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0N CHARACTERIZATION OF RIEMANNIAN MANIFOLDS

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Abstract

This survey, present some results about characterization of Riemannian manifolds by using notions of convexity. The first part deals with immersed manifolds and the second part gives a characterization for the Euclidean space and for the Euclidean sphere.

Keywords : *geodesics, convexity, axiomatic geometry, isosceles triangles.*

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1. Introduction

In 1897 Hadamard, J., proved the following fundamental theorem, [10]: "If there exists an isometric immersion from a $n-$dimensional connected and compact Riemannian manifold M into the Euclidean space $R^{n+1}, (n \geq 2)$, in such a way that the sectional curvatures K of M (or the eigenvalues of the Gauss normal application) are strictly bigger than zero, therefore the image of M in R^{n+1} is the boundary of a convex body. Precisely, M is diffeomorfic to a sphere"

Several hypothesis on the sectional curvatures, or on the eigenvalues of the second quadratic form, or even on different notions of convexity give rise to new versions of this theorem. In the sequel, we mention some of these generalizations. In 1936 Stokes J. J. [19] proved an analogous results when M is complete instead of compact. In 1960, Sacksteader, R. [17] proved that: "If $f : M^n \to R^{n+1}$ is an isometric inmersion from a $n-$dimensional connected, compact and orientable manifold in R^{n+1}, $(n \geq 2)$, such that the sectional curvature K of M is non negative and there exists a point $p \in M$ with $K_p > 0$, then, f is an imbedding and $f(M)$ is the boundary of a convex body".

By using differential topology, do Carmo, M. and Lima, E. [4] proved in a independent way an analogous results of Sacksteader. This Theorem was published only in 1972.

In 1970, do Carmo, M. and Warner, F. [5] obtain a new generalization of the Hadamard´s Theorem by replacing the Euclidean space by a sphere or even by the hyperbolic space and adapting the hypothesis on the curvatures.

In 1977 Alexander, [1] obtain a new generaliaztion replacing R^{n+1} by a simply connected Riemannian manifold H of dimension $n+1$ $(n \geq 2)$, where the sectional curvatures are non positives (Hadamard manifold), as follows: "Let x : M $\longrightarrow$ H be a hypersurface inmersion of a compact, connected, orientable manifold M of dimension n ≥ 2, and ξ be a continuous unit normal. If ξ may be chosen so that S_ξ is positive definite, then M is imbedded in H as the boundary of a convex body".

In 1978, Tribuzy, I., [22] obtained a new generalization of the Hadamard Theorem by considering a connected, non compact, complete, orientable Riemannian manifold N of dimension $n+1$, $(n \geq 2)$ with sectional curvatures $k \geq K_N > 0$ where k is a constant. Due the existence of cut locus, in this case it was neccesary to impose restriction on the curvature of the inmersion. The result reeds as follows:

Let x: M $\longrightarrow$ N be an isometric immersion of a Riemannian orientable manifold M of dimension n. Suppose that is possible to choose a unit normal vector field ξ in M so that each eigenvalue λ of the second fundamental form of x satisfies $\lambda \geq 2\sqrt{k}$. Therefore, x is embedding and $x(M)$ is the boundary of a convex body in N. In particular, M is diffeomorphic to a sphere.

In order to obtain this extension there was neccesary to establish the following results:

Theorem 1. Suppose that N is simply connected manifold with $K_N \leq 0$ and M is a compact hypersurface of N such that $K_M > K_N$. Then, there exists a point $p \in M$ and orthonormal vectors V and W in T_pM such that $K_M(V,W)_p > 0$.

Theorem 2. Let M be a convex and compact submanifold of N. Assume that N is not compact and $K_N > 0$. Then, M is a homologic sphere.

On the other hand, it was obtained a characterization of the Euclidean space R^n among the Hadamard manifolds, in the following sense: if a straight line r of R^n meets the point A of the segment AB and forms with AB an angle θ with $0 \leq \theta < \frac{\pi}{2}$, therefore, there exists just one point C in r such that the triangle with vertices ABC is isosceles with base the segment AB. It was proved that R^n is the only one Riemannian complete manifold with the mention property.

In the same spirit, it was proved that the sphere S^n, is the only one $n-$dimensional Riemannian complete manifold in R^{n+1}, $(n \geq 2)$ which allows to construct two triangle isosceles.

The considerations stated below can be founded in, [3], [7], [12].

1.1 - Let N be a Riemannian manifold. We say that $K \subset N$ is strongly convex if for any pair of points $p, q \in K$ there exists a unique minimal geodesic γ of N connecting p to q and γ is contained in K. We say that $K \subset N$ is convex, if for each point p of the closure $\overline{K}$ of K there exists a number $0 < r(p) \leq c(p)$ such that $K \cap B_{r(p)}(p)$ is strongly convex; here $c(p)$ is the convexity radius and $B_{r(p)}(p)$ denotes the open ball with center in p and radius $r(p)$. We say that K is totally convex if whenever $p, q \in K$ and γ is a geodesic segment from p to q, then γ is contained in K. If K is convex and its interior, $int(K)$, is non empty we say that K is a convex body. The fundamental properties about convex sets can be found in [7].

1.2 - We will represent by $\langle , \rangle$ and $\overline{\nabla}$ the Riemannian an metric and Riemannian connexion of N, respectively. We will denote by $K_N(X,Y)_p$ the sectional curvature of N at the point p relative to the plane generated by the vectors X and Y of the tangent space T_pN of N. When clear from the context, we will only use K_N.

Let $x : M \to N$ be a isometric of a Riemannian manifold M into N. We will identify a vector V of T_pM with $dx_p(V)$ of $T_{x(p)}N$, and for V, W in T_pM we will identify $K_N(V,W)_{x(p)}$ with $K_N(dx_p(V), dx_p(W))_{x(p)}$. The notation $K_M > K_N$ will express that for every point $p \in M$ and for every pair of linearly independent vectors $V, W \in T_pM$ we have that $K_M(V,W)_p > K_N(V,W)_{x(p)}$.

1.3 Let $g(t)$ be a geodesic in M such that $g(t_0) = p$ and $g(t_1) = q$, where $t_0 < t_1$. We will represent the segment $g([t_0, t_1])$ of $g(t)$ by $[p,q]_g$; if $g(t') = p'$ and $t_0 < t' < t_1$, we will say that p' ocurres after p and before q along g.

In this work we will also assume all geodesics are parametrized by arc lenght.

Three geodesic segments $[p,q]_\gamma$, $[q,r]_\sigma$ and $[r,p]_g$ connecting distinct points p, q and r in M make a figure that we call a geodesic triangle which will be simply represented by $\{[p,q]_\gamma; [q,r]_\sigma; [r,p]_g\}$.

We say that a geodesic triangle is simple when the union of its sides is a curve homeomorphic to S^1, or when its vertexes lie in a unique segment free of self-intersections. A simple geodesic triangle is isosceles when it has two sides with same length, in this case the third side which could eventually have different size is called the base.

We notice that if r is the medium point of a geodesic segment $[p,q]_g$ free of self-intersections, then the triangle $\{[p,r]_g, [r,q]_g, [q,p]_g\}$ is an isosceles simple triangle.

1.4 Let $g(t) = exp(tv)$ the geodesic in M which goes through the point $p \in M$, in the direction of the unit vector $v \in T_pM$. The set $C_g(p) = \{t \in [0,\infty); d(p, g(t)) = t\}$ can be $[0,\infty)$ or $[0,t_0]$ for some $t_0 > 0$. When $C_g(p) = [0,\infty)$, we say that $g(t)$ is a geodesic ray, in the other case we will say that $q = g(t_0)$ is the minimal point of p along the geodesic g.

Geometrically, this means that if $r = g(t_1)$ with $t_1 > t_0$ then the segment $[p,r]_g$ is not minimal. The set made up of the minimal points of p along all geodesics that pass through p is called the cut locus of p and is represented by $C(p)$.

1.5 Let g and γ geodesics of M parameterized by the arc length and having a common point $p \in M$. Without lost of generality we can assume $g(0) = p = \gamma(0)$ and the angle between the geodesics being the angle θ between the tangent vectors $g'(0)$ and $\gamma'(0)$.

The figure made up from the geodesic g and the geodesic segment of γ linking the point p to a point $q = \gamma(t)$ with $t > 0$, is called a configuration. If θ is the angle between g and γ in the point p then the configuration is represented by $\{g, \gamma, \theta\}_p$.

1.6 M and N will indicate orientable complete and connected C^∞-Riemannian manifold with dimensions n and $n+1$ $(n \geq 2)$, respectively.

Our results is as follows

Theorem A. ([22]) Let $x : M \to N$ be a isometric immersion. Suppose that N is noncompact and that there exist a constant K such that $K \geq K_N > 0$. Suppose further that it is possible to choose a unit normal vector field ξ in M so that each eigenvalue λ of the second fundamental form of x with respect to ξ satisfies $\lambda \geq 2\sqrt{K}$. Then x is a embedding, and $x(M)$ is the boundary of a convex body in N. In particular, M is diffeomorphic to a sphere.

In order to state the Theorem B, is required the following axiom:

First Isosceles Triangle Axiom - FITA

For every configuration $\{g, \gamma, \theta\}_p$ such that $0 \leq \theta < \frac{\pi}{2}$ and for every point $q = \gamma(s_0)$ with $s_0 > 0$, there exists a unique point $r = g(t_0)$ with $t_0 > 0$ and a unique geodesic segment $[q,r]_\sigma$ linking the point q to the point r in such a way that $\{[p,q]_\gamma, [q,r]_\sigma, [r,p]_g\}$ is the unique isosceles triangle whose basis is $[p,q]_\gamma$.

