SCHOTTKY UNIFORMIZATIONS AND RIEMANN MATRICES OF MAXIMAL SYMMETRIC RIEMANN SURFACES OF GENUS 5[*]

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Abstract

In this note we consider pairs (S, τ), where S is a closed Riemann surface of genus five and $\tau : S \to S$ is some anti-conformal involution with fixed points so that $K(S, \tau) = \{h \in Aut_\pm(S) : h\tau = \tau h\}$ has the maximal order 96 and S/τ is orientable. We observe that there are exactly two topologically different choices for τ. They give non-isomorphic groups $K(S, \tau)$, each one acting topologically rigid on the respective surface S. These two cases give then two (connect) real algebraic sets of real dimension one in the moduli space of genus 5. In this note we describe these components by classical Schottky groups and with the help of these uniformizations we compute their Riemann matrices.

Keywords : *Schottky groups, Riemann surfaces, Riemann matrices.*

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1. Introduction

A closed Riemann surfaces S of genus $g \geq 2$ is called symmetric if it admits an anticonformal involution $\tau : S \to S$, called a symmetry of S. If the symmetry τ has fixed points, then we refer to it as a reflection. We denote by $K(S, \tau)$ the total group of conformal/anticonformal automorphisms of S commuting with τ and by $K^+(S, \tau)$ its index two subgroup of conformal automorphisms. The quotient $R = S/\tau$ is a compact (possible bordered) Klein surface of algebraic genus g. The number of possible border components is equal to number of fixed point components of τ. The group $H(S, \tau) = K(S, \tau)/\tau$ is isomorphic to the full group of automorphisms of R. In the case τ is a reflection, we have that S/τ has non-empty boundary and, as consequence of [18] and [19], we have that the order of $K(S, \tau)$ is at most $24(g - 1)$. We say that (S, τ) is maximal symmetric if the maximal order $24(g - 1)$ is attained for $K(S, \tau)$; the surface S is then said to be maximal symmetric and the reflection τ to be a maximal reflection (we must not get confused with the definition of a M-curve which means τ to be a reflection with the maximal number of components of fixed points). It is also important to note that the bound $24(g - 1)$ is attained by an infinite number of values of g and also it is not attained for an infinite values of g. It is a well known fact that any closed symmetric Riemann surface S with a reflection $\tau : S \to S$ can be uniformized by a classical Schottky group [7], [16], [21] and [24]. In fact, such a Schottky group can be chosen to keep invariant a circle C so that the reflection on C is a lifting of τ [16] and, in particular, $K(S, \tau)$ lifts to such an Schottky uniformization of S (see section 2). This type of Schottky uniformizations and Burnside's arguments [2], which we recall in section 4, we may be able to compute the respective Riemann matrices of the surface S. In the case that (S, τ) is maximal symmetric we know that $S/K^+(S, \tau)$ is the Riemann sphere branched at four values with orders 2, 2, 2 and 3, respectively [12]. The reflection $\tau : S \to S$ induces a reflection of the sphere $S/K^+(S, \tau)$ permuting these branch values and, in particular, it follows that $S/K(S, \tau)$ is either: (i) a closed disc with four branched values, of orders 2, 2, 2 and 3, on its border or (ii) a closed disc with one branch value in its interior, of order 2, and two branched values on its border of orders 2 and 3. Using the fact that

the NEC group uniformizing $S/K(S,\tau)$ contains a bordered surface group F, uniformizing S/τ, as normal subgroup, we have that case (ii) is not possible. As a consequence, for a maximal symmetric Riemann surface S with a maximal reflection $\tau : S \to S$ we must have that $S/K(S,\tau)$ is a closed disc with exactly four branched values of orders 2, 2, 2 and 3 on its border.

In this note we consider maximal symmetric Riemann surfaces S of genus five with a maximal reflection $\tau : S \to S$ for which S/τ is orientable, that is, the set of fixed points of τ disconect S into two surfaces. In section 5 we observe that there are exactly two different topological action of $K(S,\tau)$. Each one of these topological actions describes a real one dimensional locus in moduli space. In each of these two cases, there is a normal subgroup $H \cong \mathcal{Z}_2 + \mathcal{Z}_2$ of $K(S,\tau)$, containing only conformal automorphisms and acting freely. The quotient S/H is a closed Riemann surface of genus two admitting the group $K(S,\tau)/H$ as group of automorphisms. This group is isomorphic to $\mathcal{Z}_2+\mathcal{Z}_2+D_3$, where D_3 is a conformal group isomorphic to the dihedral group of order 6, one of the $\mathcal{Z}_2$ is generated by the hyperelliptic involution and the other $\mathcal{Z}_2$ is generated by the reflection induced by τ. We obtain in this way the exactly two possibilities for such an action in genus two. In section 6 we describe a particular uniformization of the closed unit disc with four branch values of orders 2, 2, 2 and 3 on its border. This uniformization is given by a Kleinian group $\widehat{K}$, generated by four reflections, having a connected region of discontinuity. In section 7 we describe two Schottky normal subgroup of genus two G_2 and F_2 of index 24 in $\widehat{K}$. The Schottky groups G_2 and F_2 uniformize the closed Riemann surfaces of genus two obtained in section 3. We also describe an algebraic limit giving a noded Schottky group. With the help of these Schottky uniformizations and results due to W. Burnside [2] we compute the Riemann matrices of these surfaces. In section 8 we proceed to find a Schottky normal subgroup G_5 (respectively, F_5) of the Schottky group G_2 (respectively, F_2) of index 4, which is also normal in the group $\widehat{K}$. These groups uniformize all the Riemann surfaces of genus 5 described in section 5. We also describe some algebraic limits giving a couple of noded Schottky groups. These noded Schottky groups then uniformize boundary points in Moduli space of the above Riemann surfaces with automorphism. We proceed again

to use these Schottky uniformizations and results due to W. Burnside
[2] to compute the Riemann matrices of these surfaces.

2. Schottky Uniformizations

Let g be some positive integer. Assume we have a collection of $2g$ pair-
wise disjoint simple loops, say $\alpha_1,..., \alpha_g, \widetilde{\alpha}_1,..., \widetilde{\alpha}_g$, bounding a common
region $\mathcal{D}$ of connectivity $2g$. Let us also assume we have a collection
of Moebius transformations $A_1,..., A_g$ (necessarily loxodromic ones) so
that $A_j(\alpha_j) = \widetilde{\alpha}_j$ and $A_j(\mathcal{D}) \cap \mathcal{D} = \emptyset$, for all $j = 1,..., g$. The group
J generated by $A_1,..., A_g$ turns out to be a Kleinian group, purely
loxodromic, isomorphic to a free group of rank g and with connected
region of discontinuity [15], called a Schottky group of genus g (for
another equivalent definitions of Schottky groups one can check [17]
and [5]). The domain $\mathcal{D}$ is called a standard fundamental domain for
J, the loops $\alpha_1,..., \alpha_g, \widetilde{\alpha}_1,..., \widetilde{\alpha}_g$ are called a fundamental set of loops.

For a Schottky group J, Ω/J is a closed Riemann surface of genus
g, where Ω denotes the region of discontinuity of J. The classical
retrosection theorem states that if S is a closed Riemann surface of
genus g, then there is a Schottky group J with region of discontinuity
Ω and $P : \Omega \to S$ a normal covering map, with J as covering group.
We say that $(J, \Omega, P : \Omega \to X)$ is a Schottky uniformization of S.
A simple proof of this fact was also given by L. Bers in [1] using
quasiconformal mappings.

A Schottky group J is called hyperelliptic if there is a Moebius
transformation C of order two such that $CJC^{-1} = J$ and the group
generated by J and C uniformizes an sphere (necessarily with $2(g+1)$
branched points of order two). Equivalently, there is a set of free
generators $A_1,..., A_g$ for G so that $CA_jC = A_j^{-1}$ for $j = 1,..., g$.
The hyperelliptic Schottky groups uniformize hyperelliptic Riemann
surfaces and, reciprocally, every hyperelliptic Riemann surface can be
uniformized by such a group [13].

