of the following example

Example 2.9 : Let $S = a, b, c, d, e, f$ Define . on 'S' as follows : .. a b c d e f a b b e e e b b b b e e e b c f f d d d f d f f d d d f e b b e e e b f f f d d d f In this example, the above condition is not satisfied one can easily verify that '$\leq$' is compatible with multiplication and $b < a, d < c$ where a and c are not regular elements of 'S'. In this example, idempotents form a subsemigroup of 'S'.

Theorem 2.10 : Suppose 'S' is a regular semigroup then Lallement order 'λ' on 'S' is a compatible partial order on 'S'.

Proof : We have $a\lambda b$ if and only if $x\mathbf{R}xa \Rightarrow xa = xb$ and $yLay \Rightarrow ay = by$ for $x, y \in S$; we have for any $a \in S$, x **R** x a $\Rightarrow xa = xa$ and $yLay \Rightarrow ay = ay$ so that $a\lambda a$ for $aya \in S$. Now, let $a\lambda b$ and $b\lambda a(a, b \in S)$, so that $x\mathbf{R}xa \Rightarrow xa = xb$ and $yLay \Rightarrow ay = by$ and $x\mathbf{R}xa \Rightarrow xb = xa$, $y\mathcal{L}byy \Rightarrow by = ay$, taking $x = a'$ (where a' is inverse of a), we have $a'a = a'b$, and by taking $y = b$, we have $ab = bb$ and therefore $a = aa'a = aa'b$ (as $a'a = a'b$) $= aa'bbb$ (where $b$is any inverse of b) $= aa'abb$ (as $a'b = a'a$) $=$ $abb = bbb$ (as $ab = bb$) $= b$. Now, let $a\lambda b$ and $b\lambda c$ for any $a, b, c \in S$. Let $4x\mathbf{R}xa$ and y Lay for $x, yx \in S$, then since $a\lambda b$, we have $xa = xb$ and $ay = by$. Since $b\lambda c$, we have $xb = xc$ (since xa = xb and x **R** xa) and there fore $xa = xc$. We also have $by = cy$ (since ay = by and $yLay$). Now, let $a\lambda b$ for any $c \in S$, We have to show that a $c\lambda bc$ and $ca\lambda cb$. Let $x\mathbf{R}xac$ and $y\mathcal{L}acys!$ o that $xac = xbc$. $y\mathcal{L}acy$ we have $Sy = Sacy$, so that y= zacy for $z \in S$, then $cy = czacy$ so that $Scy = Sczacy \subseteq Sacy \subseteq Scy$. Therefore $Scy = Sacy$ and hence $cy\mathcal{L}acy$ implies that $acy = bcy$ so that $ac\lambda bc$. Now let $x\mathbf{R}xca$ and $y\mathcal{L}cay$, so that $cay = cby$. Since $x\mathbf{R}xca$, we have $x = xcaz$ for $z \in S$, so that $xc = xcazc, impliesthatxcS = xcazcS \subseteq xcaS \subseteq xcS$ and hence $xcS = xcaS$ so that $x\mathbf{R}xca$. Therefore $xc\mathbf{R}xca$ implies $xca = xcb$ and hence $ca\lambda cb$, so that 'λ' is compatible on 'S' under multiplication.

Corollary 2.11 : Suppose 'S' is a regular semigroup and if $(e, f) \in \lambda$ for $e, f \in E(S)$, then $e \leq f$ i.e, $e.f = f.e = e$.

Proof : Let $e\lambda f$ for $e, f \in E(S)$, so that $x\mathbf{R}xe$ implies $xe = xf$ and $xLey$ implies $ey = fy$. Taking $x = y = e$, then $e = ef = fe$.

The following is an example to show that $(e, f) \notin \lambda$, even though $e.f = f.e = e$.

Example 2.12 : Let $S = a, b, c, d$, Define . On 'S' by the composition table as follows : $E(S) = a, b, c, d$, since $b.d = d.b = b$ so that $b \leq d$, but $a = bc \neq dc = d$, and hence $(b, d) \notin \lambda$. .. a b c d a a b a a b a b a b c a b c c d a b d d

Lemma 2.13 : Suppose ale for e $\in E(S)thena \in E(S)$

Proof : It is obvious The following theorem is due to [2]. For the sake of definiteness, we stated the following.

Theorem 2.14 : Suppose S is a regular semigroup with set E of Idempotents and let the relation λ' defined by the rule that $a\lambda' b$ iff
$\forall a' \in V(a)$, $\forall e \in E \cap aa'S$, $\forall f \in ESa'a$, $ea = eb$, $af = bf4$, then the following holds
a. $\lambda \subseteq \lambda'$
b. Suppose $(a,b) \in \lambda'$, then $x\mathbf{R}xa$ implies that $ga = gb$ where g is the sandwich set $S(x'x, aa')$ and $y\mathcal{L}ay$ implies that $ah = bh$ where $h \in S(a'a, yy')$
c. $\lambda = \lambda'$
d. If $\forall S \forall$ is orthodox then $\lambda = (a,b) \in SS : \forall a' \in V(a), \forall b' \in V(b)$, $a'ea = a'eb$ and $aea' = bea'$.