Theorem B. ([20],[23]) If M satisfies the first isosceles triangle axiom then M is isometric to the Euclidean space $\mathbf{R}^n$.

In order to state the Theorem C, is required the following axiom:

Second Isosceles Triangle Axiom - SITA

For every configuration $\{g, \gamma, \theta\}_p$ and for each point $q = \gamma(s) \neq p = g(0) = \gamma(0)$ there exist only two real numbers t_1 and t_2 with $t_2 < 0 < t_1$ such that the points $r_1 = g(t_1)$ and $r_2 = g(t_2)$ determine the segments $[q, r_1]_\sigma$ and $[q, r_2]_\tau$ in such a way that the triangles $\{[p, q]_\gamma [q, r_1]_\sigma [r_1, p]_g\}$ and $\{[p, q]_\gamma [q, r_2]_\tau [r_2, p]_g\}$ are isosceles triangles whose common basis is $[p, q]_\gamma$.

Remark. In the case of SITA the angle θ can be given arbitrarily, thus we our notation for a configuration will dismiss the angle θ, that is $\{g, \gamma\}_p$.

Theorem C. ([23])If M satisfies the second isosceles triangle axiom then M is isometric to the Euclidean sphere S^n.

2. Proof of Theorem A

Lemma 2.1 Let A be a convex body of a Riemannian manifold L such that its boundary S is a submanifold of L. If γ(t) is a geodesic of L tangent to S in $p = \gamma(0)$, there exists $\delta > 0$ such that $\gamma(t) \in L - A$ for all t $\in (-\delta, \delta)$.

Proof : Let ξ_p be the unit normal vector of S at p, such that for $s > 0$ and sufficiently small $exp_p(s\ \xi_p) \in L - A$. Suppose that for all $\delta > 0$, there exists $t \in (-\delta, \delta)$ such that $\gamma(t) \in A$. Since A is a convex body of L, there exist a number $r = r(p) > 0$ such that $C = B_r(p) \cap A$ is open and strongly convex. Let $\gamma(t_0)$ be a point of γ inside C. Since C is open, there exists $\epsilon > 0$ such that $B_\epsilon(\gamma(t_0)) \subset C$. By continuity, there exists a vector v in the 2-plane generated by the vectors ξ_p and $\gamma'(0)$ such that $\langle v, \xi_p \rangle > 0$, and the geodesic $\sigma(t) = exp_p tv$ has a point $q_1 = \sigma(t_1)$ in the ball $B_\epsilon(\gamma(t_0))$. By construction, σ is transverse to S in p. Therefore, there exists a neighborhood $(-\tau, \tau)$ of $0 \in R$, such that $\sigma(0, \tau)$ is outside C, and $\sigma(-\tau, 0)$ is inside C. In particular if $t_2 \in (-\tau, 0)$, the point $q_2 = \sigma(t_2) \in C$. Then σ connects q_1 to q_2 of C, but it is not contained in C. This contradicts the fact that C is strongly convex, and completes the proof.

Proposition 2.1 Assume that M is submanifold of N and that M separates N in two connected components. Assume further that the eigenvalues of the second fundamental form of M do not change sign. Then M is the boundary of convex body in N.

Proof : Let A and B be the connected components of $N - M$. We can choose an unit normal vector field in M such that the second fundamental form is semidefinite positive. By [2], M is locally convex. This means that for every $p \in M$ there exists a neighborhood V_p of the origin in T_pN such that $exp_p(V_p \cap T_pM)$ is contained in the closure of one of the two connected components of $N - M$,(here exp_p denotes the exponential map of N). Let us assume that this connected component is B. In this case, we will show that $\overline{A}$ is a convex body of N. In fact, it is enough to show that $\overline{A}$ is convex.

The argument to be used is an adaptation of the method used by E. Schmidt to show that the simple locally convex curves of the plane are boundaries of convex bodies.

If $\overline{A}$ is not convex, then there exists a point $p \in \overline{A}$ such that, for every $\epsilon > 0$ $\overline{A} \cap B_\epsilon(p)$ is not strongly convex. It is clear that such p must be in M. Let $\epsilon_0 > 0$ be such that $B_{\epsilon_0}(p)$ is strongly convex and that $C = \overline{A} \cap B_{\epsilon_0}(p)$ is connected. Then there are points $\overline{p}$ and $\overline{q}$ in C that cannot be connected by a minimal geodesic contained in C. Since $int\ C \neq$, there exists distinct points $p_1 = \overline{p}, p_2, ..., p_m = \overline{q}$ in $int\ C$ and there exists a unique minimal geodesic joining p_i to p_{i+1} which is contained in C. However, there exists an index k such that for $i \leq k$, p_1 can be joined to p_i by a minimal geodesic contained in $int\ C$ but p_1 cannot be joined to p_{k+1} by a minimal geodesic contained in $int\ C$. Let $g(t)$ be the minimal geodesic joining $p_k = g(0)$ to $p_{k+1} = g(l)$, and let $\gamma_t(s)$ be the minimal geodesic joining p_1 to $g(t)$. Set $L = \{t \in [0, l] \mid \gamma_t(s)$ is contained in $int\ C\}$. Since L is bounded and nonempty, there exists t_0 such that $t_0 = \sup L$. The geodesic $\gamma_0 = \gamma_{t_0}$ connecting p_1 to $g(t_0)$ is contained in $\overline{C}$, because γ_0 is limit of geodesics contained in $int\ C$. Furthermore, γ_0 is tangent to M. In fact, since $t_0 = \sup L$, γ_0 has a point in common with the boundary ∂C of C. Since $B_{\epsilon_0}(p)$ is strongly convex and γ_0 has points in $int\ B_{\epsilon_0}(p)$, by Lemma 2.1, cannot be tangent to $\partial B_{\epsilon_0}(p)$. Therefore γ_0 is tangent to M. Let $q = \gamma_0(s_1)$ be the first point of M where γ_0, issuing from p_1 is tangent M. Then the geodesic $\sigma(s) = \gamma_0(s_1 - s)$ that starts at q and passes through p_1 is contained in A, for $0 < s \leq s_1$. This contradicts the fact that M is locally convex. Therefore $\overline{A}$ is a convex body. This completes the proof of Proposition 2.1.

Proposition 2.2 Let A be a convex body in N. Suppose that the boundary $M = \partial A\ of\ A$ is a compact and connected submanifold of N. If M is contained in a normal neighborhood of an interior point of A, then M is diffeomorphic to a sphere.

Proof: Let U be a normal neighborhood of a point $p \in int\ A$, such that $M \subset\ U$. Then, any geodesic that issues from p leaves U, hence $\overline{A}$. Since M is the boundary of a convex body, by Lemma 2.1, the geodesics that issue from p must meet M transversely. On the other hand, since U is a normal neighborhood of the point p, the geodesics that issue from p do not meet in U. Thus, we can define a map

$$\phi : M \ \rightarrow\ S^n\ \subset\ T_pN$$

by

$$\phi(q) = \frac{\exp_p^{-1}(q)}{|\exp_p^{-1}(q)|}.$$

Clearly ϕ is a diffeomorphism, and this concludes the proof.

The Proposition 2.2 has how consequence the THEOREM 1, in fact,

Corollary 2.1. Suppose that N is simply connected and $K_N \leq 0$. If M is a compact hypersurface of N such that $K_M > K_N$ then, there exists a point $p \in M$ and orthonormal vectors V and W in T_pM such that $K_M(V, W)_p > 0$.

Proof: Since $K_M > K_N$, the eigenvalues of the second fundamental form do not change sign. Since N is simply connected and M is a compact hypersurface of N, M separates N in two connected components. By Proposition 2.1, M is the boundary of a convex body and by Proposition 2.2, M is diffeomorphic to a sphere. If $K_M \leq 0$, there M is covered by $\mathbf{R}^n$, which is a contradiction.

Let L be an orientable $(n+1)$-dimensional Riemannian manifold and let $f : L \rightarrow \mathbf{R}$ be a differentiable functions without critical points. We will denote by $S_t = f^{-1}(t)$ the level hypersurface of f at t. We will denote by n_t a unit normal vector field of S_t, and by $\mu_t(p)$ the greatest eigenvalue of the second fundamental form of S_t at p along η_t. Let H be an orientable $n-$dimensional Riemannian manifold, and let $x : H \rightarrow L$ be an isometric immersion. We will denote by ξ a unit normal vector field of H, and by λ_p the smallest eigenvalue of the second fundamental form of x at p along ξ.

Proposition 2.3. With the above notation, assume that at each critical point p of $f \circ x$

$$\lambda_p > \mu_x(p).$$

Then, $f \circ x$ is a Morse function that has no saddle points.

Proof. We denote by $h = f \circ x$ the restriction of f to $x(H)$. If h has no critical points the result is trivial. Assume that $p_0 \in H$ is critical point of h. Let S_{t_0} be the level hypersurface of h which passes through $x(p_0)$. We must show that p_0 is a nondegenerate critical point of h and that p_0 is not a saddle point of h.

By Nash's Theorem [15], we may assume that L is isometrically embedded in $\mathbf{R}^r$, for large. We consider the orthogonal decomposition of $\mathbf{R}^r$ given by

$$\mathbf{R}^r = T_{x(p_0)}L \oplus (T_{x(p_0)}L)^{\perp}$$

and let $P : \mathbf{R}^r \to T_{x(p_0)}L$ be the corresponding orthogonal projection. Because the result is local, we can restrict ourselves to a neighborhood V of $x(p_0)$ in L where the restriction $P|_V$ is a diffeomorphism onto $P(V)$. To simplify the notation, we will assume that x is an embedding and we will identify H with $x(H)$. We will also denote $H = H \cap V$ and $S_{t_0} = S_{t_0} \cap V$.