In the case we can choose the above loops α_j and $\widetilde{\alpha}_j$ (for all j)
as Euclidean circles, for some set of free generators of the Schottky
group, we say that J is a classical Schottky group. The existence of
non-classical Schottky groups is well known [14]. An open problem is
to determine if every closed Riemann surface can be uniformized by

a classical Schottky group. Related to this problem, B. Maskit has proved the following:

Theorem 2.1 [16]. *Let S be a closed Riemann surface admitting a reflection $\tau : S \to S$. Then there is a classical Schottky group preserving an Euclidean circle $\mathcal{C}$ on the Riemann sphere, that is, either is a Fuchsian group or is an extended Fuchsian group (it is not Fuchsian but contains an index two subgroup which is Fuchsian) uniformizing S. The reflection on $\mathcal{C}$ is a lifting of the reflection τ.*

In the case the reflection τ has the maximal number of components of fixed points, the proof of Maskit's result is simple. In fact, set $R = S/\tau$ and $P : S \to R$ the anti-holomorphic branched covering induced by the action of τ. The surface R is an orientable closed surface of genus p with l boundary components, where $g + 1 = 2p + l$ and l denotes the number of connected components of fixed points of τ. Let X be the interior of R. The set of fixed points of τ disconnect S into two surfaces (open surfaces), say S_1 and S_2, with common boundary (the set of fixed points of τ). The restriction of P to S_1 produces a conformal homeomorphism between X and S_1. Let us uniformize X by a Fuchsian group F acting on the unit disc Δ, that is, there is a conformal covering $Q : \Delta \to X = \Delta/F$. The group F is a Fuchsian group of the second kind and, in particular, a classical Schottky group of genus g. If we denote by Ω the region of discontinuity of F, then $\widehat{S} = \Omega/F$ is a closed Riemann surface of genus g admitting a symmetry $\eta : \widehat{S} \to \widehat{S}$ so that $S/\tau = \widehat{S}/\eta$. Set $\widehat{S}_1 = Q(\Delta)$ and $\widehat{S}_2 = Q(\Omega - \overline{\Delta})$. If we denote by $L : \widehat{S} \to S/\tau$ the anticonformal branched covering induced by the action of η on $\widehat{S}$, then the map $T : S \to \widehat{S}$ defined by $T|_{\overline{S_1}} = L|_{\widehat{S}_1}^{-1} P|_{\overline{S_1}}$ and $T|_{S_2} = L|_{\widehat{S}_2}^{-1} P|_{S_2}$ turns out to be a biholomorphic map.

A lifting problem concerning Schottky uniformizations of closed Riemann surfaces is the following. Assume we have a closed Riemann surface S and a group H of conformal/anticonformal automorphisms of it. In general, there is no Schottky uniformization of S for which the group H lifts. It is then natural to ask for conditions in order to ensure the existence of a Schottky uniformization of S for which this lifting is possible. When this is the case, we say that H is of Schottky type.

In [8] we have found easy necessary conditions to be satisfied by H, a group of conformal automorphisms, in order for it to be of Schottky type (condition (A)). These conditions are trivially satisfied by groups isomorphic to a dihedral group and for groups acting free fixed points. In [8], [9], [10] and [11] we have shown that these necessary conditions turn out to be sufficient for Abelian groups, dihedral groups, the alternating groups $\mathcal{A}_4$ and $\mathcal{A}_5$ and for the symmetric group $\mathcal{S}_4$. In the case of groups admitting orientation-reversing transformations, we have some results for the cyclic case in [4]. Maskit's result asserts the following

Corollary 2.2. *Let S be a closed Riemann surface with a reflection τ. Then $K(S, \tau)$ is of Schottky type.*

The reason of this is the following. Let us assume we have a closed Riemann surface S and a reflection $\tau : S \to S$. Let G be a Schottky group uniformizing S so that the unit circle S^1 is invariant under G and σ, the reflection on S^1, is a lifting of τ (as consequence of the above Maskit's result). Let us denote by Δ the unit disc. Set Ω the region of discontinuity of G and denote by $P : \Omega \to S$ the holomorphic covering induced by G.

(i) Assume that the fixed points of τ divide S into two surfaces, say S_1 and S_2. In this case, we have that G is a fuchsian group. We may assume $S_1 = P(\Delta)$. Set K the index two subgroup of $K(S, \tau)$ that keeps S_1 invariant. We have then that $K(S, \tau)$ is generated by K and τ. Since $P : \Delta \to S_1$ is a universal covering, we can lift K. Such a lifting is a fuchsian group $\widehat{K}$. It is now clear that the group generated by σ and $\widehat{K}$ is the lifting of $K(S, \tau)$.

(ii) Assume that the fixed points of τ do not divide S. Set as $\widehat{G}$ the group generated by G and σ. Since the unit circle S^1 is invariant under G, we have that the order two subgroup F of $\widehat{G}$ generated by σ is a normal subgroup. We consider the branched covering $Q : \Omega \to \overline{\Delta}$ induced by the action of F. The quotient group $H = \widehat{G}/F$ acts as group of conformal automorphisms of the unit disc Δ. We have that the holomorphic covering $Q : \Delta \to (S/\tau)^*$, induced by H is a universal covering, where $(S/\tau)^* = (S - \mathrm{Fix}(\tau))/\tau$. In particular, we can lift

$K(S, \tau)/\tau$ to Δ by this universal covering. Such a lifting is given by Moebius transformations keeping invariant the unit disc and they extend in natural manner to act on the unit circle boundary. Now, it is easy to see how to lift any of these automorphisms of $\overline{\Delta}$ by the (branched) covering Q to obtain in this way the lifting of $K(S, \tau)$ as desired.

In sections 6,7,and 8 we deal with the explicit construction of classical Schottky groups of genus $g = 5$ which uniformize maximal symmetric Riemann surfaces S admitting a maximal reflection τ for which S/τ is orientable and $S/K(S, \tau)$ is a closed disc with branch values of orders 2, 2, 2 and 3 on its border. These Schottky groups have the property that they uniformize all the surfaces with such conditions.

3. Riemann Matrices

Let us assume we have a closed Riemann surface S of genus g together a group K of automorphisms of it (maybe containing orientation reversing ones).

A symplectic basis of S is a basis of the first homology group of S, with integer coefficients, whose intersection matrix is

$$ J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} $$

where I denotes the identity matrix of size $g \times g$.

To each symplectic basis of S, say $\{\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g\}$, we have associated a dual basis of holomorphic one forms $\{w_1, ..., w_g\}$,that is,$\int_{\alpha_i} w_j = \delta_{ij}$.

The Siegel space $\mathcal{H}_g$ is the space consisting of the complex symmetric matrices of size $g \times g$ with positive imaginary part. The Riemann matrix $Z = (\int_{\beta_j} w_i)$ belongs to $\mathcal{H}_g$ as consequence of Riemann's bilinear relations.

Each matrix $Z = X + iY \in \mathcal{H}_g$ produces a lattice L_Z in $\mathbf{C}^g$: the lattice generated by integer linear combinations of the columns of the period matrix $(I \quad Z)$. Its Gram matrix is given by $H = P + iJ$, where

$$ P = \begin{pmatrix} Y^{-1} & Y^{-1}X \\ XY^{-1} & XY^{-1}X + Y \end{pmatrix}, $$

defines a principal polarization on the complex torus $\mathbf{C}^g/L_Z$ and, in particular, we obtain a principally polarized Abelian variety.

The group K induces a natural faithful representation $\theta : K \rightarrow \widetilde{\mathrm{Sp}_{2g}}(\mathbf{Z})$, where $\widetilde{\mathrm{Sp}_{2g}}(\mathbf{Z})$ denotes the set of integer matrices N of size $2g \times 2g$ such that ${}^t N J N = \pm J$ (the extended symplectic group). Its index two subgroup $\mathrm{Sp}_{2g}(\mathbf{Z})$, consisting of those $N \in \widetilde{\mathrm{Sp}_{2g}}(\mathbf{Z})$ so that ${}^t N J N = J$, is called the symplectic group.

The group $\widetilde{\mathrm{Sp}_{2g}}(\mathbf{Z})$ acts on the Siegel space $\mathcal{H}_g$ as follows: Let $Z \in \mathcal{H}_g$ and

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widetilde{\mathrm{Sp}_{2g}}(\mathbf{Z}), \quad \text{then}$$

$$N(Z) = \begin{cases} (A + ZC)^{-1}(B + ZD), & \text{if } N \in \mathrm{Sp}_{2g}(\mathbf{Z}) \\ \\ (A + \overline{Z}C)^{-1}(B + \overline{Z}D), & \text{if } N \notin \mathrm{Sp}_{2g}(\mathbf{Z}) \end{cases}$$

The matrix $Z = (\int_{\beta_j} w_i)$ is fixed by the symplectic transformation $\theta(k)$ for each $k \in K$.

