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AN ASYMPTOTIC FORMULA FOR THE NUMBER OF EIGENVALUES OF A DIFFERENTIAL OPERATOR

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Abstract

In this work, it is proved that the spectrum of an differential operator with unbounded operator coefficients in elliptic type with partial derivatives is pure discrete and an asymptotic formula is found for the number of eigenvalues of this operator.

1. Introduction

The asymptotic behavior of the spectrum of Sturm-Liouville operator with operator coefficient had been firstly studied by A.G. Kostyuchenko and B.M. Levitan, [1]. Some works about this subject had been done later by M.L. Gorbacuk ,[2], V.. Gorbacuk and M.L. Gorbacuk ,[3], [4], M. Otelbayev ,[5], M.Z. Solomyak ,[6], F.G. Maksudov, M. Bairamoglu and E.E. Adiguzelov ,[7], and etc. In this work, the spectrum of self adjoint operator L forming with differential expression

$$(1.1) \quad l(u) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + Q(x)u, \quad (x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n)$$

defined in $H_1 = L_2(H; \mathbb{R}^n)$ is studied and an asymptotic formula for the number of eigenvalues of L is found , where H denotes the Hilbert space. Here, we will accept that the coefficients $a_{ij}(x)$ ($i, j = 1, \dots, n$) and $Q(x)$ in (1.1) satisfy the following conditions:

- 1) Real valued functions $a_{ij}(x) = a_{ji}(x)$ have bounded derivatives $\frac{\partial a_{ij}(x)}{\partial x_k}$ ($i, j, k = 1, \dots, n$) in $\mathbb{R}^n$.
- 2) There are positive numbers ν and μ such that

$$\nu \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \mu \sum_{i=1}^n \xi_i^2.$$

- 3) $Q(x) : D(Q(x)) \rightarrow H$ is a self adjoint operator, where $D(Q(x)) \subset H$ for each $x \in \mathbb{R}^n$. The set $D = \cap_{x \in \mathbb{R}^n} D(Q(x))$ is dense in H . $Q(x) \geq I$ and $Q^{-1}(x) \in \sigma_\infty(H)$.
- 4) There are constants $0 < a < \frac{3}{2}$ and $B > 0$ such that

$$||[Q(x) - Q(\xi)]Q^{-a}(x)|| \leq B|x - \xi|$$

for $|x - \xi| \leq 1$.

- 5) There is a constant $l > 0$ satisfying $Q^{-l}(x) \in \sigma_1(H)$ and $\int_{\mathbb{R}^n} ||Q^{-l}(x)||_1 dx < \infty$ for $x \in \mathbb{R}^n$.
- 6) There is a positive valued function f satisfying

$$||e^{-ctQ(\xi)}||_1 \leq ||e^{-f(c)tQ(x)}||_1$$

for $c > 0$, $t > 0$ and $|x - \xi| \leq 1$.

7) For every $M > 0$

$$\int_{\mathbb{R}^n} \text{tr} e^{-MtQ(x)} dx = O(1) \int_{\mathbb{R}^n} \text{tr} e^{-tQ(x)} dx.$$

Here, $\text{tr} A = \text{trace} A$ is the sum of the eigenvalues of a kernel operator A .

8) Let $\alpha_1(x) \leq \alpha_2(x) \leq \dots \leq \alpha_m(x) \leq \dots$ be eigenvalues of operator $Q(x)$. Assume that the functions $\alpha_1(x), \alpha_2(x), \dots$ are measurable functions. Let

$$\rho(\lambda) = \sum_i \int_{\alpha_i(x) < \lambda} \phi(x) (\lambda - \alpha_i(x))^{\frac{n}{2}} dx,$$

$$\phi(x) = \int_{\mathbb{R}^n} e^{-\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j} d\xi, \quad (\xi = (\xi_1, \dots, \xi_n)).$$