By projecting orthogonally V onto T_{p_0} by P, we will obtain submanifolds $\tilde{H} = P(u)$ and $\tilde{S}_{t_0} = P(W)$ in $T_{p_0}L$, where u and W are, respectively, neighborhoods of p_0 in H and S_{t_0}, with the property that the restrictions $P|_u$ and $P|_W$ are embeddings. Since p_0 is a critical point of $h, T_{p_0}H = T_{p_0}S_{t_0}$. Thus is clear that $\tilde{H}$ and $\tilde{S}_{t_0}$ are contained in $T_{p_0}H \oplus \{t\xi_{p_0} \mid t \in \mathbf{R}\}$.

Denote by $\tilde{\lambda}_{p_0}$ the smallest eigenvalue of the second fundamental form of $\tilde{H}$ at p_0 along ξ_{p_0}, and by $\tilde{\mu}_{p_0}$ the greatest eigenvalue of $\tilde{S}_{t_0}$ at p_0, with respect to ξ_0. Since $\lambda_{p_0} > \mu_{X(p_0)}$, we have that $\tilde{\lambda}_{p_0} > \tilde{\mu}_{p_0}$.

Consider the function $F = f \circ P^{-1} : P(V) \to \mathbf{R}$. It is clear that F is differentiable. Moreover, the level hypersurfaces of F are manifolds $\tilde{S}_t = P(V \cap S_t)$.

Claim 1. If $X \in T_{p_0}H$, then $d^2 f_{p_0}(X, X) = d^2 F_{p_0}(X, X)$.

In fact, by the definition of F,

$$dF_{p_0}(X) = df_{P^{-1}(p_0)}.dP_{p_0}^{-1}(X)$$

and

$$d^2F_{p_0}(X,X) = d^2 f_{P^{-1}(p_0)}(dP_{p_0}^{-1}(X), dP_p^{-1}(X)) + df_{P^{-1}(p_0)}\, d^2P_{p_0}^{-1}(X,X).$$

Since p_0 is a critical point of h, $dh_{p_0}(v) = df_{x(p_0)}dx_{p_0}(v) = 0$ for every vector $v \in T_{p_0}H$. But $x(p_0) = P^{-1}(p_0) = p_0$. Then $df_{p_0}(w) = 0$ for every $w \in T_{p_0}H$. Therefore,

$$d^2F_{p_0}(X,X) = d^2 f_{p_0}(X,X).$$

Claim 2. $p_0 = P(p_0)$ is a nondegenerate critical point of $F|_{\tilde{H}}$, which is not saddle point.

Since p_0 is a critical point of h, $T_{p_0}\tilde{H} = T_{p_0}\tilde{S}_{t_0}$. We may assume that $\tilde{H}$ and $\tilde{S}_t$ are graphs of functions α and β defined in $T_p\tilde{H}$, respectively. Thus,

$$\tilde{H} = \{(x_1, ..., x_n, x_{n+1}) \mid x_{n+1} = \alpha(x_1, ..., x_n)\}$$

$$\tilde{S}_{t_0} = \{(x_1, ..., x_n, x_{n+1}) \mid x_{n+1} = \beta(x_1, ..., x_n)\}$$

Now, we will express the second derivative of F at the point p_0, by computing $\frac{\partial^2 F}{\partial x^2}$ with respect to $\tilde{H}$ and $\tilde{S}_{t_0}$.

Along $\tilde{H}$, we obtain:

$$\frac{\partial^2}{\partial x_i^2} F(x_1, ..., x_n, (x_1, ..., x_n)) = \frac{\partial^2 F}{\partial x_i^2} + \frac{\partial^2 F}{\partial x_{n+1} \partial x_i} . \frac{\partial \alpha}{\partial x_i} + \frac{\partial F}{\partial x_{n+1}} . \frac{\partial^2 \alpha}{\partial x_i^2}$$

But, at p_0, $\frac{\partial \alpha}{\partial x_i} = 0$. Therefore

$$\frac{\partial^2}{\partial x_i^2} F(x_1, ..., x_n, \alpha(x_1, ..., x_n)) = \frac{\partial^2 F}{\partial x_i^2} + \frac{\partial F}{\partial x_{n+1}} \frac{\partial^2 \alpha}{\partial x_i^2} \tag{2.1}$$

Similarly, along $\tilde{S}_t$, we have

$$\frac{\partial^2}{\partial x_i^2} F(x_1, ..., x_n, \beta(x_1, ..., x_n)) = \frac{\partial^2 F}{\partial x_i^2} + \frac{\partial F}{\partial x_{n+1}} \frac{\partial^2 \beta}{\partial x_i^2} \tag{2.2}$$

Since $F(\tilde{S}_{t_0})$ is constant, because $\tilde{S}_{t_0}$ is a level hypersurface of F, $\frac{\partial^2}{\partial x_i^2} F(x_1, ..., x_n, \beta(x_1, ..., x_n)) = 0$. Thus, (2.2) becomes

$$\frac{\partial^2 F}{\partial x_i^2} + \frac{\partial F}{\partial x_{n+1}} \frac{\partial^2 \beta}{\partial x_i^2} = 0. \tag{2.3}$$

It follows from (2.1) and (2.3), that, at the point p_0,

$$\frac{\partial^2 F}{\partial x_i^2} = \frac{\partial F}{\partial x_{n+1}} \left(\frac{\partial^2}{\partial x_i^2} (\alpha - \beta) \right) = 0.$$

Since f has no critical point in V, F has no critical point in $P(V)$. Since $\frac{\partial F}{\partial x_i}(p_0) = 0$, for $i = 1, 2, ..., n$, we have that $\frac{\partial F}{\partial x_{n+1}}(p_0) \neq 0$.

Now, observe that

$$\frac{\partial^2 \alpha}{\partial x_i^2} = B^1 \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right)_{p_0}$$

and

$$\frac{\partial^2 \beta}{\partial x_i^2} = B^2\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right)_{p_0}$$

where $B^1\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right)_{p_0}$ (resp. $B^2\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right)_{p_0}$) denotes the value for the pair $\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right)$ of the second fundamental form of $\tilde{H}$ (resp. $\tilde{S}_{t_0}$) at p_0, along ξ_{p_0} (resp. η_{p_0}).

Since $\tilde{\lambda}_{p_0} > \tilde{\mu}_{p_0}$,

$$\frac{\partial^2}{\partial x_i^2}(\alpha - \beta) > 0.$$

This completes the proof of Proposition 2.3.

Lemma 2.2. $K_M > 4K$.

Proof. It's a straightforward consequence.

We denote by $i(N)$ the injectivity radius of N, that is to say, $i(N)$ is the largest number $\rho > 0$ such that, for all $p \in N$, the exponential map, exp_p, is an embedding in the open ball of radius ρ in T_pN. In [14], M. Maeda proved that, under the hypothesis of Teorema A, $i(N) \geq \frac{\pi}{\sqrt{K}}$.

Let $\mathcal{D}$ be a compact totally convex set of N, such that

$$\mathcal{D} \supset \bigcup_{p \in M} B_{\frac{\pi}{\sqrt{K}}}(x(p)).$$

(the proof of existence of such sets can be found in [7].

Set

$$a = inf\{K_N(X,Y)_p \mid p \in \mathcal{D};\ X, Y \in T_pN \text{ and } \langle X, Y\rangle = 0\}.$$

Since $K_N > 0$ and $\mathcal{D}$ is compact, $a > 0$.

Now , we will make use of the following fact, whose proof can be found in [11].

Lemma 2.3. Let $\gamma(t)$ a geodesic in *int* $\mathcal{D}$ with $|\gamma'(t)| = 1$, and let $Y(t)$ be a Jacobi field along γ, such that $Y(0) = 0$ and $\langle Y(t), \gamma'(t)\rangle = 0$. Then, for all $0 \leq t < \frac{\pi}{\sqrt{K}}$ one has:

$$\sqrt{a}\frac{cos\sqrt{at}}{sin\sqrt{at}} \geq \frac{|Y(t)|'}{|Y(t)|} \geq \sqrt{K}\frac{cos\sqrt{kt}}{sin\sqrt{kt}}.$$

Proof. See [11]

We will denote by $B(p)$ the open ball of N with center at p and radius equal to $\frac{\pi}{2\sqrt{K}}$, and by $S(p)$ the geodesic sphere which is the boundary of $B(p)$.

Lemma 2.4. We can choose a unit normal vector field η in $S(p)$, such that each eigenvalues μ of the second fundamental form of $S(p)$ with respect to η satisfies

$$\sqrt{K} > \mu \geq 0.$$

Proof. We can consider $\mathcal{D}$ sufficiently large, so that $S(p) \subset int\ \mathcal{D}$. Let X be a differentiable unit tangent vector field in $S(p)$ defined in a neighborhood of a point q. Let $\alpha : (-\epsilon, \epsilon) \to S(p)$ be the solution of X such that $\alpha(0) = q$ and $\alpha'(0) = X_q$.

Let $\sigma : (-\epsilon, \epsilon) \times [0, \frac{\pi}{2\sqrt{K}}] \to N$ be the variation defined by

$$\sigma(s,t) = exp_p\ t\tilde{\alpha}(s) \text{ where } \tilde{\alpha}(s) = \frac{exp_p^{-1}(\alpha(s))}{|exp_p^{-1}(\alpha(s))|}.$$

Since $B(p)$ is contained in a normal neighborhood, $\tilde{\alpha}$ is well-defined and σ is differentiable.