For $Z \in \mathcal{H}_g$ fixed by the symplectic transformations $\theta(k)$, for all $k \in K$, the lattice L_Z induced by Z has $\theta(K)$ has group of isometries (for the positive Hermitian form H). In particular, they define principally polarized Abelian varieties with a group of automorphisms isomorphic to K.

Remark. If we have $Z = X + iY \in \mathcal{H}_g$ and consider $Y^{-1} = {}^t B B$, then the lattice $L_{Z,B} = (B \quad BZ)$ is isomorphic to L_Z. The Hermitian form in this new lattice is the standard one. A main problem is to determine those lattices for which the minimum non-zero norm is maximal [3] [22] [23]. One may try to use the explicit families in this work to consider the above extremal problem.

4. Schottky Uniformizations and Riemann Matrices

In this section we describe how Schottky uniformizations of Riemann automorphisms with a Schottky type automorphism group can be used to compute principally polarized Abelian varieties.

Assume we have a Kleinian group K (admitting anticonformal Moebius transformations) containing a Schottky group J of genus $g \geq 2$, with region of discontinuity Ω, as a normal subgroup of finite index. We have then that K has also Ω as region of discontinuity [16].

Let us denote by S the closed Riemann surface of genus g uniformized by G, that is, $\Omega/G = S$, and denote by $P : \Omega \to S$ the natural holomorphic covering with G as covering group.

Let us fix some fundamental set of loops for J, say $\alpha_1,..., \alpha_g, \widetilde{\alpha}_1,..., \widetilde{\alpha}_g$, and let $A_1,..., A_g$ a set of free generators of J respect to these loops.

We give orientation on the loops $\alpha_1,..., \alpha_g$. Also we consider suitable oriented paths $\beta_1,..., \beta_g$, pairwise disjoint, so that β_k connects equivalent points in α_k and $\widetilde{\alpha}_k$, and β_k is disjoint from α_t and $\widetilde{\alpha}_t$ for all $t \neq k$. Let us denote again by α_k and β_k the projections on S of the above respective loops. We have then that they define a symplectic base for S and, in particular, a Riemann matrix Z_0 for S.

The group K induces a finite group of conformal automorphisms on S, the group K/G. In particular, the choice of the symplectic base gives us a faithful representation $\rho : K/J \to \widetilde{\mathrm{Sp}_{2g}}(\mathbf{Z})$. Since the normalizer of the projected loops $\alpha_1,..., \alpha_g$ is kept invariant under the action of K/J (because of the lifting property of K/J by $P : \Omega \to S$), we have that:

(1) if $\gamma \in K$ is conformal:

$$\rho(\gamma) = \begin{pmatrix} \Theta(\gamma) & B_\gamma \\ 0 & {}^t\Theta(\gamma) \end{pmatrix}$$

(2) if $\gamma \in K$ is anti-conformal:

$$\rho(\gamma) = \begin{pmatrix} \Theta(\gamma) & B_\gamma \\ 0 & -{}^t\Theta(\gamma) \end{pmatrix}$$

where $\Theta : K/J \to GL(g, \mathbf{Z})$ is the isomorphism induced at the level of homology on the submodule generated by the loops $\alpha_1, ..., \alpha_g$.

On the other hand, since J is normal subgroup of K, we have that each $\gamma \in K$ induces an isomorphism $\eta(\gamma) : J \to J$ and an isomorphism $\eta(\gamma)^{ab} : J/[J,J] \to J/[J,J]$. It is not hard to check that:

(1) if $\gamma \in K$ is conformal, then $\Theta(\gamma) = {}^t\eta(\gamma)^{ab}$; and

(2) if $\gamma \in K$ is ant-conformal, then $\Theta(\gamma) = -{}^t\eta(\gamma)^{ab}$

4.1. Schottky Uniformization With Invariant Circles

In the particular case that K keeps invariant an Euclidean circle C, the reflection σ on C will commute with every element of K. In particular, the surface S uniformized by J has an anticonformal involution with fixed points and commuting with each automorphism of S induced by K. In this case, the above procedure asserts that $\eta(\sigma)^{ab}$ is the identity matrix, that is,

$$\rho(\sigma) = \begin{pmatrix} -I & B_\sigma \\ 0 & I \end{pmatrix}$$

If $Z \in \mathcal{H}_g$ is fixed by the group $\rho(K/J)$, then in particular is fixed by $\rho(\sigma)$. This follows that $Re(Z) = \frac{1}{2}B_\sigma$. From this also follows that:

(1) if $\gamma \in K$ is conformal, then $B(\gamma) = \frac{1}{2}(B(\sigma)\eta(\gamma)^{ab} - {}^t\eta(\gamma)^{ab}B(\sigma))$; and

(2) if $\gamma \in K$ is anti-conformal, then $B(\gamma) = \frac{1}{2}(B(\sigma)\eta(\gamma)^{ab} + {}^t\eta(\gamma)^{ab}B(\sigma))$

The computation of $B(\sigma)$ is simple in this case. In particular, we get a faithful symplectic representation $\rho : K/J \to \widetilde{\mathrm{Sp}_{2g}}(\mathbf{Z})$.

In sections 8 we use the constructed Schottky uniformizations to describe faithful symplectic representations of the groups $K(S, \tau)$ in the respective considered cases. We compute their fixed points in the respective Siegel space, that is, principally polarized Abelian varieties with group of automorphisms isomorphic to $K(S, \tau)$.

4.2. Burnside's Remark [2]

Let us assume that G is a Schottky group keeping invariant some Euclidean circle on the Riemann sphere (a line is a circle through

infinite) and for which $\infty \in \Omega(G)$. Denote by $G(\infty)$ the orbit of ∞. Burnside has remarked that the series

$$\sum_{g \in G} g'(z)$$

converges uniformly on compact subsets of $\Omega(G)$ to a meromorphic map which is holomorphic on $\Omega(G) - G(\infty)$ and it has doble poles at each point in $G(\infty)$. The proof for the case that G keeps invariant a line L is rather simple. In fact, let $K \subset \Omega(G)$ be some compact set and $M = \{g_1, ..., g_n\} \subset G$ be so that $g_j^{-1}(\infty) \in K$. Then there is a positive constant $A > 0$ so that

$$\sum_{g \in G - M} |g'(z)| \le A \sum_{g \in G} \frac{1}{|c|^2},$$

where $g(z) = \frac{az+b}{cz+d}$, $ad - bc = 1$.

Let us choose a point $q \in L$ and a positive number $R > 1$ so that $B = \{z \in \mathcal{C} : |z - q| > R\} \cup \{\infty\}$ is precisely invariant under the identity in G. In this case, we have that the sets $g(B)$ are discs orthogonal to L and, in particular,

$$\sum_{g \in G, g \ne I} \operatorname{diam}(g(B)) \le 2R,$$

where diam denotes the Euclidean diameter. We may use the inequality given in C.7 of [15] which in our case reads

$$\operatorname{diam}(g(B)) \ge \frac{1}{|c|^2} \frac{1}{\operatorname{dist}(g^{-1}(\infty), B)} \ge \frac{1}{|c|^2} \frac{1}{R},$$

for $g \ne I$.

Since for $K \subset \Omega(G) - G(\infty)$ we have $M = \emptyset$, the above says that

$$\sum_{g \in G} g'(z)$$

converges uniformly on K to a holomorphic map. Now, around a point $g_1^{-1}(\infty)$ we have that the series

$$\sum_{g \in G - \{g_1\}} g'(z)$$

converges locally to a holomorphic map and, since $g_1'(z)$ has a doble pole in $g_1^{-1}(\infty)$, we are done in this case.

The case of circles can be carried to that situation in the following way. Assume that C is the invariant circle. Choose some point $q \in C \cap \Omega(G)$ (then $q \neq \infty$). Let $T(z) = \dfrac{1}{z-q}$. Set $H = TGT^{-1}$ and $L = T(S)$. Then H is a Schottky group keeping invariant the line L and for which $\infty \in \Omega(H)$. In this case we have the convergence property for $\sum_{h \in H} h'(w)$ on compact sets of $\Omega(H)$. Let K be a compact subset of $\Omega(G)$ so that $q \notin K$. Then for $z \in K$ we have

$$\sum_{h \in H, h \neq I} |h'(T(z))| = \sum_{g \in G, g \neq I} |g'(z)| \frac{1}{|g(z) - q|^2 |T(z)|^2}.$$

The fact that $q \notin K$ asserts that

$$\text{Inf}\{\frac{1}{|T(z)|^2}; z \in K\} = M > 0.$$

Also, except for a finite number of elements of G, say $g_1, ..., g_n$, we have that

$$\text{Inf}\{\frac{1}{|g(z) - q|^2}; z \in K, g \in G - \{g_1, ..., g_n\}\} = N > 0.$$

Now it follows that

$$\sum_{g \in G, g \neq I} |g'(z)| \frac{1}{|g(z) - q|^2 |T(z)|^2} \geq NM \sum_{g \in G, g \neq I} |g'(z)|,$$

obtaining the desired convergence.