Suppose that there is a constant $a_0 > 0$ such that

$$\lambda \rho'(\lambda) \leq a_0 \rho(\lambda)$$

for positive big values of λ . Let D' denotes the set of elements in the form $\sum_{k=1}^m \varphi_k(x) f_k$ of the space H_1 , where $\varphi_1(x), \varphi_2(x), \dots, \varphi_m(x)$, are scalar functions having compact support and continuous derivatives of second order defined in $\mathbb{R}^n$ and $\{f_k\}_{k=1}^m \subset D$. It is known that $\overline{D'} = H_1$. The linear operator L' defined in the form

$$L'u = l(u) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + Q(x)u, \quad u \in D'$$

is a symmetric and positive operator in H_1 . Suppose that the operator L which is closure of L' is pure discrete. Let

$$L_0(\eta, \frac{\partial}{\partial x}) = -\sum_{i,j=1}^n a_{ij}(\eta) \frac{\partial^2}{\partial x_i \partial x_j}$$

$$L_1(x, \frac{\partial}{\partial x}) = -\sum_{i,j=1}^n \frac{\partial a_{ij}(x)}{\partial x_i} \frac{\partial}{\partial x_j}.$$

To solve our main problem, we shall firstly study the Green function of Cauchy problem in the following:

$$\begin{cases} \frac{\partial u}{\partial t} = -Lu = -[L_0(x, \frac{\partial}{\partial x}) + L_1(x, \frac{\partial}{\partial x}) + Q(x)]u \\ u(x, 0) = \psi(x); \quad u = u(x, t), \quad \psi(x) \in L_2(H, \mathbb{R}^n) \end{cases} \quad (1.2)$$

We shall look this for the Green function given by the form

$$(1.3) \quad G(x, y, t) = G_1(x - y, y, t)e^{-tQ(y)} + \int_0^t d\tau \int_{\mathbb{R}^n} G_1(x - \xi, \xi, t - \tau)e^{-(t-\tau)Q(\xi)} \varphi(\xi, y, \tau) d\xi$$

We will use parametric method for this. Here $G_1(x - y, y, t)$ is a function of the form

$$(1.4) \quad G_1(x - y, y, t) = R(x - y)G_0(x - y, y, t),$$

which has continuous derivatives of second order, where

$$R(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > 1 \end{cases} \quad (1.5)$$

And $G_0(x - y, \eta, t)$ denotes the Green function of the problem

$$\begin{cases} \frac{\partial u}{\partial t} = -L_0(\eta, \frac{\partial}{\partial x})u \\ u(x, 0) = \psi(x); \quad \psi(x) \in L_2(H, \mathbb{R}^n) \end{cases} \quad (1.6)$$

The function $\varphi(x, y, t)$ in (1.3) is an unknown operator function that it must be here found. If the equality

$$(1.7) \quad \frac{\partial G}{\partial t} = -LG = -[L_0(x, \frac{\partial}{\partial x}) + L_1(x, \frac{\partial}{\partial x}) + Q(x)]G$$

is true, then we can take $G(x, y, t)$ as a Green function of problem (1.2). And so, the integral equation

$$(1.8) \quad \varphi(x, y, t) = K(x, y, t) + \int_0^t d\tau \int_{\mathbb{R}^n} K(x, \xi, t - \tau) \varphi(\xi, y, \tau) d\xi$$

is obtained, where

$$\begin{aligned}
 K(x, y, t) = & -e^{-tQ(y)} \left[\frac{\partial}{\partial t} + L_0(x, \frac{\partial}{\partial x}) \right] G_1(x - y, y, t) + \\
 (1.9) \quad & + [Q(y) - Q(x)] e^{-tQ(y)} G_1(x - y, y, t) - \\
 & -e^{-tQ(y)} L_1(x, \frac{\partial}{\partial x}) G_1(x - y, y, t).
 \end{aligned}$$

2. Preliminaries

Now, before we give main results, we remind some of its definitions used in this paper as below:

About Measurability

Let $f(x)$ and $\{f_i(x)\}_{i=1}^{\infty}$ be any functions defined on Euclidean space $\mathbb{R}^n$ that their values belong to a separable Hilbert space H .