Denote by $J(t) = \frac{\partial\sigma}{\partial s}(0,t) = (d\ exp_p)_{t\tilde{\alpha}(0)} t\tilde{\alpha}'(0)$ the Jacobi field along the geodesic $\sigma(0,t)$. It is clear that $J(0) = 0$ and $J(\frac{\pi}{2\sqrt{K}}) = X_q$. Denote by $Z(t) = \frac{\partial\sigma}{\partial t}(0,t) = (d\ exp_p)_{t\tilde{\alpha}(0)}\tilde{\alpha}(0)$ the velocity vector of the geodesic $\sigma(0,t)$.

Choose a unit normal vector field η such that

$$\eta_q = -Z(\frac{\pi}{2\sqrt{K}}).$$

Then

$$\begin{aligned}
\mu(q) &= \langle \overline{\nabla}_X X, \eta\rangle_q = -\langle \overline{\nabla}_X \eta, X\rangle_q = \langle \overline{\nabla}_X(-\eta), X\rangle_q = \\
&= \langle \frac{\overline{D}}{ds}\frac{\partial\sigma}{\partial t}, \frac{\partial\sigma}{\partial s}\rangle_{(0,\frac{\pi}{2\sqrt{K}})} = \langle \frac{\overline{D}}{dt}\frac{\partial\sigma}{\partial s}, \frac{\partial\sigma}{\partial s}\rangle_{(0,\frac{\pi}{2\sqrt{K}})} = \\
&= \frac{1}{2}\frac{d}{dt}\langle \frac{\partial\sigma}{\partial s}, \frac{\partial\sigma}{\partial s}\rangle_{(0,\frac{\pi}{2\sqrt{K}})} = \frac{1}{2}\langle J(t), J(t)\rangle_{\frac{\pi}{2\sqrt{K}}}
\end{aligned}$$

(where $\overline{D}$ is covariant derivative of N).

Observe that

$$\frac{|J(t)|'}{|J(t)|} = \frac{\langle J(t), J'(t)\rangle}{\langle J(t), J(t)\rangle} = \frac{1}{2}\frac{\langle J(t), J(t)\rangle'}{\langle J(t), J(t)\rangle}.$$

and that in $t = \frac{\pi}{2\sqrt{K}}$, $\langle J(t), J(t)\rangle = 1$. It follows from Lemma 3.2 that

$$\sqrt{a}\ cot\sqrt{a}\frac{\pi}{2\sqrt{K}} \geq \ \langle \overline{\nabla}_X X, \eta\rangle_q \ \geq 0, \quad 0 < a < K.$$

By taking $u = \frac{\sqrt{a}}{\sqrt{K}}\frac{\pi}{2}$, one has

$$\frac{2\sqrt{K}}{\pi} u\ cot\ u \geq \ \langle \overline{\nabla}_X X, \eta\rangle_q \ \geq 0, \quad 0 < u < \frac{\pi}{2}.$$

Now, set $f(u) = u\ cot\ u, \quad 0 < u < \frac{\pi}{2}$. Observe that

i) $1 = \lim_{u\to 0}\ f(u)$

ii) $f'(u) = \frac{sin\ 2u - \ 2u}{2sin^2 u} < 0, \quad$ if $u > 0$.

Hence, $1 \geq u\ cot\ u$, and therefore,

$$\frac{2}{\pi}\sqrt{K} \geq \langle \overline{\nabla}_X X, \eta\rangle_q \geq 0.$$

We finally conclude that

$$\sqrt{K} > \frac{2}{\pi}\sqrt{K} \geq \mu \geq 0,$$

and this completes the proof of Lemma 2.4.

Lemma 2.5. For all $p \in N$ the open ball $B(p)$ is strongly convex.

Proof: Since $i(N) \geq \frac{\pi}{\sqrt{K}}$, $S(p)$ is contained in a normal neighborhood u of p. Furthermore, if q_1 and q_2 are points of $B(p)$ there exists a unique minimal geodesic connecting q_1 to q_2. Since u is simply connected, $S(p)$ separates u into two connected components ([13]). By Lemma 3.4, the eigenvalues of the second fundamental of $S(p)$ do not change sign. By Proposition 2.1, $S(p)$ is then a boundary of a convex body of N.

It is enough to show that the minimal geodesic that joins two points of $B(p)$ is contained in $B(p)$. This follows by using the same adaptation of the E. Schimidt's method used in the proof of Proposition 2.2. This concludes the proof of Lemma 2.5.

Assertion 1. There exists a Morse function defined in M that has only two critical points, one maximum and one minimum.

Let p_0 be a point of N, and let $\gamma(t)$ be a geodesic of N passing through p_0. Reparametrize γ so that $|\gamma'(t)| = 1$ and $\gamma(\frac{\pi}{\sqrt{K}}) = p_0$.

We will denote by $T_{\gamma(t)}$ the parallel translation of N along from $\gamma(0)$ to $\gamma(t)$. Consider the set:

$$\tilde{\Sigma}_\gamma(0) = \{v \in T_{\gamma(0)N} \mid \langle v, \gamma'(0)\rangle > 0 \text{ and } |v| = \frac{\pi}{2\sqrt{K}}\}.$$

Thus, $\Sigma_\gamma(t) = exp_{\gamma(t)} T_t(\tilde{\Sigma}_\gamma(0))$ is a hemisphere of the geodesic sphere with center in $\gamma(t)$ and radius $\frac{\pi}{2\sqrt{K}}$.

Lemma 2.6. For $0 < t < \frac{\pi}{\sqrt{K}}$, the family $\{\Sigma_\gamma(t)\}$ is a foliation of $B(p_0)$.

Proof. First, we claim that if $0 < t_1 < t_2 < \frac{\pi}{\sqrt{K}}$, then $\Sigma_\gamma(t_1) \cap \Sigma_\gamma(t_2) \cap B(p_0) = \emptyset$. In fact, Suppose there exists $q \in \Sigma_\gamma(t_1) \cap \Sigma_\gamma(t_2) \cap B(p_0)$. Then $d(q, \gamma(t_1)) = d(q, \gamma(t_2)) = \frac{\pi}{2\sqrt{K}}$, and $d(q, p_0) < \frac{\pi}{2\sqrt{K}}$.

Consider the open ball $B(q)$ with center in q and radius $\frac{\pi}{2\sqrt{K}}$. By Lemma 2.5, $B(q)$ is strongly convex. It is clear that $p_0 \in B(q)$. Let $\sigma_i(s)$ $(i = 1, 2)$ be the minimal geodesic connecting $\gamma(t_i)$ $(i = 1, 2)$ to q. By definition of $\Sigma_\gamma(t)$, $\langle \sigma_i'(0), \gamma'(t_i)\rangle > 0$, hence, γ is transverse at $\gamma(t_i)$ to the geodesic sphere $S(q)$, boundary of $B(q)$, $(i = 1, 2)$. This implies that there exist disjoint neighborhoods V_1 and V_2 of t_1 and t_2, respectively, such that $\gamma(V_i)$ has points inside $B(q)$ and outside $B(q)$ near $\gamma(t_i)$ $(i = 1, 2)$. Now, let $\gamma(t_0)$ be a point of $\gamma(v_1) \cap B(q)$. Then $\gamma(t)$, $t_0 \leq t \leq \frac{\pi}{\sqrt{K}}$, is a segment of a minimal geodesic connecting $\gamma(t_0)$ to p_0 inside $B(q)$, and $\gamma(t)$ leaves $B(q)$. This contradicts the fact that $B(q)$ is strongly convex, and proves our claim.

Now, let q be any point of $B(p_0)$. Consider the geodesic sphere $S(q)$. Since p_0 is inside $B(q)$, the geodesic $\gamma(t)$ has points inside $B(q)$. By ([7]), γ goes to infinite, hence it leaves the closure $\overline{B(q)}$ of $B(q)$.

Let $\gamma(t_1)$ be the point where γ enters $B(q)$ for first time before passing through p_0. Then, $q \in \Sigma_\gamma(t_1)$. In fact, by construction, $d(q, \gamma(t_1)) = \frac{\pi}{2\sqrt{K}}$. Furthermore, since γ is transverse to $S(q)$ at $\gamma(t_1)$, if $\sigma(s)$ is the minimal geodesic joining $\gamma(t_1)$ to q, then $\langle \sigma'(0), \gamma'(t_1)\rangle > 0$. This fact completes the proof of Lemma 2.6.

Let $f_\gamma : B(p_0) \to \mathbf{R}$ be the function defined by

$$f_\gamma\ (q) = t \ \Leftrightarrow\ q \in \Sigma_\gamma(t).$$

By Lemma 2.6, f_γ is well-defined and by definition of the family $\{\Sigma_\gamma(t)\}$ f_γ is differentiable.

Since $K_M > 4K > 0$, by Bonnet-Myers' Theorem, M is compact and $diam\ M \leq \frac{\pi}{2\sqrt{K}}$ ($diam\ M$ denotes diameter of M). Since $K_M > K_N$, no curve of $x(M)$ can be a geodesic in N, and so

$$diam\ x(M) < diam\ M \leq \frac{\pi}{2\sqrt{K}},$$

then, for every point $p \in M$, $x(M) \subset B(x(p))$. Now, by fixing $p \in M$ and a geodesic γ in N passing through $x(p)$; we can construct a function f_γ as above. Therefore, we can define the function $h_\gamma : M \to R$ by $h_\gamma = f_\gamma \circ x$.

Lemma 2.7. h_γ is a Morse function that has two critical points, one maximum and one minimum.

Proof: It is clear that h_γ is well-defined and is differentiable. Observe now, that f_γ has no critical points in $B(x(p))$. On the other hand, the maximum eigenvalues μ_t of the second fundamental form of each level surface $\Sigma_\gamma(t)$, with respect to the unit normal vector field as in Lemma 2.4, is strictly less that the minimum eigenvalue of the second fundamental form of x with respect to ξ according to Lemma 2.4. By Proposition 2.3, h_γ is a Morse function without saddle points. Since M is compact, h_γ has only two critical points, one maximum and one minimum ([4]). This completes the proof of the Lemma 2.7 and of the Assertion 1.