As a consequence of the above, we have the following. Let us consider a set of free generators $A_1, ..., A_g$ for the Schottky group G. Let $\alpha_1, ..., \alpha_g, \widetilde{\alpha_1}, ..., \widetilde{\alpha_g}$ be a set of simple loops defining a standard fundamental domain for G, respect to the above generators. If we give the counterclockwise orientation to the loops α_j, then

$$w_j(z) = \frac{1}{2\pi i} \sum_{g \in G} \frac{g'(z)}{g(z) - A_j^{-1}(\infty)} dz,$$

for $j = 1, ..., g$, are the lifting to $\Omega(G)$ of the holomorphic one-forms dual to the homological half-symplectic basis on $S = \Omega(G)/G$ determined by the projections of the oriented loops $\alpha_1, ..., \alpha_g$.

5. Maximal Symmetric Riemann Surfaces of genus 5

Let us consider a maximal symmetric Riemann surface S of genus five together a maximal reflection $\tau : S \to S$ so that S/τ is orientable. We already noted that $S/K(S,\tau)$ is the closed disc with exactly four branch values of orders 2, 2, 2 and 3 in its border.

Theorem 5.1. *In genus 5 there are exactly two different topological maximal reflections τ with the condition that S/τ is orientable. They give rise to topologically rigid and non-isomorphic groups $K(S,\tau)$. In particular, we have exactly two algebraic real sets in the moduli space of genus five, each one of real dimension one, determining the surfaces above in consideration. In each case, there is a normal subgroup L of $K(S,\tau)$, isomorphic to $\mathcal{Z}_2 + \mathcal{Z}_2$, of conformal automorphisms acting freely and S/L is a closed Riemann surface of genus two.*

Proof.　Since S/τ is orientable, we have exactly three possibilities:

(1) S/τ is a genus one Riemann surface with four boundary components;

(2) S/τ is a genus zero Riemann surface with six boundary components; and

(3) S/τ is a genus two Riemann surface with two boundary components.

　　The fact that τ belong to the centralizer of $K(S,\tau)$, we have that $K(S,\tau)$ descends to a group of automorphisms H_{48}, of order 48, on the quotient S/τ.

(1) S/τ **has genus one and four boundary components.** Denote by H_{48} the group of order 48 of automorphisms of S/τ induced by H_{96}. If H_{48} only has conformal automorphisms, then this will imply the existence of a group of conformal automorphisms of order 48 acting on a genus one Riemann surface keeping transitively invariant a set of 4 points. This implies that the stabilizer in such group of any of these points is a

cyclic group of order 12, which is impossible. It follows that H_{48} must have orientation reversing automorphisms. Let H_{24} be the index two subgroup of orientation preserving automorphisms of H_{48}. The action of such a group corresponds (topologically) to the action of a conformal group of order 24 on a genus one Riemann surface with four punctures. This action is the unique action of the group generated by an alternating group $\mathcal{A}_4 = \langle W, T \rangle$ and an involution J. Each puncture is fixed by J. In this case, the genus one surface corresponds to the curve $y^2 = x^3 - 1$ and the group H_{48} is generated by the transformations $J : (x, y) \rightarrow (x, -y)$, $S : (x, y) \rightarrow (\overline{x}, \overline{y})$, $T : (x, y) \rightarrow \left(\frac{x+2}{x-1}, \frac{3y}{(x-1)^2}\right)$ and $W : (x, y) \rightarrow (\rho x, y)$, where $\rho = e^{\frac{2\pi i}{3}}$, where the group H_{24} is the alternating group generated by W and T. In particular, the topological action of $K(S, \tau)$ is unique in this situation and has presentation: $K(S, \tau) = \langle T, W, J, \tau : T^2 = W^3 = J^2 = \tau^2 = (TW)^2 = (TJ)^2 = (T\tau)^2 = \tau W \tau W^{-1} = (\tau J)^2 = (JWJW^{-1})^2 = (\tau W)^6 = 1 \rangle$.

Let us observe at this point that $K(S, \tau)$ has to involutions (the corresponding to J and τST) generate the Klein group $L = \mathbb{Z}_2 + \mathbb{Z}_2$ acting free fixed points, which is normal subgroup. The surface S/L is a genus two Riemann surface with group of automorphisms $\mathbb{Z}_2 + \mathbb{Z}_2 + D_3$, where D_3 denotes a dihedral group of order 6. One of the $\mathbb{Z}$ is a reflection (induced by τ) with exactly one component of fixed points, a dividing closed simple geodesic γ. Such a closed geodesic divides S/L into two one holed tori, say T_1 and T_2. The other $\mathbb{Z}_2$ is the hyperelliptic involution and D_3 is the dihedral group generated by an automorphisms of order 3 keeping invariant each T_i and a conformal involution permuting T_1 and T_2. In section 4 we produce the Schottky uniformizations of these surfaces of genus two.

(2) **S/τ is the Riemann sphere with six boundary components.** Let H_{48} be the induced group of automorphisms by H_{96} on this surface. If H_{48} only has conformal automorphisms, then it implies the existence of a group of Möbius transformations of order 48 keeping transitively invariant a set of 6 points, which is impossible. It follows that H_{48} has anticonformal automor-

phisms. Let H_{24} be the index two subgroup of H_{48} of conformal automorphisms. Topologically, this corresponds to the action of the symmetric group $\mathcal{S}_4 = \langle a, b : a^4 = b^2 = (ab)^3 = 1 \rangle$, where the six special points are exactly the six fixed points of the elements of order 4. The extra anticonformal automorphism is in fact a reflection which fixes four of these points and interchange the other two. In particular, the topological action of $K(S, \tau)$ is unique and has presentation: $K(S, \tau) = \langle a, b, \mu, \tau : a^4 = b^2 = \mu^2 = \tau^2 = (ab)^3 = \tau a \tau a^{-1} = (\tau b)^2 = (\mu \tau)^2 = \mu a \mu a^{-1} = \mu b \mu a^2 b a^2 = 1 \rangle$.

The subgroup $L = \langle a^2, ba^2b \rangle \cong \mathcal{Z}_2 + \mathcal{Z}_2$ is a normal subgroup of $K(S, \tau)$ acting freely. The quotient surface S/L is a genus two Riemann surface admitting the group $\mathcal{Z}_2 + \mathcal{Z}_2 + D_3$. In this case, one of the $\mathcal{Z}_2$ is generated by a reflection (the one induced by τ) with exactly three components of fixed points (an M-symmetry). These three components divide S/L into two three holed spheres, say S_1 and S_2. The other $\mathcal{Z}_2$ is generated by the hyperelliptic involution and D_3 is generated by an automorphism of order three keeping invariant each S_i and a conformal involution also keeping invariant each S_i. In section 4 we produce the Schottky uniformizations of these surfaces of genus two.

(3) S/τ **is a genus two Riemann surface with two boundary components.** Again, we denote by H_{48} the induced automorphism group. Either this has only conformal automorphisms or there is an index two subgroup of conformal automorphisms. In any of this cases, it will imply the existence of a group of conformal automorphisms of either order 48 or 24 on a closed Riemann surface of genus two keeping invariant two points. The stabilizer of any of this points will be either a cyclic group of order 24 or 12, which is impossible.

In any of the two possible cases (1) and (2), the family depends on one real dimensional parameter. This because the Teichmüller space of the closed unit disc with four marked points on the boundary has real dimension one. □

We must remark at this point that in genus two, there are exactly two different topological actions of a group $\mathcal{Z}_2 + \mathcal{Z}_2 + D_3$ as group

of automorphisms where one of the $\mathcal{Z}_2$ is generated by a reflection. These two are the ones described in (1) and (2), respectively.

6. Uniformization of the closed disc with branch values of orders 2, 2, 2 and 3 on its border

For each $p \in (2 - \sqrt{3}, 1)$, let us consider the group $\widehat{K}$ generated by the following reflections: $\sigma_1(z) = \overline{z}$, $\sigma_2(z) = e^{\frac{2\pi i}{3}}\overline{z}$, $\sigma_p(z) = \frac{(1+p^2)\overline{z}-2p}{2p\overline{z}-(1+p^2)}$ and $\sigma(z) = \dfrac{1}{\overline{z}}$ (see figure 1).