Definition 2.1 A function defined on $\mathbb{R}^n$ whose range is a countable set, is said to be countable valued if it takes on a measurable set its all values except zero.

Definition 2.2 A function $f(x)$ is said to be strong measurable if there exists a sequence $\{f_i(x)\}_{i=1}^{\infty}$ of countable valued functions such that $\lim_{i \rightarrow \infty} \|f_i(x) - f(x)\|_H = 0$ in almost every where on $\mathbb{R}^n$.

The space $H_1 = L_2(H; \mathbb{R}^n)$

Let us denote with $H_1 = L_2(H; \mathbb{R}^n)$ the set of strong measurable functions f of satisfying the condition

$$\int_{\mathbb{R}^n} \|f(x)\|_H^2 dx < \infty \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

If the inner product of two element f and g of the set H_1 is defined by the formula

$$(f, g) = \int_{\mathbb{R}^n} (f(x), g(x))_H dx$$

then the set H_1 forms a separable Hilbert space, [8].

The spaces $\sigma_{\infty}(H)$ and $\sigma_1(H)$

The set of all complete continuous operators from H to H is denoted with σ_{∞} . Let $A \in \sigma_{\infty}(H)$ be an operator which is different from zero and A^* be adjoint operator of A . Let $s_1 \geq s_2 \geq \dots \geq s_i \geq \dots$ be the eigenvalues which is different from zero of positive self-adjoint operator

$(A^*A)^{1/2}$. Here every eigenvalues is repeated according to multiplicity number. Moreover, $s_1, s_2, \dots, s_i, \dots$ are the positive numbers. These are called the s -numbers of operator A . We can write the s -numbers of A in the form $s_i(A)$ and show the number of the s -numbers of A with $\nu(A)$. Not that $\nu(A)$ can be finite or infinite. We shall denote the union of the operator "0" and the set of all operators $A \in \sigma_\infty(H)$ satisfying

$$\sum_{i=1}^{\nu(A)} s_i(A) < \infty$$

with σ_1 or $\sigma_1(H)$. σ_1 is a Banach space, [9]. In this space, every operator $A \neq 0$ is defined by

$$\|A\|_{\sigma_1(H)} = \sum_{i=1}^{\nu(A)} s_i(A)$$

and we accept that $\|0\|_{\sigma_1(H)} = 0$, [9].

Along this study, we shall denote positive constants with c which may not be same and the norm of the kernel operator from H to H in space $\sigma_1(H)$ with $\|\cdot\|_1$ as in [9]. Moreover, here we shall use the inequalities

$$\|AB\|_1 \leq \|A\|_1 \|B\|$$

$$\|BA\| \leq \|A\|_1 \|B\| \quad (A \in \sigma_1(H), B \in L(H, H))$$

and

$$(2.1) \quad \|D_x^{(k)} G_0(x-y, y, t)\| \leq ct^{\frac{n+k}{2}} \exp(-c \frac{|x-y|^2}{t}), \quad (k = 0, 1, 2)$$

which were proved in [9,10] respectively and we shall suppose $t \in (0, 1)$.

3. The spectrum of operator L asymptotic formula for the number of its eigenvalues

In this section, the spectrum of operator L is studied and an asymptotic formula is found for the number of eigenvalues which is less than λ , where λ is a positive variable. First of all, to solve integral equation (1.8), we need to estimate the operator function $K(x, y, t)$.

Theorem 3.1. If the conditions 1), 4) are satisfied and for every $t > 0$, $x \in \mathbb{R}^n$ $e^{-tQ(x)} \in \sigma_1(H)$ then

$$\|K(x, y, t)\|_1 \leq \text{const.} t^{-b} \|\exp(-ctQ(y))\|_1 \exp[-c \frac{|x-y|^2}{t}],$$

where $b = \max\{\frac{n+1}{2}, \frac{n-1}{2} + a\}$.