Assertion 2. x is a embedding.

Proof of Assertion 2: Suppose, by contradiction, that x is not an embedding. Then, there exists distinct points p and q of M, such that $x(q) = x(p)$.

Consider the geodesic $\gamma(t)$ that passes through $x(p) = \gamma(\frac{\pi}{\sqrt{K}})$ and that $\gamma'(\frac{\pi}{\sqrt{K}}) = \xi_p$ is the unit normal vector field ξ of M at p.

Now, consider the function $h_\gamma = f_\gamma \circ x$. By Lemma 2.7 h_γ is a Morse function that has only two critical points, one maximum and one minimum.

By construction of h_γ, p is a critical point of h_γ, which we assume to be a point of minimum, with $h_\gamma(p) = t_0$. (the case where p is a point of maximum can be treated similarly).

Let u and v be disjoint neighborhoods of p and q, respectively, such that x restricted to u or to v is an embedding. we will consider two cases:

1^{st} **case.** $x(u)$ is not transverse to $x(v)$ at $x(p)$. In this case, q is also critical point of h_γ and so, is a point of maximum. Furthermore, $h_\gamma(q) = h_\gamma(p) = t_0$. Since q is a point of maximum of h_γ, there exists a neighborhood v_1 of q in M such that if $r \in v_1$ and $r \neq q$, then $h_\gamma(r) < t_0$. This implies that there exists a point of minimum if h_γ in M distinct of p. This contradicts Lemma 2.7.

2^{nd} **case.** $x(u)$ is transverse to $x(v)$ at $x(p)$. In this case, there exist points of $x(V)$ contained in the level below $x(p)$. This implies that there exists another point of minimum distinct from p. This contradicts Lemma 2.7.

Then, x is embedding, thereby proving Assertion 2.

Now, since $B(x(p))$ is simply connected and x is an embedding, $x(M)$ separates $B(x(p))$ in two connected components ([13], p. 72). Since the eigenvalues of the second fundamental form do not change sign, by Proposition 2.1, $x(M)$ is the boundary of a convex body of N. Since $x(M)$ is contained in a normal neighborhood of p_0, by Proposition 2.2, $x(M)$ is diffeomorphic to a sphere. Therefore M is diffeomorphic to a sphere. This completes the proof of Theorem A.

In 1978 was proved in [21] Theorem 2. The proof is based on a series of lemmas. In the context of this survey we are going to prove just some of them.

Theorem 2. Let M be a convex and compact submanifold of N. Assume that N is not compact and $K_N > 0$. Then, M is a homological sphere.

Lemma 2.8. Let A be a convex body of N with non empty boundary ∂A. Let $\gamma : [0, l] \to N$ a geodesic of N such that $\gamma(t) \in intA$ for $t \in [0, l)$ and $\gamma(l) \in \partial A$. Then, there exists $\epsilon > 0$ such that for every $s : 0 < s < \epsilon$, the curve $\gamma(l + s)$ it does not´belongs to $\overline{A}$.

Lemma 2.9. Let M be a convex submanifold of N and A convex body of N with boundary M. Let $\gamma(t)$ be a geodesic of N which is tangent to M at the point $p = \gamma(0)$. Therefore, there exists $\epsilon > 0$ such that $\gamma(-\epsilon, \epsilon)$ is contained in the closure of $N - A$.

Lemma 2.10. Let A and B convex bodies of N. Assume A is a strongly convex set and $A \cap B$ is not empty. Then, any connected component of $A \cap B$ is a strongly convex set.

Proof. Let U be a connected component of $A \cap B$. Since U is a subset of A, given two points in U there exists just one geodesic of N, entirely contained in A, joining them. Suppose U is not a strongly convex set. There are points p, $q \in U$ such that the geodesic γ join p with q leaves U. It is possible to assume p, $q \notin \partial U$, the boundary of U. In fact, if $p \in \partial U$ since B is convex there are positive numbers $0 < \epsilon(p) < r(p)$ such that $B \cap B_{\epsilon(p)}(p)$ is strongly convex. Furthermore, since B and $B_\epsilon(p)$ are open sets it follows that $B \cap B_\epsilon(p)$ is open. By extending γ we are able to obtain points in $B \cap B_\epsilon(p) \cap \gamma$ which are not in ∂U. Since M is a convex and open set we can join p to q by a broken geodesic in $U - \partial U$. By the hypothesis of local convexity, we get a geodesic in $\overline{B}$ with ending points in the interior of B and with a common point with ∂B. This fact is in contradiction with Lemma 2.8 and the proof is complete.

Lemma 2.11. Let M be a submanifold of N, such that both connected components of $N - M$ are convex sets. Then, M is totally geodesic.

Proof. Let us denote by A and B the connected components of $N - M$. Let p be an arbitrary point of M. Since A and B are convex sets, their closures $\overline{A}$ and $\overline{B}$ are also convex. In particular, there are positive numbers $0 < \epsilon(p) < r(p)$ such that $\overline{A} \cap B_{\epsilon(p)}(p)$ and $\overline{B} \cap B_{\epsilon(p)}(p)$ are strongly convex sets. Let $q \neq p$ a point in $M \cap B_{\epsilon(p)}(p)$. By the definition of ϵ, there exists just one minimal geodesic γ of N joining p to q. Since $\overline{A} \cap B_{\epsilon(p)}(p)$ and $\overline{B} \cap B_{\epsilon(p)}(p)$ are strongly convex sets, γ must be contained in their intersection. Then, γ is included in $M \cap B_\epsilon(p)$. Since q is arbitrary, it follows that $M \cap B_\epsilon(p)$ is totally geodesic in p. Since p is arbitrary, M is totally geodesic in N.

Now, we are able to prove Theorem A.

Proof. Denotes by A the convex component of $N - M$ and by B the other component. First, we show that A is bounded. Since M is compact and N is diffeomorphic to $\mathbf{R}^{n+1}$, either A or B is bounded. Let us denote by X the bounded component. We need to prove $X = A$. For that assume X is convex and $X = B$. By Lemma 2.11, M is totally geodesic. Since M is compact, it must exists a closed geodesic in M which is also a closed geodesic in N. But, this is a contradiction. Next, we show that X is convex. Since $\overline{X}$ is compact, there exists a compact subset C_0 of N which contains $\overline{X}$ with the following property: each geodesic joining point of C_0 is contained in C_0. This kind of sets are called totally convex. We observe that a totally convex set is convex. From the next Theorem it turns out that there exists a totally convex subset $C_0^{a_0}$ of N, in such a way that $C_0^{a_0}$ contains $\overline{X}$ and $\partial C_0^{a_0}$ intersect the boundary of $\overline{X}$ in M .

Proposition 2.4. (Cheeger and Gromoll). Let N be a Riemannian manifold with non negatives sectional curvatures. Let C be a convex subset (totally convex) closed in N such that the boundary ∂C of C is not empty. Therefore,

1) For each a, the set

$$C^a = \{p \in C; d(p, \partial C) \geq a\}$$

is convex (totally convex).

2) If $C^{max} = \cap_{C^a \neq \emptyset} C^a$ then $dim C^{max} < dim C$.

Proof. See [7]. Consider the set

$$L = \{a \in [0, l]; \overline{X} \subset C_0^a\}.$$

Since L is not empty and bounded there exists $inf L = a_0$. By definition, $\overline{X} \subset C_0^{a_0}$ and $M_0 = \partial C_0^{a_0} \cap M \neq \emptyset$. If $M_0 = M$ then $X = C_0^{a_0}$. Thus X is convex. If $M_0 \neq M$ consider a point $q \in M_0 - int M_0$.

Since $C_{a_0}^0$ is convex, every geodesic of N which is tangent to M at q has a neighborhood of q in the closure of $N - C_0^{a_0}$. But $C_0^{a_0}$ contains $\overline{X}$, so there exists a geodesic $\gamma(t)$ of N, tangent to M at $q = \gamma(0)$: for $0 < t < \epsilon$, $\gamma(t) \notin \overline{X}$. In particular, from Lemma 2.9, $N - \overline{X}$ can not be convex. Then, X is a convex set.

Now, the closed set $\overline{A}$ is convex, thus by applying Theorem 2.1 to $\overline{A}$ we obtain

$$\overline{A}^a = \{p \in \overline{A}; d(p, M) \geq a\}$$

is convex. Furthermore, if $\overline{A}_0 = \cap_{A^a \neq \emptyset} \overline{A}^a$ then $dim\overline{A}_0 < dim\overline{A}$. Next, we show that $\overline{A}$ reduces to the singleton $\{p_0\}$. For that, we need the following results:

Let $\psi : C \to \mathbf{R}$ a function defined by $\psi(p) = d(p, \partial C)$. Then, for any geodesic segment γ contained in C the function $\psi \circ \gamma$ is weakly convex. In other words

$$\psi \circ \gamma(\alpha t_1 + \beta t_2) \geq \alpha \psi \circ \gamma(t_1) + \beta \psi \circ \gamma(t_2),$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. On the other hand, let us assume $\psi \circ \gamma(s) \equiv d$ is constant on the interval $[a, b]$. Denotes by $V(s)$ the parallel vector field throughout $\gamma_{|_{[a,b]}}$ such that $V(a) = \gamma_a(0)$. Here, γ_a is a minimal geodesic from $\gamma(a)$ to ∂C. Therefore, for every s

$$\exp_{\gamma(s)} tV(s)_{|_{[0,d]}}$$

is a minimal geodesic from $\gamma(s)$ to ∂C. The rectangle

$$\varphi : [a, b] \times [0, d] \to N$$

defined by

$$\varphi(s, t) = \exp_{\gamma(s)t} \circ V(s)$$

is flat and totally geodesic.