If we set $W(z) = \sigma_2\sigma_1(z) = e^{\frac{2\pi i}{3}}z$, $T(z) = \sigma\sigma_1(z) = \dfrac{1}{z}$ and

$$J(z) = \sigma_p\sigma_1(z) = \frac{(p + \frac{1}{p})z - 2}{2z - (p + \frac{1}{p})},$$

then the group $\widehat{K}$ is also generated by the transformations T, W, J and σ. As consequence of Klein-Maskit's combination theorems [15], we have

$$\widehat{K} = \langle T, W, J, \sigma : T^2 = W^3 = (WT)^2 = J^2 = (TJ)^2 = \sigma^2 = (\sigma T)^2 = \sigma W \sigma W^{-1} = (\sigma J)^2 = 1\rangle.$$

The Kleinian group $\widehat{K}$ has a connected region of discontinuity Ω and a fundamental domain for it is determined the the bounded region determined by the unit circle, the two rays of arguments $\frac{\pi i}{3}$ and $\frac{-\pi i}{3}$, respectively, and the circle C orthogonal to the unit circle and containing the points p and $\dfrac{1}{p}$ (see figure 1). The Klein surface uniformized by $\widehat{K}$ is the closed unit disc with exactly four branch values of orders 2, 2, 2 and 3 on its border.
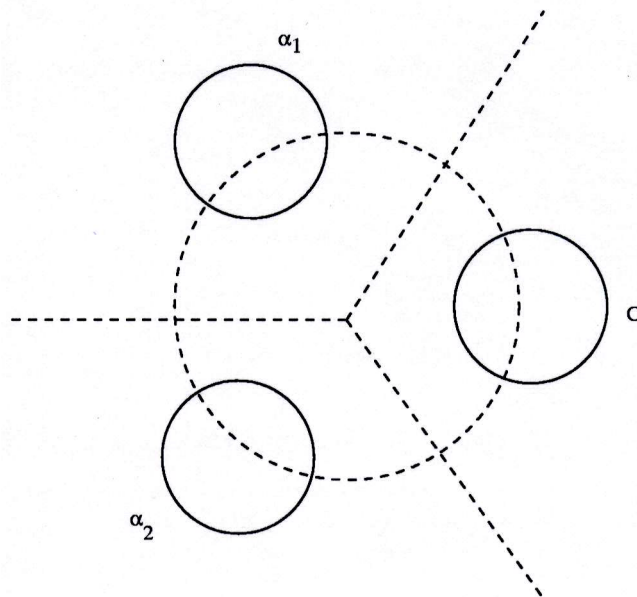
Figure 1

Theorem 6.1. *Every Klein surface which is the unit closed disc with exactly four branch values of orders* 2, 2, 2 *and* 3 *on the border can be uniformized by a group* $\widehat{K}$ *for some* $p \in (2 - \sqrt{3}, 1)$.

Proof.   This is just consequence of quasiconformal deformation and the fact that every reflection on the Riemann sphere has an Euclidean circle as fixed points. □

The index two subgroup K of $\widehat{K}$ of the orientation preserving transformations is generated by the transformations T, W and J and the presentation of K is

$$K = \langle T, W, J : T^2 = W^3 = (WT)^2 = J^2 = (TJ)^2 = 1 \rangle,$$

that is, isomorphic to the free amalgamated product of the dihedral group $D_3 = \langle T, W \rangle$ with the Klein group $\mathcal{Z}_2 + \mathcal{Z}_2 = \langle T, J \rangle$ over the cyclic group of order two generated by T. The Riemann surface Ω/K is the Riemann sphere with four branched values of order 2, 2, 2 and 3, all of them fixed by the reflection induced by σ on it. A fundamental domain of K can be considered as the union of the above fundamental domain of $\widehat{K}$ with its image under σ together the points of the unit circle located between $e^{\frac{\pi i}{3}}$ and $e^{\frac{-\pi i}{3}}$, respectively.

Remark. (1) The boundary case $p = 2 - \sqrt{3}$ corresponds to the case that $\widehat{K}$ uniformizes a hyperbolic closed triangle with one vertex at infinity, one vertex with angle $\frac{\pi}{3}$ and the other with angle $\frac{\pi}{2}$. The corresponding group K uniformizes an sphere with a puncture and two branched values of orders 2 and 3, respectively. (2) The boundary case $p = 1$ corresponds to the case that $\widehat{K}$ is only generated by the reflections σ_1, σ_2 and σ, but acting on the three-punctured Riemann sphere. In this case, the group $\widehat{K}$ uniformizes the closed unit disc with three branched values at the border with orders 2, 3 and ∞, respectively.

7. Schottky uniformizations of the genus two maximal symmetric Riemann surfaces

Let us consider the Möbius transformations $A_1 = JWJW^{-1}$, $A_2 = JW^{-1}JW$,
$B_1 = (\sigma_3 \sigma_2)^2$ and $B_2 = \sigma_1 B_1 \sigma_1$. The groups $G_2 = \langle A_1, A_2 \rangle$ and $F_2 = \langle B_1, B_2 \rangle$ are classical Schottky groups of genus two and a standard fundamental domain for both G_2 and F_2 is given by the circles $W(C)$, $J(W(C))$, $W^{-1}(C)$ and $J(W^{-1}(C))$.

Since $WA_1W^{-1} = A_1^{-1}A_2$, $WA_2W^{-1} = A_1^{-1}$, $JA_1J = A_1^{-1}$, $JA_2J = A_2^{-1}$, $TA_1T^{-1} = A_2$, $TA_2T = A_1$, $\sigma A_1 \sigma = A_1$, $\sigma A_2 \sigma = A_2$, $WB_1W^{-1} = B_1^{-1}B_2$, $WB_2W^{-1} = B_1^{-1}$, $JB_1J = B_2^{-1}$, $JB_2J = B_1^{-1}$, $TB_1T = B_2$, $TB_2T = B_1$, $\sigma B_1 \sigma = B_1$ and $\sigma B_2 \sigma = B_2$, we have that both G_2 and F_2 are normal subgroups of $\widehat{K}$. Moreover, for $Q \in \{G_2, F_2\}$, $\widehat{K}/Q \cong D_3 + \mathcal{Z}_2 + \mathcal{Z}_2$ and $K/Q \cong D_3 + \mathcal{Z}_2$. In particular, the Riemann surfaces of genus two $S_2 = \Omega/G_2$ (respectively, $R_2 = \Omega/F_2$) admit the group $\widehat{K}/G_2$ (respectively, $\widehat{K}/F_2$) as group of symmetries, with K/G_2

(respectively, K/F_2) as its index two subgroup of orientation preserving symmetries. The quotient $S_2/(\widehat{K}/G_2)$ (respectively, $R_2/(\widehat{K}/F_2)$) is the unit closed disc with four branch values in its boundary of orders 2, 2, 2 and 3.

Theorem 7.1. *If X is a maximal symmetric Riemann surface of genus two with a maximal reflection $\rho : X \to X$ so that X/ρ is orientable, then it can be uniformized by either G_2 or F_2 for a suitable $p \in (2 - \sqrt{3}, 1)$.*

Proof. Fix a value of $p \in (2-\sqrt{3}, 1)$ and let K be the corresponding group above constructed. Let X be a maximal symmetric Riemann surface of genus two together a maximal reflection $\tau : X \to X$ as in the hypothesis. The topological action of $K(X, \tau)$ is either reflected by G_2 or F_2, since there are exactly two possible actions. Without lost of generality, we may assume that this is reflected by G_2. Set $S_2 = \Omega/G_2$ and let $f : S_2 \to X$ be an orientation preserving homeomorphisms so that $f\widehat{K}/G_2 f^{-1} = K(X, \tau)$. We may assume that f is quasiconformal homeomorphisms with complex Beltrami differential μ. We lift μ to Ω and extend it as zero to the limit set of $\widehat{K}$. Let $W : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$ be a μ-quasiconformal homeomorphisms. Then $W\widehat{K}W^{-1}$ is again a group generated by reflections. We may assume that W fixes 0, 1 and ∞. Since a reflection has as fixed points only Euclidean circles on the Riemann sphere, we have that the group $W\widehat{K}W^{-1}$ is again one of our groups $\widehat{K}$ for a suitable value of p. If we denote by $\pi : \Omega \to \Omega/G_2$ the holomorphic covering induced by G_2, then we have that $f\pi W^{-1} : \Omega(W\widehat{K}W^{-1}) \to X$ an uniformization of X by the corresponding Schottky group G_2, so that the lifting of $K(X, \tau)$ is exactly $W\widehat{K}W^{-1}$. $\square$

7.1. The Schottky Group G_2

In this subsection we proceed to see the following:

Theorem 7.2. *Let X be a maximal symmetric Riemann surface of genus 2 with a maximal reflection $\rho : X \to X$ so that X/ρ is a genus one surface with one hole. Let $p \in (2 - \sqrt{3}, 1)$ be so that $X = \Omega/G_2$.*

Then a Riemann matrix of X is given by

$$Z = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i \begin{bmatrix} w & \frac{w}{2} \\ \frac{w}{2} & w \end{bmatrix},$$

where

$$w = \frac{1}{2\pi} Log \left(\prod_{\gamma \in G_2} \frac{\gamma(q_2) - A_1^{-1}(\infty)}{\gamma(q_1) - A_1^{-1}(\infty)} \right),$$

$$q_1 = e^{(\frac{2\pi}{3} - \theta)i}, \quad q_2 = J(q_1)$$

and $\theta \in (0, \frac{\pi}{2})$ is such that $\cos\theta = \frac{2p}{1+p^2}$.