Proof. From the relations (1.4), (1.6), (2.1) and the functions $a_{ij}(x)$ satisfy the condition 1), for $|x-y| \leq \frac{1}{2}$ we obtain

$$\begin{aligned} \left[\frac{\partial}{\partial t} + L_0(x, \frac{\partial}{\partial x}) \right] G_1(x-y, y, t) &= \left\| L_0(x, \frac{\partial}{\partial x}) - L_0(y, \frac{\partial}{\partial x}) \right\| G_0(x-y, y, t) \\ &= \left\| \sum_{i,j=1}^n [a_{ij}(x) - a_{ij}(y)] \frac{\partial^2 G_0(x-y, y, t)}{\partial x_i \partial x_j} \right\| \\ &\leq \text{const.} t^{-\frac{n+2}{2}} |x-y| \exp[-c \frac{|x-y|^2}{t}] \\ &\leq \text{const.} t^{-\frac{n+2}{2}} \exp[-\frac{c}{2} \frac{|x-y|^2}{t}]. \end{aligned} \quad (3.1)$$

If we use again (1.4) and (2.1), for $\frac{1}{2} < |x-y| \leq 1$ we find

$$\|L_0(x, \frac{\partial}{\partial x}) G_1(x-y, y, t)\| \leq \text{const.} t^{-\frac{n+2}{2}} \exp[-c \frac{|x-y|^2}{t}],$$

$$\begin{aligned} \left\| \frac{\partial}{\partial t} G_1(x-y, y, t) \right\| &= |R(x, y)| \left\| \frac{\partial}{\partial t} G_0(x-y, y, t) \right\| \\ &= |R(x, y)| \left\| L_0(y, \frac{\partial}{\partial x}) G_0(x-y, y, t) \right\| \\ &\leq \text{const.} t^{-\frac{n+2}{2}} \exp[-c \frac{|x-y|^2}{t}], \end{aligned} \quad ,$$

$$(3.2) \quad \left\| \left[\frac{\partial}{\partial t} + L_0(x, \frac{\partial}{\partial x}) \right] G_1(x-y, y, t) \right\| \leq \text{const.} \exp[-\frac{c}{2} \frac{|x-y|^2}{t}].$$

On the other hand, recall for $|x-y| > 1$ $G_1(x-y, y, t) \equiv 0$ and from (3.1) and (3.2) we obtain

$$(3.3) \quad \left\| \left[\frac{\partial}{\partial t} + L_0(x, \frac{\partial}{\partial x}) \right] G_1(x-y, y, t) \right\| \leq \text{const.} t^{-\frac{n+1}{2}} \exp[-\frac{c}{2} \frac{|x-y|^2}{t}].$$

By the hypothesis, since for $t > 0$, $x \in \mathbb{R}^n$ $e^{-tQ(x)} \in \sigma_1(H)$ we write

$$\begin{aligned} & \| (Q(y) - Q(x))e^{-tQ(y)}G_1(x - y, y, t) \|_1 \leq \text{const.} \| (Q(y) - Q(x)) \cdot \\ (3.4) \quad & e^{-\frac{t}{2}Q(y)} \| \| e^{-\frac{t}{2}Q(y)} \|_1 t^{-\frac{n}{2}} \exp[-c \frac{|x - y|^2}{t}]. \end{aligned}$$

Since $Q(x)$ satisfies the condition 4) we have

$$\begin{aligned} \| (Q(y) - Q(x))e^{-\frac{t}{2}Q(y)} \| & \leq \| (Q(y) - Q(x))Q^{-a}(y) \| \| Q^a(y)e^{-\frac{t}{2}Q(y)} \| \\ & \leq B|x - y|t^{-a} \| t^a Q^a(y)e^{-\frac{t}{2}Q(y)} \| \\ & \leq \text{const.} t^{-a} |x - y|. \end{aligned}$$