If $\overline{A}_0$ contains more than one point, by an convexity argument there exists a geodesic segment σ in $\overline{A}_0$. By definition of $\overline{A}_0$ it turns out that $\psi \circ \sigma \equiv$ constant. By Theorem 2.1, there exists a totally geodesic flat rectangle in $\overline{A}$, which is a contradiction with the fact $K_N > 0$.

Lemma 2.12. $\overline{A}_0$ is a retract of deformation of $\overline{A}$.

Proof. Since $\overline{A}$ is compact and convex there exists a positive number ϵ_1 such that for any $p \in \overline{A}$, the set $\overline{A} \cap B_{\epsilon_1}(p)$ is strongly convex. On the other hand, there exists $\epsilon_2 > 0$ such that if $B_r(q)$ is the open ball of N, with center $q \in \overline{A}$ and radio $0 < r \leq \epsilon_2$, the curve $C : [0, \eta] \to B_r(q)$ is a non constant geodesic and $C_0 : [0, 1] \to B_r(q)$ is a minimal geodesic from q to $C(0)$ with $\langle C(0), C_0(1) \rangle \geq 0$. In particular, the function $S \to d(C(s), q)$ is strictly increasing in $[0, \eta]$.

Let $0 < \epsilon < min\{\epsilon_1, \epsilon_2\}$.

We claim: if $p \in \overline{A}$ and $\overline{A}^b \cap B_\epsilon(p) \neq \emptyset$ for some $b > 0$, then $\overline{A}^b \cap B_\epsilon(p)$ is strongly convex. In fact, by Lemma 2.9 and Proposition 2.4, it is enough to prove that $\overline{A}^b \cap B_\epsilon(p)$ is connected. If not, let r, s two points of different connected component of $\overline{A}^b \cap B_\epsilon(p)$. Let γ a geodesic segment joining r to s and

$$C = \{a \in [0, b] : \gamma \subset \overline{A}^a\}.$$

Since C is not empty and bounded there exists $c = supC$. So, γ is contained in $\overline{A}^c$. Furthermore, since $c = supC$, γ has a common point with the boundary of $\overline{A}^c$. But this is a contradiction with Lemma 2.8. In fact, $\overline{A}^c$ is convex and r and s belong to the interior of $\overline{A}^c$. Thus, $\overline{A}^b \cap B_\epsilon(p)$ is connect which prove our claim.

Let $b > a$ and $b - a < \epsilon$, then $\overline{A}^b$ is a retract of deformation of $\overline{A}^a$. In fact, let

$$f_a^b : \overline{A}^a \to \overline{A}^b$$

defined by $f_a^b(p) = \widetilde{p}$, $p \in \overline{A}^a$, where $\widetilde{p}$ satisfy

$$d(p, \overline{A}^b) = d(p, \widetilde{p}).$$

The function f_a^b is well defined. Let $p \in \overline{A}^a$ then $f_a^b(p) = p$. If $p \in \overline{A}^a - \overline{A}^b$, assume the existence of two different points $\widetilde{p}_1$ and $\widetilde{p}_2$ in $\overline{A}^b$ which realize the distance from p to $\overline{A}^b$. Since $b - a < \epsilon$, $d(p, \overline{A}^b) < \epsilon$. So, $\widetilde{p}_1$ and $\widetilde{p}_2$ belong to the ball $B_\epsilon(p)$. But, $\overline{A}^b \cap B_\epsilon(p)$ is not empty, it follows that $\overline{A}^b \cap B_\epsilon(p)$ is strongly convex. Thus, there exists just one minimal geodesic $\gamma(t)$ in N joining $\widetilde{p}_1 = \gamma(0)$ to $\widetilde{p}_2 = \gamma(l)$ and $\gamma(t)$ is contained in $\overline{A}^b \cap B_\epsilon(p)$. Let

$$h : [0, l] \to \mathbf{R}$$

defined by

$$h(t) = d^2(p, \gamma(t)).$$

Thus,

$$h(0) = d^2(p, \widetilde{p}_1) = d^2(p, \widetilde{p}_2) = h(l).$$

Since h is differentiable, there exists $t_0 \in (0, l)$ such that $h'(t_0) = 0$. Since $\gamma(t)$ is contained in $\overline{A}^b \cap B_\epsilon(p)$, it follows that t_0 is the only one minimum of h. So, $h(t_0) < h(0)$, which is in contradiction with the fact *that* $\widetilde{p}_1$ *realizes the distance from* p *to* $\overline{A}^b \cap B_\epsilon(p)$. *Therefore,* f_a^b *is well defined.*

Next, we prove that f_a^b *is a continuous function. Let* p *be an arbitrary element of* $\overline{A}^a$ *and* (p_n) *a convergent sequence of points in* $\overline{A}^a$ *such that:*

$$\lim p_n = p.$$

Denotes by $\widetilde{p}_n = f_a^b(p_n)$ *and for* $\widetilde{p} = f_a^b(p)$. *We show that* $\lim \widetilde{p}_n = \widetilde{p}$.

By the own definition of f_a^b, *we get*

$$| d(p_n, \widetilde{p}_n) - d(p, \widetilde{p}) | = | d(p_n, \overline{A}^b) - d(p, \overline{A}^b) | \leq d(p_n, p).$$

Thus, $\lim | d(p_n, \widetilde{p}_n) - d(p, \widetilde{p}) | = 0$.

Since $\overline{A}^b$ *is compact, the sequence* $(\widetilde{p}_n)$ *admit a convergent subsequence* $(\widetilde{p}_{nk})$. *Let* $\widetilde{p}_0 = \lim \widetilde{p}_{nk}$. *Then,*

$$\lim | d(p_{nk}, \widetilde{p}_{nk}) - d(p, \widetilde{p}_0) | = 0.$$

So, for any n_k

$$| (p, \widetilde{p}_0) - d(p, \widetilde{p}) | \leq | d(p, \widetilde{p}_0) - d(p_{1k}, p_{1k}) | + | (p_{nk}, \widetilde{p}_{nk}) - d(p, \widetilde{p}) |$$

Then,

$$d(p, \widetilde{p}_0) = d(p, \widetilde{p}).$$

Since $\widetilde{p}_0 \in \overline{A}^b$ *and* f_a^b *is well defined it follows that* $\widetilde{p}_0 = \widetilde{p}$. *So,* f_a^b *is continuos.*

Let $i : \overline{A}^b \to \overline{B}^a$ *the inclusion application. It is clear that* $i \circ f_a^b$ *is a identity* $id_{\overline{A}^b}$ *in* $\overline{A}^b$. *So,* $\overline{A}^b$ *is a retract of deformation of* $\overline{A}^a$. *Let*

$$F : [0,1] \times \overline{A}^a \to \overline{A}^a$$

the application defined by

$$F(t,p) = \exp_p t \exp_p^{-1}(f_a^b(p)).$$

We know that $b - a < \epsilon$ *and* f_a^b *is continuous. So,*

F *is well defined and*

$$F(0,p) = p, \;\; F(1,p) = f_a^b(p).$$

Thus, $i \circ f_a^b$ *is homotopic to the identity of* $\overline{A}^a$*, which prove that* $\overline{A}^b$ *is a retract of deformation of* $\overline{A}^a$*, as we claimed.*

Consider the ball $B_\epsilon(p_0)$*, where* $\{p_0\} = \overline{A}_0$*. Since* $\{p_0\} = \cap_{A^a \neq \emptyset} \overline{A}^a$*, there exists a positive value* c *such that* $\overline{A}^c$ *is contained in* $B_\epsilon(p_0)$*. Clearly,* $\{p_0\}$ *is a retract of deformation of* $\overline{A}^c$*. It is possible, to decompose the interval* $[0, c]$ *in a finite number of points:* $0 = t_0 < t_1 < ... < t_k = c$ *in such a way that* $t_i - t_{i-1} < \epsilon$*. Therefore, since* $\overline{A}^{t_{i-1}}$ *is a retract of deformation of* $\overline{A}^{t_i}$*, by transitivity the singleton* $\{p_0\}$ *is a retract of deformation of* $\overline{A}$*, which ends the proof.*

Remark. *By the Poincaré-Lefschetz Duality Theorem, we have*
$H^k(\overline{A}) \cong H_{n-k+1}(\overline{A}, M)$*. Since* $\{p_0\}$ *is a retract of deformation of* $\overline{A}$*, we get that*

$$H^k(\overline{A}) \cong H^k(\{p_0\}).$$

It follows that

$$H_{n+1}(\overline{A}, M) \cong Z$$

and $H_q(\overline{A}, M) \cong 0, q < n + 1$*. By considering the exact sequence* $M \longrightarrow \overline{A} \longrightarrow (\overline{A}, M)$*, we get:*

$$... \to H_{q+1}(\overline{A}, M) \to H_q(M) \to H_q(\overline{A}) \to ...$$

So, for $0 < q < n$*, we have*

$$0 \to H_q(M) \to 0.$$

Therefore, $H_q(M) \cong 0$*, for* $0 < q < n$*. Since* M *is a connected manifold* $H_0(M) \cong Z$*. Since* M *is compact, orientable without boundary we have*
$H_n(M) \cong Z$*. Thus,* $H_*(M) \cong H_*(S^n)$*.*

At the present, the Poincare's Conjecture has already been solved and consequently this fact proves that M *is homeomorphic to a sphere.*

3. Proof of Theorem B

Lemma 3.1. If M satisfies FITA then every metric ball is strongly convex

Proof. Let us suppose by contradiction that there exists a point $p_0 \in M$ and a real number $\rho > 0$ such that the open ball $B = B_\rho(p_0)$ is not strongly convex. Then there are points m_1 and m_2 such that the segment $[m_1, m_2]$

of the geodesic (t) joining the points m_1 and m_2 has points outside the closure $\overline{B}$ of the set B. Let p and q be the points in $\overline{B}$ where $\gamma(t)$ get in and get out respectively.