Proof.   Set $\alpha_1 = W(C)$, $\alpha_1' = J(\alpha_1)$, $\alpha_2 = W^{-1}(C)$ and $\alpha_2' = J(\alpha_2)$. If we orient the circles α_1, α_2, α_1' and α_2' and consider the oriented paths β_1 and β_2 as shown in figure 2, then the projection of these loops and paths determine a symplectic basis on the closed Riemann surface S_2.
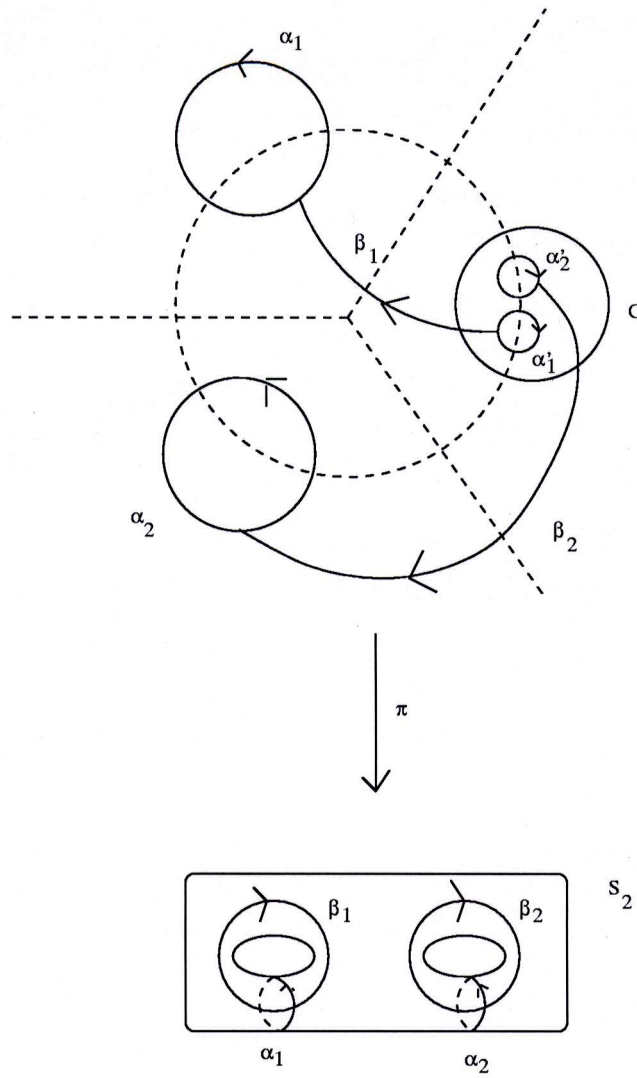
Figure 2

In this way, we obtain a faithful symplectic representation

$$\rho : \widehat{K}/G_2 \to \mathrm{Sp}\widetilde{(4; \mathcal{Z})}$$

defined by

$$\rho(\sigma) = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\rho(J) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \rho(W) = \begin{bmatrix} 0 & -1 & a & b \\ 1 & -1 & c & d \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Riemann matrix $Z = X + iY \in \mathcal{H}_2$ of S_2 defined by the above symplectic basis is a fixed point of the symplectic group $\rho(\widehat{K}/G_2)$. It follows that: $a = -1$, $b = 0$, $c = 0$, $d = 1$, and

$$Z = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i \begin{bmatrix} w & \frac{w}{2} \\ \frac{w}{2} & w \end{bmatrix},$$

where $w > 0$.

The results due to Burnside [2] described at the end of section 4 can be used in this case to the group G_2 to compute the value of w in function of p as described in the theorem. $\square$

Remark. In the case $p = 2 - \sqrt{3}$, the group G_2 is a noded Schottky group of genus two uniformizing an stable Riemann surface of genus two with exactly one dividing node. In the case $p = 1$, we can think of the group G_2 as the trivial group acting on the three-punctured sphere, that is, uniformizing the three-punctured sphere.

7.2. The Schottky Group F_2

In this subsection we proceed to see the following:

Theorem 7.3. *Let X be a maximal symmetric Riemann surface of genus 2 with a maximal reflection $\rho : X \to X$ so that X/ρ is a genus zero surface with three holes. Let $p \in (2 - \sqrt{3}, 1)$ be so that $X = \Omega/F_2$. Then a Riemann matrix of X is given by*

$$Z = i \begin{bmatrix} w & \frac{w}{2} \\ \frac{w}{2} & w \end{bmatrix},$$

where

$$w = \frac{1}{2\pi} Log \left(\prod_{\gamma \in F_2} \frac{\gamma(p_2) - B_1^{-1}(\infty)}{\gamma(q_1) - B_1^{-1}(\infty)} \right),$$

$$q_1 = e^{(\frac{2\pi}{3} - \theta)i}, \quad p_2 = \sigma_3(q_1)$$

and $\theta \in (0, \frac{\pi}{2})$ *is such that* $\cos\theta = \frac{2p}{1+p^2}$.

Proof. Set $\alpha_1 = W(C)$, $\alpha_1' = \sigma_3(\alpha_1)$, $\alpha_2 = W^{-1}(C)$ and $\alpha_2' = \sigma_3(\alpha_2)$. If we orient the circles α_1, α_2, α_1' and α_2' and consider the oriented paths β_1 and β_2 as shown in figure 3, then the projection of these loops and paths determine a symplectic basis on the closed Riemann surface R_2.
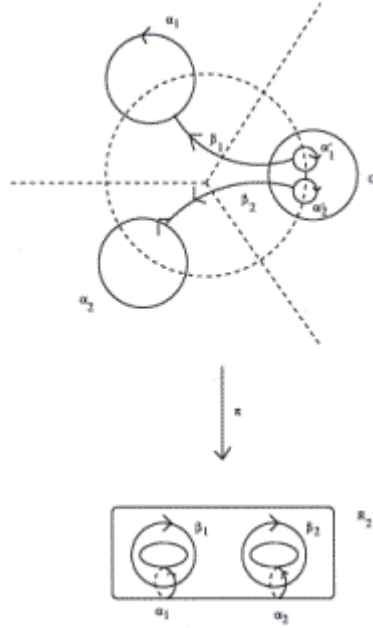

Figure 3

In this way, we obtain a faithful symplectic representation

$$\rho : \widehat{K}/F_2 \to \mathrm{Sp}(\widetilde{4; \mathcal{Z}})$$

defined by

$$\rho(\sigma) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \rho(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\rho(J) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \rho(W) = \begin{bmatrix} 0 & -1 & a & b \\ 1 & -1 & c & d \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The Riemann matrix $Z = X + iY \in \mathcal{H}_2$ of R_2 defined by the above symplectic basis is a fixed point of the symplectic group $\rho(\widehat{K}/F_2)$. It follows that: $a = b = c = d = 0$, and

$$Z = i \begin{bmatrix} w & \frac{w}{2} \\ \frac{w}{2} & w \end{bmatrix},$$

where $w > 0$.

Again the results due to Burnside [2] described at the end of section 4 can be used in this case to the group F_2 to compute the value of w in function of p, as desired. $\square$

Remarks.

(1) In the case $p = 2 - \sqrt{3}$, the group F_2 is a noded Schottky group of genus two uniformizing an stable Riemann surface of genus two with exactly three non-dividing nodes. In the case $p = 1$, we can think of the group F_2 as the trivial group acting on the three-punctured sphere, that is, uniformizing the three-punctured sphere.