Theorem 2.15 : Suppose S is an orthodox semigroup then the restriction of Lallement order 'λ' to $B(S)$ is the usual order on $B(S)$ iff $B(S)$ is a normal band.

Proof : Suppose S is an orthodox semigroup and the restriction to $B(S)$ is the usual order on $B(S)$. As 'λ' is compatible partial order and hence the restriction to $B(S)$ is a compatible partial order. So we have A band B(S) is a normal band iff it is compatible w.r.t multiplication, so that $B(S)$ is a normal band. Hence restriction of 'λ' to B(S) is the usual order on $B(S)$. Conversly suppose that the restriction 'λ' to B(S) is a normal band and if for any a, b $\in E(S) with a.b = b.a = a$.

Now we claim that $a\lambda b$ i.e. $aea' = bea'$ and $a'ea = a'eb \forall e \in E$, $a' \in V(a)$. Consider $bea' = bea'aa' = bea'(ab)a'(asab = a) = [b(ea')ab]a' = b(a)(ea')ba'(asabca = acba) = aea'ba'$ $(asb.a = a) = aea'ba'aa' = aea'ba'aa' = aba'ea'aa'$ (by using normality) $= aba'ea' = aa'ea'(asa.b = a) = aa'ea'aa' = aa'ea'aa' = aea'a'aa'$ (by using normality) $= aea'a' = aea'asaea' \in E$. Hence for all $e \in E$, $aea' = bea'$. Now we have to show that $a'ea = a'eb$.

Consider $a'eb = a'aa'eb = a'aba'eb(asab = a) = a'aba'eb = a'baa'eb$ (normality) $= a'ba'eab = a'ba'ea(asa.b = a) = a'ba'eaa'a = a'ba'eaa'a = a'aba'ea'a$ (normality) $= a'aa'ea'a(asa.b = a) = a'ea'e = a'aa'ea'a = a'aa'ea'a = a'eaa'a'a$ (by using normality) $= a'eaa'a = a'ea$. Hence $\lambda = (a,b) \in SS : \forall a' \in V(a), \forall b' \in V(b), a'ea = a'eb$ and $aea' = bea'$ Hence λ restricted to $B(S)$ is the usual order on $B(S)$ (by using theorem 2.14)

Corollary 2.16 : A band B is a normal band iff λ is equal to usual partial order on $B(S)$.

Proof : Suppose B is a normal band, then B is an orthodox and by theorem 2.15 λ restricted to $B(S)$ is the usual partial order on $B(S)$.

Conversely if λ restricted to $B(S)$ is the usual partial order on $B(S)$ and λ is compatible, hence the usual partial order on $B(S)$ is also compatible. Hence $B(S)$ is a normal band.

Theorem 2.17 : Suppose R is a partial order relation on a semigroup 'S'. Then Rb is defined by the rule that a R$^b b$ if $(xay, xby) \in R \forall x, y, \in S^1$, then R^b is the largest compatible relation contained in R.

Proof : We have a $R^b b$ for any $a, b \in S \Rightarrow (xay, xby) \in R \forall x, y \in S^1$.

Reflexive : We have $(a, a) \in R$ imply that $(a, a) \in R^b$ as $a = 1.a.1$ and $b = 1.b.1$. Antisymmetric : Let $aRbb$ and $bRba$, for any $a, b \in S$. Since $aR^b b \Rightarrow (xay, xby) \in R \forall x, y \in S1. Since$b R$^b a \Rightarrow (xby, xay) \in R$. In particular if $x = y = 1$, then $(a, b) \in R(b, a) \in R$ so that $a = b$ as R is antisymmetirc.

Transitive : Let $(a, b) \in Rb$ and $(b, c) \in R^b$ for any $a, b, c \in S$. Since $(a, b) \in R^b$ so that $(xay, xby) \in R$ and $(b, c) \in R^b$ so that
$(xay, xby) \in R \forall x, y \in S^1$. As R is transitive, $(xay, xcy) \in R$. Hence R is a partially order relation on 'S'. For $(a, b) \in R^b$ imply that $(xay, xby) \in R \forall x, y \in S^1$ so that $(a, b) \in R for x = y = 1$, Hence $R^b \subseteq R$.

Compatibility: Let $(a, b) \in R^b$ so that $(xay, xby) \in R$ for all $x, y \in S^1$ imply that $(xcay, xcby) \in R$ and hence $(ca, cb) \in R^b$ so that left compatibility holds. Similarly for $(a, b) \in R$ imply that $(xay, xby) \in R \forall x, y \in S^1$ so that $(xacy, xbcy) \in R$. Hence $(ac, ca) \in R^b$. Here $R^b$is a compatible relation on 'S' contained in R. Let be any compatible partial order relation on 'S' which is contained in R. For $(a, b) \in r \Rightarrow (a, b) \in R$. Since r in compatible so that $(xay, xby) \in r$ and hence $(xay, xby) \in R$. Hence $(a, b) \in R^b$ so that $r \subseteq R^b$. Hence R^b is the largest compatible partial order relation contained in R.