Consider the configuration $\{g, \gamma, \theta\}_p$ given by the geodesics g and which get out of the point p_0 and pass through p and q respectively. We consider them parameterized so that $g(-\rho) = p_0 = (-\rho)$, $g(0) = p$ and $(0) = q$. The angle between $[p, q]_\gamma$ and g is θ.

By the Gauss lemma we have $\theta < \frac{\Pi}{2}$, thus by FITA there is a point $r = g(t)$ with $t > 0$ such that $\{[p, q]_\gamma, [p, r]_g, [r, q]_\tau\}$ is an isosceles triangle with basis $[p, q]_\gamma$. On the other hand, $\{[p, q]_\gamma, [p_0, p]_g, [p_0, q]_\sigma\}$ is also an isosceles triangle and this contradicts the FITA.

The following results are immediate consequences of Lemma 3.1

Lemma 3.2. *If M satisfies FITA then every geodesic of M realizes the distance between every pair of its points.*

Lemma 3.3. *If M satisfies FITA then for every $p \in M$ the exponential map $exp_p : T_pM \to M$ is a homeomorphism. This means that M is diffeomorphic to $\mathbf{R}^n$ and in particular M is simply connected and so is orientable.*

Lemma 3.4. *If M satisfies FITA then every geodesic of M cannot lie inside any compact set.*

Lemma 3.5. *Let M satisfies FITA. If distinct metric spheres S_1 and S_2 of M are tangent to each other then the set $S_1 \cap S_2$ is unitary.*

The following two lemmas are immediate consequences.

Lemma 3.6. *If M satisfies FITA then the closure of a strongly convex body in M is strongly convex.*

Lemma 3.7. *Let M satisfies FITA. If H and K are strongly convex intersecting subsets of M then $H \cap K$ is also strongly convex.*

Proof of Theorem B:

By using that M satisfies FITA and Lemma 3.2 we have that all geodesic of M are lines and consequently for every point $p \in M$, the exponential map $exp_p : T_pM \to M$ is a difeomorfism (Lemma 3.3) .

Given an arbitrary point $p \in M$ and a unit vector $v \in T_pM$, let us consider the sets:

$$L_p = \{w \in T_pM; \langle w, v\rangle = 0\},$$

$$L_p^+ = \{w \in T_pM; \langle w, v\rangle \geq 0\},$$

$$L_p^- = \{w \in T_pM; \langle w, v\rangle \leq 0\}.$$

These sets allow us to define the following subsets of M:

$$\Sigma = exp_p(L_p),$$

$$H^+ = exp_p(L_p^+),$$

$$H^- = exp_p(L_p^-).$$

We consider the geodesic ray $r(t) = exp_p(tv)$ starting at p in the direction v and $B_r = \cup_{t>0} B_t(r(t))$, where $B_t(r(t))$ is the open ball centered at the point $r(t)$ and radius t.

The Lemma 3.5 assures that if $t_1 < t_2$ then $B_{t_1}(r(t_1)) \subset B_{t_2}(r(t_2))$. Moreover, as for each t the set $B_t(r(t))$ is strongly convex (Lemma 3.1) and $B_{t_1}(r(t_1)) \subset B_{t_2}(r(t_2))$ when $t_1 < t_2$, we have that B_t is strongly convex (c.f. [12])

Let us denote by $\overline{B}_r$ the closure of B_r. We will prove that $\overline{B}_r = H^+$. Using that for each $t > 0$, $\overline{B_t(r(t))} \subset H^+$, by convexity and the equality

$$\overline{B}_r = \overline{\cup_{t>0} B_t(r(t))} = \cup_{t>0} \overline{B_t(r(t))}\,,$$

we conclude that $\overline{B}_r \subset H^+$.

Let $q \in H^+$ be an arbitrary point and let q_n be a convergent sequence made up of interior points in H^+ such that $\lim q_n = q$. Let ρ_n be the geodesic segment connecting the points p and q_n. As q_n is an interior point then $\langle r'(0), \rho_n'(0)\rangle > 0$. The manifold M satisfies FITA so there exists $r_n = r(t_n)$ in such a way that the geodesic triangle whose vertices are the points p, q_n, and r_n is an isosceles triangle with basis ρ_n. Therefore

$$q_n \in \overline{B_{t_n}(r(t_n))} \subset \overline{B}_r.$$

As $\overline{B}_r$ is closed we have $q \in \overline{B}_r$. Thus, $H^+ \subset \overline{B}_r$. This way we have proved that $\overline{B}_r = H^+$.

According to Lemma 3.6 the set B_r is strongly convex and consequently H^+ is also strongly convex.

By using a similar construction with the radius $s(t) = exp_p t(-v)$, we obtain that the set $H^- = \overline{B}_s$ is strongly convex.

According to the Lemma 3.7 the set $\Sigma = H^+ \cap H^-$ is strongly convex. Since exp_p is a diffeomorphism we have that Σ is a complete submanifold of M without boundary with dimension $n-1$. This means that Σ is a totally geodesic submanifold of M.

Let us assume that $n \geq 3$. Since the points p and q are given arbitrarily, the manifold M satisfies p axiom of r-planes, for $r = n-1 \geq 2$. It follows from the r-planes Theorem due to Cartan (see [6]) that M has constant sectional curvature (see [16]). As M is not compact it can only be isometric to the Euclidean space $\mathbf{R}^n$ or to the hyperbolic space H^n.

Since the set $\Sigma = \partial\overline{B}_r$ is a horosphere in M and M is a space form, then all sectional curvatures of Σ vanish ([18]). On the other hand, the set Σ is totally geodesic and consequently all sectional curvatures of M must vanish. Therefore M is isometric to $\mathbf{R}^n$.

Let us assume now that $n = 2$. Let g be a geodesic (a line) in M. We say that a geodesic is an asymptote at g passing through the point $q = \gamma(0)$ if there exists a sequence of minimal geodesics $\sigma_n : [0, s_n] \to M$ such that for every real value s, the sequence $\sigma_n(s)$ converges to the restriction of to the interval $[0, \infty)$ and we have $\sigma_n(s_n) = g(t_n)$ with $t_n \to \infty$.

When there exists another sequence $\tau_n : [0, s_n] \to M$ such that for every real value s, the sequence $\tau_n(s)$ converges to the restriction of $\gamma(s)$ to the interval $(-\infty, 0]$ and we have $\tau_n(s_n) = g(t_n)$ with $t_n \to -\infty$, we say that $\gamma(s)$ is a bi-asymptote at g passing through the point q.

Let g be a geodesic of M and $p = g(t_1)$ and $q = g(t_2)$ points on the geodesic g. By constructing horospheres Σ_p and Σ_q starting from the geodesic g, we notice that they both meet g orthogonally. On the other hand as FITA is satisfied we can immediately conclude that Σ_q is a bi-asymptote at Σ_p.

Eschenburg proves that there is an isometric immersion $F : [t_1, t_2] \times \mathbf{R} to M$ such that $\Sigma_p = F|_{t_1} \times \mathbf{R}$ and $\Sigma_q = F|_{t_2} \times \mathbf{R}$ (see [8]). This implies that the region of M limited by Σ_p and Σ_q has curvature zero. Since M is simply connected, the curve g is an arbitrary geodesic and the points $g(t_1)$ and $g(t_2)$ are also arbitrary, we conclude that $M = \mathbf{R}^2$.

4. Proof of Theorem C

Proposition 4.1. If the Riemannian manifold M satisfies SITA than it is compact.

Proof. To show the compactness, it suffices to find a point p_0 of M such that every geodesic $g : [0, \infty) \to M$, leaving p_0, has a cut point with respect to p_0, see [7]. Let us fix a point $p_0 \in M$. Choose $\delta > 0$ such that the open ball $B_\delta(p_0)$ with center p_0 and radius δ is convex. Denote by $S_r(p_0)$ the geodesic sphere of center p_0 and radius r, for some $\delta > r > 0$.

Let g be a geodesic leaving p_0 and p the point where g first crosses $S_r(p_0)$, we reparameterize g in such a way that $g(0) = p$ and $g(r) = p_0$. Finally, we construct a configuration $\{g, \gamma\}_p$ by choosing a point $q \neq p$ in $S_r(p_0)$ and γ a geodesic arc within $B_\delta(p_0)$ joining $p = \gamma(0) = g(0)$ to $q = \gamma(s)$. SITA assures the existence of exactly two real numbers $t_2 < 0 < t_1$ such that $r_j = g(t_j)$ determine geodesic segments $[q, r_1]_\sigma$ and $[q, r_2]_\tau$ that are the sides of two simple isosceles triangles whose common basis is $[p, q]_\gamma$. By construction, $t_1 = r$, i.e., $p_0 = g(r) = g(t_1) = r_1$, since p and q belong to the geodesic sphere $S_r(p_0)$. Therefore,

$$l([p_0, r_2]_g) = l([p_0, q]_\sigma) + l([q, r_2]_\tau) \,.$$

Hence, the geodesic g has a cut point $p' = g(t')$ with respect to p_0, which concludes the proof.