(2) Let us note at this point that the results in [20] (see also the work of González-Díez [6]) permit us to describe the algebraic curve

(using theta functions) of the surfaces S_2 and R_2 in function of w and, in particular, of p. In this way, we can obtain an explicit relation between the Schottky groups G_2, F_2 and the respective algebraic curves.

8. Schottky uniformizations of the genus five maximal symmetric Riemann surfaces

Let G_5 (respectively, F_5) be the subgroup of G_2 (respectively, F_2) generated by the square of all the elements in G_2 (respectively, F_2), that is:

$$G_5 = \langle x^2 : x \in G_2 \rangle$$

$$F_5 = \langle x^2 : x \in F_2 \rangle.$$

Since G_2 (respectively, F_2) is normal subgroup of index 24 in $\widehat{K}$ and G_5 (respectively, F_5) is normal of index 4 in G_2 (respectively, F_2), we have that G_5 (respectively, F_5) is a normal subgroup of $\widehat{K}$ of index 96. The group G_5 (respectively, F_5) is a free group of index four in the Schottky group G_2 (respectively, F_2) and then it is a Schottky group of genus 5. Free generators for G_5 (respectively, F_5) are given by $D_1 = A_1^2$, $D_2 = A_2^2$, $D_3 = A_1^{-1}A_2^2A_1$, $D_4 = A_1^{-1}A_2^{-1}A_1A_2$ and $D_5 = A_2^{-1}A_1A_2A_1$ (respectively, $E_1 = B_1^2$, $E_2 = B_2^2$, $E_3 = B_1^{-1}B_2^2B_1$, $E_4 = B_1^{-1}B_2^{-1}B_1B_2$ and $E_5 = B_2^{-1}B_1B_2B_1$).

Theorem 8.1. *If S is a maximal symmetric Riemann surface of genus five admitting a maximal reflection $\tau : S \to S$ so that S/τ is orientable, then it can be uniformized by either G_5 or F_5 for a suitable $p \in (2 - \sqrt{3}, 1)$.*

Proof. The proof is exactly the same as done for theorem 7.1. □

Remark. In the case $p = 2 - \sqrt{3}$, the group G_5 is a noded Schottky group uniformizing a stable Riemann surface of genus five with exactly four non-dividing nodes and two components, each one of genus one. Similarly, the group F_5 is a noded Schottky group uniformizing a stable Riemann surface of genus five with exactly six non-dividing nodes.

A standard fundamental domain for G_5 (respectively, F_5) respect to the above free generators is determined by the circles $\theta_1 = A_1^{-1}(\alpha_1)$, $\theta_1' = \alpha_1'$, $\theta_2 = A_2^{-1}(\alpha_2)$, $\theta_2' = \alpha_2'$, $\theta_3 = A_1^{-1}A_2^{-1}(\alpha_2)$, $\theta_3' = A_1^{-1}(\alpha_2')$, $\theta_4 = A_2^{-1}(\alpha_1)$, $\theta_4' = A_1^{-1}A_2^{-1}(\alpha_1')$, $\theta_5 = A_1^{-1}A_2^{-1}(\alpha_1)$ and $\theta_5' = A_2^{-1}(\alpha_1')$ (respectively, $\theta_1 = B_1^{-1}(\alpha_1)$, $\theta_1' = \alpha_1'$, $\theta_2 = B_2^{-1}(\alpha_2)$, $\theta_2' = \alpha_2'$, $\theta_3 = B_1^{-1}B_2^{-1}(\alpha_2)$, $\theta_3' = B_1^{-1}(\alpha_2')$, $\theta_4 = B_2^{-1}(\alpha_1)$, $\theta_4' = B_1^{-1}B_2^{-1}(\alpha_1')$, $\theta_5 = B_1^{-1}B_2^{-1}(\alpha_1)$ and $\theta_5' = B_2^{-1}(\alpha_1')$).

We can give to these θ-loops the orientations determined by the orientations we have given to α_1, α_2, α_1' and α_2'. Let us choose the following paths: $\delta_1 = \beta_1 \cup A_1^{-1}(\beta_1)$, $\delta_2 = \beta_2 \cup A_2^{-1}(\beta_2)$, $\delta_3 = A_1^{-1}(\delta_2)$ (respectively, $\delta_1 = \beta_1 \cup B_1^{-1}(\beta_1)$, $\delta_2 = \beta_2 \cup B_2^{-1}(\beta_2)$, $\delta_3 = B_1^{-1}(\delta_2)$), with the orientations determined by the orientations given to β_1 and β_2. For $j = 4, 5$, we choose oriented paths δ_j contained inside the unit disc starting at θ_j and ending at θ_j' and disjoint from δ_1 and δ_{9-j}. The projections of the above loops and paths determine on the Riemann surface S_5 (respectively, R_5) a symplectic basis. Since we have the following relations:

(a) $JD_1J = D_1^{-1}$, $JD_2J = D_2^{-1}$, $JD_3J = D_1D_3^{-1}D_1^{-1}$, $JD_4J = D_1D_3D_5^{-1}D_2^{-1}$, $JD_5J = D_2D_4^{-1}D_3^{-1}D_1^{-1}$, $JE_1J = E_2^{-1}$, $JE_2J = E_1^{-1}$, $JE_3J = E_2E_4^{-1}E_5^{-1}E_2^{-1}$, $JE_4J = E_2E_5E_3^{-1}E_1^{-1}$, $JE_5J = E_1E_5^{-1}E_2^{-1}$;

(b) $WD_1W^{-1} = D_3D_5^{-1}$, $WD_2W^{-1} = D_1^{-1}$, $WD_3W^{-1} = D_4^{-1}D_5^{-1}$, $WD_4W^{-1} = D_5D_1^{-1}$, $WD_5W^{-1} = D_2D_4^{-1}D_5^{-1}$, $WE_1W^{-1} = E_3E_5^{-1}$, $WE_2W^{-1} = E_1^{-1}$, $WE_3W^{-1} = E_4^{-1}E_5^{-1}$, $WE_4W^{-1} = E_5E_1^{-1}$, $WE_5W^{-1} = E_2E_4^{-1}E_5^{-1}$;

(c) $TD_1T = D_2$, $TD_2T = D_1$, $TD_3T = D_5D_4$, $TD_4T = D_4^{-1}$, $TD_5T = D_3D_4$, $TE_1T = E_2$, $TE_2T = E_1$, $TE_3T = E_5E_4$, $TE_4T = E_4^{-1}$, $TE_5T = E_3E_4$;

(d) $\sigma D_j\sigma = D_j$, $\sigma E_j\sigma = E_j$, $j = 1, ..., 5$,

we obtain faithful symplectic representations

$$\rho : \widehat{K}/G_5 \to \mathrm{Sp}(\widetilde{10}; \mathcal{Z})$$

and

$$\eta : \widehat{K}/F_5 \to \mathrm{Sp}(\widetilde{10}; \mathcal{Z})$$

defined by

$$\rho(\sigma) = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\eta(\sigma) = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$\rho(T) = \eta(T) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & t_{11} & t_{12} & t_{13} & t_{14} & t_{15} \\
1 & 0 & 0 & 0 & 0 & t_{21} & t_{22} & t_{23} & t_{24} & t_{25} \\
0 & 0 & 0 & 1 & 1 & t_{31} & t_{32} & t_{33} & t_{34} & t_{35} \\
0 & 0 & 0 & -1 & 0 & t_{41} & t_{42} & t_{43} & t_{44} & t_{45} \\
0 & 0 & 1 & 1 & 0 & t_{51} & t_{52} & t_{53} & t_{54} & t_{55} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}$$

$$\rho(J) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & j_{11} & j_{12} & j_{13} & j_{14} & j_{15} \\ 0 & -1 & 0 & 0 & 0 & j_{21} & j_{22} & j_{23} & j_{24} & j_{25} \\ 0 & 0 & -1 & 0 & 0 & j_{31} & j_{32} & j_{33} & j_{34} & j_{35} \\ 1 & -1 & 1 & 0 & -1 & j_{41} & j_{42} & j_{43} & j_{44} & j_{45} \\ -1 & 1 & -1 & -1 & 0 & j_{51} & j_{52} & j_{53} & j_{54} & j_{55} \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\eta(J) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & j_{11} & j_{12} & j_{13} & j_{14} & j_{15} \\ -1 & 0 & 0 & 0 & 0 & j_{21} & j_{22} & j_{23} & j_{24} & j_{25} \\ 0 & 0 & 0 & -1 & -1 & j_{31} & j_{32} & j_{33} & j_{34} & j_{35} \\ -1 & 1 & -1 & 0 & 1 & j_{41} & j_{42} & j_{43} & j_{44} & j_{45} \\ 1 & -1 & 0 & 0 & -1 & j_{51} & j_{52} & j_{53} & j_{54} & j_{55} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix}$$

$$\rho(W) = \eta(W) = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & w_{11} & w_{12} & w_{13} & w_{14} & w_{15} \\ 0 & 0 & -1 & 0 & 1 & w_{21} & w_{22} & w_{23} & w_{24} & w_{25} \\ 1 & -1 & 0 & 1 & 0 & w_{31} & w_{32} & w_{33} & w_{34} & w_{35} \\ 0 & 1 & -1 & -1 & 0 & w_{41} & w_{42} & w_{43} & w_{44} & w_{45} \\ 0 & -1 & 0 & 1 & 0 & w_{51} & w_{52} & w_{53} & w_{54} & w_{55} \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 \end{bmatrix}$$