Theorem 2.18 : Suppose 'S' is a regular semigroup, then $\lambda = \nu^b$ where 'ν' is Nambooripads order on 'S'.

Proof : Let $(a, b) \in nb$ then $(xay, xby) \in \nu$ for all $x, y \in S^1$. Now, for $x = u, y = 1$, $(ua, ub) \in \nu$ and so that $ue = eub$ and $e \in \mathbf{R}_u a$...(1). We have $R_u = R_u a \leq R_u$ so that $R_u = R_u a$ and hence $uS = uaS$. Since $e \in Rua$ so that Rua = Re as Ru = Rua = Re, we have uS = uaS = ubS $\subseteq uS. So that$uS = uaS = ubS.$Hence$Rub = Re$imply that$(e, ub) $\in \mathbf{R}$ so that $eub = eb$. But from (1), $eub = ua$ and hence $ua = ub$. Similarly $v\mathcal{L}av \Rightarrow av = bv$. Hence

$(a, b) \in \nu^b$, we have $(a, b) \in \lambda$ so that $nb \subseteq \lambda$...(2). On the other hand, let $(a, b) \in \lambda$, we have $x\mathbf{R}xa \Rightarrow xa = xb$ For $x = a'$, we have $(a', a'a) \in \mathbf{R}$. Now $xa = xb$ so that $a'a = a'b$. Since $y\mathcal{L}ay \Rightarrow ay = by$, Choose $y = a'$, so that $aa' = ba'$. Now, a =aa'a=ba'a$\in bS, so that a \in bS. But for$a $\in bS$ and $a' \in V(a)$. So that $(a, b) \in \nu$. Hence $\lambda \subseteq \nu$. Hence 'λ' is compatible partial order which is contained in ν. But nb is the largest compatible partial order relation which is contained in ν. Hence $\lambda \subseteq \nu^b$...(3). From (2) and (3) we have $\lambda = \nu^b$

Theorem 2.19 : Suppose 'S' is a regular semigroup, then the following conditions are equivalent. (1) $(xey, xfy) \in \nu$ where $(e, f) \in \nu \cup (E(S) \times E(S))$ for all $x, y \in S^1$. (2) 'ν' is compatible with multiplication. (3) 'S' is a locally inverse semigroup. (4) $\lambda = \nu$. (5)Restriction of Lallement order λ to $E(S)$ is the usual order on $E(S)$

Proof : (1) $\Rightarrow$ (2)$Assume (1) holds i.e.$(xey, xfy)$\in \nu$ whenever $(e, f) \in \nu E(S)E(S)$ for all $x, y \in S^1$. Let $(a, b) \in \nu$, by using $[N1]$, for every $f \in E(Rb)$, there exists $e \in Ra$ such that $e.f = f.e = e$ and $a = eb$. By the assumption $(xey, xfy) \in \nu$ for all $x, y \in S^1$ so that $(xeb, xfb) \in \nu$ for all $x, y \in S^1$ so that $(xeb, xfb) \in \nu$ (by taking y = b)$and hence$(xa, xb) $\in \nu$...$(*)$. And also we have $(a, b) \in \nu$, by using proposition [3] for each $f \in E(Lb)$ there exists $e' \in E(La)$ such that $e'.f' = e' = f'e'$ and $a = be'$ since $(e', f') \in \nu$ and $(xe'y, xf'y) \in \nu$ such that $(be'y, bf'y) \in \nu$ (by taking $x = b$) and hence $(ay, by) \in \nu$ (Since $a = be'$ and $f' \in E(Lb)$). Therefore $(ay, by) \in \nu$...$(**)$. From (*) and (**) we have $(xay, xby) \in \nu$ for any $(a, b) \in \nu$ and hence 'ν' is compatible with multiplication.

(2) $\Rightarrow$ (3) : (2) and (3) are equivalent from [3] (3) $\Rightarrow$ (4) Assume (3), since 'S' is a locally inverse semigroup by using (Exercise 6.4; 3 of [2]) $\lambda = nb$ where ν^b is the largest compatible relation on 'S' which is contained in ν and by using [2] $\lambda = \nu$. (4) $\Rightarrow$ (5)$Suppose \lambda = \nu$, since the restriction of $\nu to E(S)$ is the usual order on $E(S)$ and hence the restriction of $\lambda to E(S)$ is also usual order on $E(S)$. (5)$\Rightarrow$ (1). Assume (5), i.e the restriction of to $E(S)$ in the usual order on $E(S)$ and let (e, f) $\in \nu \cap E(S)xE(S)$ so that $e.f = f.e = e$ and hence $e \leq f in \lambda$. Since the restriction of λ to $E(S)$ is the usual order on $E(S)$, we have $(e, f) \in nb$. Hence $(xey, xfy) \in \nu$ for all $x, y \in S^1$. Hence the given conditions on any regular semigroup 'S' are equivalent.

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