Proposition 4.2. *If a Riemannian manifold M satisfies SITA then it has no geodesic loop.*

Proof. *Let us suppose there exists a geodesic loop in M, that is, there exists a geodesic $g : \mathbf{R} \to M$ and points t_0 and $\overline{t}_0$ in $\mathbf{R}$, with $t_0 \neq \overline{t}_0$ such that $g(t_0) = p = g(\overline{t}_0)$ and $g'(t_0) \neq g'(\overline{t}_0)$.*

We consider a strongly convex ball $B(p)$. Let $p_i = g(t_i)$ with $i = 1, 2, 3, 4$ be the points in the boundary $\partial B(p)$ where g gets in and gets out and afterwards gets in and gets out of $B(p)$.

Let $q = g(\widetilde{t})$ with $\widetilde{t} < t_2$ be points obtained in such a way that $d(q, p_2) = d(p_2, p_3)$. Joining q to p_3 using the segment $[q, p_3]_\gamma$ we obtain the configuration $\{g, \gamma, \theta\}_q$. According to SITA there is a point $r = g(\hat{t})$ with $\hat{t} < \widetilde{t}$, in such a way that the triangles $\{p_2, q, p_3\}$ and $\{r, q, p_3\}$ are isosceles triangles whose basis is the segment $[q, p_3]$. Now, we observe that considering the medium point $\overline{p}$ of the segment $[q, p_3]_g$, the triangle $\{q, \overline{p}, p_3\}$ is also a geodesic triangle distinct of the other two. This contradicts the SITA.

Proposition 4.3. If M is a Riemannian manifold satisfying SITA then every geodesic is closed.

Proof. We will prove that every geodesic of M is closed by showing the existence of geodesics which are not closed lead us to a contradiction to the SITA. Let $c(p)$ denote the function which associates to every point $p \in M$ the convexity radius of M at the point p, that is $c(p)$ is the greatest number such that the ball $B_r(p)$ centered at p and having radius $r < c(p)$ is strongly convex. According to the Whitehead Theorem $c(p)$ is a continuous function on M, (see [24]). As M is compact and c is continuous there exists a number > 0 such that for every $p \in M$, the ball $B(p)$ is strongly convex.

Let us consider the family of open sets $\{B_r(p)\}_{p \in M}$ where $2r < \delta$. As such a family covers M and M is compact, we can find a finite cover of M, say $\{B_r(p_1), ..., B_r(p_k)\}$.

Let us assume that there exists geodesic g which is not closed. In this case, as M is complete, either g gets in and gets out twice in the same ball of the family $\{B_r(p_1), ..., B_r(p_k)\}$, or else there exists $t_0 \in \mathbf{R}$ such that for every $t > t_0$ the geodesic g is contained within open balls of the family $\{B_r(p_1), ..., B_r(p_k)\}$.

In the first case, let us suppose that the geodesic g gets in and gets out twice in the ball $B_r(p)$, as there not exist geodesic loops (Lemma 4.2), we know there exist four distinct points which we will denote by $p_i = g(t_i)$ in the boundary $\partial B_r(p)$ where g gets in and gets out and this geodesic gets in and gets out in $B_r(p)$.

Let us consider a point $q = g(t)$ with $t < t_2$ in such a way that $d(q, p_2) = d(p_2, p_3)$. Joining q to p_3 we obtain an isosceles triangle $\{q, p_2, p_3\}$ whose basis is $[q, p_3]$. Using SITA there exists a point $\overline{p} = g(\overline{t})$ with $\overline{t} < t_0$ such that $\{\overline{p}, q, p_3\}$ is an isosceles triangle whose basis is $[q, p_3]$. On the other hand, there is a point $g(\hat{t}) = \hat{p}$ in the segment $[p_2, p_3]_g$ so that $d(p_2, \hat{p}) = d(\hat{p}, p_3)$. From this we conclude that the triangle $\{p_2, \hat{p}, p_3\}$ is also isosceles whose basis is $[q, p_3]$. This contradicts SITA.

In the second case, let us assume there is a number $t_0 \in \mathbf{R}$ such that for every $t > t_0$, $g(t)$ is contained within the balls $B_r(p_i)$. Let us fix a point $q = g(\hat{t})$ in $B_r(p_i)$ and let us consider the strongly convex ball $B(q)$ which contains the set $B_r(p_i)$. This means that g passes through the center of the strongly convex ball $B_\delta(q)$ and that for $t \geq \overline{t}$, the number $g(t)$ is the radius of the ball $B_\delta(q)$. This is not possible for this ball has radius $\delta < \infty$.

Now we are able to prove theorem C.

Proof of Theorem C.

Let p_0 an arbitrary point in M and let g be an arbitrary geodesic starting at p_0 and parameterized by the arc length. Using Lemma 3.1 we have that g is a simple closed geodesic, therefore, there exists $l \in \mathbf{R}$ satisfying $g(2l) = p_0 = g(-2l)$.

The point $\overline{p}_0 = g(l)$ will be called the antipode point of p_0 with respect to the geodesic g. In order to simplify our notation, we will denote by $\overline{g}(t) = g(-t)$ the geodesic satisfying $\overline{g}(0) = p$ and $\overline{g}'(0) = -g'(0)$.

We denote by $p_0' = g(t_0)$ the cut point of g with respect to p_0. It is clear that p_0' cannot occur after the point $\overline{p}_0$ because $l([p_0, \overline{p}_0]_g) = l([p_0, \overline{p}_0]_{\overline{g}})$.

We shall prove now that $p_0 = \overline{p}_0$. Let us suppose by contradiction that $p_0' \neq \overline{p}_0$. This means that $l([p_0', \overline{p}_0]_g) > 0$ and consequently we can choose a real number $\hat{t}$ such that $t_0 < \hat{t} < l$ and the point $\hat{p} = g(\hat{t})$ occurs after p_0' and before $\overline{p}_0$. The fact that g does not minimize the distance from p_0 to $\hat{p}$ implies the existence of a minimal geodesic γ joining p_0 to $\hat{p}$ and satisfying $l([p_0, \hat{p}]_g) > l([p_0, \hat{p}]_\gamma)$. Moreover, if we denote by $[\hat{p}, p_0]_g$ the segment joining $\hat{p}$ to p_0 and passing through the point $\overline{p}_0$, we have $l([\hat{p}, p_0]) > l([p_0, \hat{p}]_\gamma)$

In this case, we can find a point p_0'' on g obtained from the point $\hat{p}$ in such a way that $l([\hat{p}, p_0'']_g) = l([p_0, \hat{p}]_\gamma)$. Let us denote by $[p_0'', p_0]_\lambda$ the segment of the geodesic λ joining p_0'' to p_0 and let us consider the configuration $\{g, \lambda\}_p$. By construction we have the isosceles triangles

$$\{[p_0'', p_0]_\lambda, [p_0, \hat{p}]_\gamma, [\hat{p}, p_0]_g\}, \{[p_0'', p_0]_\lambda, [p_0, \tilde{p}]_g, [\tilde{p}, p_0'']_g\}$$

and their base is the segment $[p_0'', p_0]_\lambda$. Moreover, we also have the isosceles triangle $\{[p_0'', p_0], [p_0, \breve{p}]_g, [\breve{p}, p_0'']_g\}$ where $\breve{p}$ is the middle point of the segment $[p_0, p_0'']_g$, joining the points p_0, p_0'' and passing through the point $\overline{p}_0$, which contradicts the SITA. Therefore we have $p_0' = \overline{p}_0$.

Let us now consider a strongly convex ball $B_r(p_0)$ chosen so that the set $\overline{B_r(p_0)}$ be also strongly convex. We will denote by $\Sigma = \partial B_r(p_0)$ the boundary of $\overline{B_r(p_0)}$ and let us consider the points $p = g(r)$ and $\overline{p} = g(-r) = \overline{g}(r)$ where the geodesic g meets Σ.

We fix a point $q \in \Sigma$ given arbitrarily and different from the points p and $\overline{p}$; We also consider the configuration $\{g, \gamma\}_p$ where γ is the geodesic joining p to q. According to the SITA, there exist segments $[p_0, q]_\sigma$ and $[\overline{p}_0, q]_\tau$ such that the triangles $\{[p_0, p]_g, [p_0, q]_\sigma, [p, q]_\gamma\}$ and $\{[p, \overline{p}_0]_g, [\overline{p}_0, q]_\tau, [p, q]_\gamma\}$ are simple and isosceles.

The fact $l([p_0,\overline{p}_0]_g) = l([p_0,p]_g) + l([p,\overline{p}_0]_g) = l([p_0,q]s) + l([q,\overline{p}_0]_\tau)$ implies that $[p_0,q]_\sigma$ and $[q,\overline{p}_0]_\tau$ are segments of the same geodesic joining p_0 to $\overline{p}_0$ and passing through the point q, which we denote by ϕ. Besides, if $l([p_0,\overline{p}_0]_g) = l([p_0,\overline{p}_0]_\phi)$ and $\overline{p}_0$ is not the cut point of ϕ with respect to p_0 there would exist a geodesic segment with length smaller than the length of $[p_0,\overline{p}_0]_g$ joining p_0 to $\overline{p}_0$ and this contradicts what we have shown before.

Using the fact that the point q was taken arbitrarily we can conclude that $\overline{p}_0$ is the cut point of all geodesics passing through p_0. Thus the cut locus $C(p_0)$ of p_0 is the set $\{\overline{p}_0\}$ and since p_0 is arbitrary we have that for every point p, the cut locus $C(p)$ is a unitary set and therefore M is a wiedersehen manifold.

If $n = 2$, the result follows from Green Theorem (see [9]) which says that M is isometric to the Euclidean sphereS^2. If $n > 2$ and is an odd number, the result follows from the Yang Theorem (see [25]) which says that M is isometric to the Euclidean sphere S^n. If $n > 2$ and is an even number, the result follows from the Kazdan Theorem (see [26]) which says that M is isometric to the Euclidean sphere S^n.

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