In the next three subsections we proceed to show the following results:

Theorem 8.2. *Let S be a maximal symmetric Riemann surface of genus 5 admitting a maximal reflection $\tau : S \to S$ so that S/τ is a genus one surface with four holes. Let $p \in (2 - \sqrt{3}, 1)$ be so that $S = \Omega/G_5$. Then a Riemann matrix of S is given by*

$$
Z = \frac{1}{2}
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

$$
+ i
\begin{bmatrix}
2u+v & u+v & u+v & v & 2(u+v) \\
u+v & 2u+v & 2u+3v & -v & u+2v \\
u+v & 2u+3v & 2u+v & v & u \\
v & -v & v & -3v & 2v \\
2(u+v) & u+2v & u & 2v & 2u
\end{bmatrix},
$$

$$
where \begin{cases}
u = \dfrac{-1}{2\pi} Log \left(\dfrac{\sqrt{\displaystyle\prod_{\gamma \in G_2} \dfrac{\gamma(q_2) - A_1^{-1}(\infty)}{\gamma(q_1) - A_1^{-2}(\infty)}}}{\displaystyle\prod_{\gamma \in G_5} \dfrac{\gamma(q_2) - A_1^{-2}(\infty)}{\gamma(A_1^{-1}(q_1)) - A_1^{-2}(\infty)}} \right) \\[40pt]
v = \dfrac{w}{2} - u
\end{cases}
$$

Theorem 8.3. *Let S be a maximal symmetric Riemann surface of genus 5 admitting a maximal reflection $\tau : S \to S$ so that S/τ is a genus zero surface with six holes. Let $p \in (2 - \sqrt{3}, 1)$ be so that $S = \Omega/F_5$. Then a Riemann matrix of S is given by*

$$
Z = i
\begin{bmatrix}
2u+v & u+v & u+v & v & 2(u+v) \\
u+v & 2u+v & 2u+3v & -v & u+2v \\
u+v & 2u+3v & 2u+v & v & u \\
v & -v & v & -3v & 2v \\
2(u+v) & u+2v & u & 2v & 2u
\end{bmatrix},
$$

$$
where \begin{cases} u &= \dfrac{-1}{2\pi} Log\left(\dfrac{\sqrt{\displaystyle\prod_{\gamma \in F_2} \dfrac{\gamma(p_2) - B_1^{-1}(\infty)}{\gamma(q_1) - B_1^{-2}(\infty)}}}{\displaystyle\prod_{\gamma \in F_5} \dfrac{\gamma(p_2) - B_1^{-2}(\infty)}{\gamma(B_1^{-1}(q_1)) - B_1^{-2}(\infty)}} \right) \\[2em] v &= \dfrac{w}{2} - u \end{cases}
$$

8.1. The case G_5

The Riemann matrix $Z = X + iY \in \mathcal{H}_5$ of S_5 defined by the above symplectic basis is a fixed point of the symplectic group $\rho(\widehat{K}/G_5)$. It follows that: $t_{nm} = j_{nm} = 0$, $w_{11} = w_{15} = -w_{21} = -w_{22} = -w_{31} = -w_{32} = w_{41} = w_{45} = 1$, $w_{22} = w_{23} = w_{24} = w_{31} = w_{24} = w_{25} = w_{31} = w_{34} = w_{35} = w_{41} = w_{42} = w_{43} = w_{44} = w_{45} = w_{52} = w_{53} = w_{54} = 0$, and

$$
Z = \frac{-1}{2} \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}
$$

$$
+ i \begin{bmatrix} 2u+v & u+v & u+v & v & 2(u+v) \\ u+v & 2u+v & 2u+3v & -v & u+2v \\ u+v & 2u+3v & 2u+v & v & u \\ v & -v & v & -3v & 2v \\ 2(u+v) & u+2v & u & 2v & 2u \end{bmatrix},
$$

where $0 < -v < u$.

8.2. The case F_5

The Riemann matrix $Z = X + iY \in \mathcal{H}_5$ of R_5 defined by the above symplectic basis is a fixed point of the symplectic group $\eta(\widehat{K}/F_5)$. It

follows that: $t_{nm} = j_{nm} = w_{nm} = 0$, and

$$Z = i \begin{bmatrix} 2u+v & u+v & u+v & v & 2(u+v) \\ u+v & 2u+v & 2u+3v & -v & u+2v \\ u+v & 2u+3v & 2u+v & v & u \\ v & -v & v & -3v & 2v \\ 2(u+v) & u+2v & u & 2v & 2u \end{bmatrix},$$

where $0 < -v < u$.

8.3. The relations between w, u and v

Let us now consider the regular (unbranched) covering $P : S_5 \to S_2$ (respectively, $P : R_5 \to R_2$), whose covering group is the Klein group $\mathcal{Z}_2 + \mathcal{Z}_2$ generated by the automorphisms of S_5 (respectively, R_5) defined by A_1 and A_2 (respectively, B_1 and B_2). We have that $H_1(P) : H_1(S_5, \mathcal{Z}) \to H_1(S_2, \mathcal{Z})$ (respectively, $H_1(P) : H_1(R_5, \mathcal{Z}) \to H_1(R_2, \mathcal{Z})$) is given by $H_1(P)(\theta_1) = H_1(P)(\theta_4) = H_1(P)(\theta_5) = \alpha_1$, $H_1(P)(\theta_2) = H_1(P)(\theta_3) = \alpha_2$, $H_1(P)(\delta_1) = H_1(P)(\delta_5) = 2\beta_1$, $H_1(P)(\delta_2) = H_1(P)(\delta_3) = 2\beta_2$ and $H_1(P)(\delta_4) = 0$.
Let us denote by w_1 and w_2 the dual holomorphic one-forms of α_1 and α_2, respectively, for the surface S_2 (respectively, R_2). Similarly, let us denote by η_1, η_2, η_3, η_4 and η_5 the dual holomorphic one-forms of θ_1, θ_2, θ_3, θ_3 and θ_5, respectively, for the surface S_5 (respectively, R_5). It follows then the pull-back of holomorphic forms $P^* : H^{1,0}(S_2) \to H^{1,0}(S_5)$ (respectively, $P^* : H^{1,0}(R_2) \to H^{1,0}(R_5)$) is defined by $P^*(w_1) = \eta_1 + \eta_4 + \eta_5$ and $P^*(w_2) = \eta_2 + \eta_3$. Using the equality

$$4(u+v)i = \int_{\delta_1} \eta_1 + \eta_4 + \eta_5 = \int_{\delta_1} P^* w_1 = 2 \int_{\beta_1} w_1 = 2iw,$$

we obtain that $w = 2(u+v)$. Again using the results due to Burnside [2] described at the end of section 4 to the groups G_5 and F_5 we obtain

$$2u + v = -i \int_{\delta_1} \eta_1 = \frac{-1}{2\pi} \mathrm{Log} \left(\prod_{\gamma \in G_5} \frac{\gamma(q_2) - A_1^{-2}(\infty)}{\gamma(A_1^{-1}(q_1)) - A_1^{-2}(\infty)} \right), \quad \text{for } G_5$$

$$2u + v = -i \int_{\delta_1} \eta_1 = \frac{-1}{2\pi} \text{Log} \left(\prod_{\gamma \in F_5} \frac{\gamma(q_2) - B_1^{-2}(\infty)}{\gamma(B_1^{-1}(q_1)) - B_1^{-2}(\infty)} \right), \quad \text{for } F_5.$$

From the above two equalities, we obtain the forms for u and v as desired.

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