If we consider this last relation for (3.4), then we obtain

$$\begin{aligned} & \| (Q(y) - Q(x))e^{-tQ(y)}G_1(x - y, y, t) \|_1 \leq \text{const.} t^{-\frac{n-1}{2}-a}. \\ (3.5) \quad & \| e^{-\frac{t}{2}Q(y)} \|_1 \exp[-\frac{c}{2} \frac{|x - y|^2}{t}]. \end{aligned}$$

Since the derivatives of second order of $R(x)$ are continuous, from (1.4) and (2.1) we find

$$\begin{aligned} & \| e^{-tQ(y)} L_1(x, \frac{\partial}{\partial x}) G_1(x - y, y, t) \|_1 \\ (3.6) \quad & \leq \text{const.} t^{-\frac{n+1}{2}} \cdot \| e^{-tQ(y)} \|_1 \exp[-c \frac{|x - y|^2}{t}]. \end{aligned}$$

From (1.9), (3.3), (3.5) and (3.6) we obtain

$$\| K(x, y, t) \|_1 \leq \text{const.} t^{-b} \| \exp(-ctQ(y)) \|_1 \exp[-c \frac{|x - y|^2}{t}].$$

This proves theorem.

In a similar form of the proof of this theorem, the inequality

$$(3.7) \quad ||K(x, y, t)|| \leq \text{const.} t^{-b} \exp[-c \frac{|x-y|^2}{t}]$$

can be shown.

From theorem 3.1, (3.7) and the inequalities

$$(3.8) \quad \int_{-\infty}^{\infty} \exp[-c \frac{(\xi-z)^2}{t-\tau} - c \frac{(z-\eta)^2}{\tau}] [(t-\tau)\tau]^{\frac{1}{2}} dz$$

$$\leq c(\epsilon) t^{\frac{1}{2}} \exp[-(c-\epsilon) \frac{(\xi-\eta)^2}{t}],$$

$$\frac{(\xi-z)^2}{t-\tau} + \frac{(z-\eta)^2}{\tau} \geq \frac{(\xi-\eta)^2}{t}$$

which were proved in [10], it can be proved that the series

$$\varphi(x, y, t) = \sum_{m=1}^{\infty} K_m(x, y, t)$$

which is a solution of the equation (1.8), is convergent with respect to the norm in $\sigma_1(H)$ of such that $Q^{-l}(x) \in \sigma_1(H)$ for every $x \in \mathbb{R}^n$ and the inequalities

$$||\varphi(x, y, t)||_1 \leq \text{const.} \exp[-c \frac{|x-y|^2}{t}]$$

$$||Q^{-l}(y)||_1 + t^{-b} ||\exp(-ctQ^{-l}(y))||_1 \} (3.9)$$

$$(3.10) \quad ||\varphi(x, y, t)||_1 \leq \text{const.} ||Q^{-l}(y)||_1 \exp[-c \frac{|x-y|^2}{t}], \quad (|x-y| > 1)$$

$$(3.11) \quad ||\varphi(x, y, t)|| \leq \text{const.} t^{-b} \exp[-c \frac{|x-y|^2}{t}].$$

are satisfied.

Theorem 3.2. If the coefficients of differential expression (1.1) satisfy the conditions 1),3),4),6) and for every $x \in \mathbb{R}^n$ $Q^{-l}(x) \in \sigma_1(H)$, then while $t \rightarrow +0$

$$G(x, y, t) = G_1(x-y, y, t)e^{-tQ(y)} + t^{1-b} \exp\left[-c \frac{|x-y|^2}{t}\right] \cdot \{ \|Q^{-l}(y)\|_1 + \\ + \| \exp(-ctQ(y)) \|_1 \} O(1),$$

where $O(1)$ is an operator function which is bounded with respect to the norm $\sigma_1(H)$ in variables x, y and t and $b = \max\{\frac{n+1}{2}, \frac{n-1}{2} + a\}$.

Proof. From (1.3), (1.4), (1.5) and (2.1) we obtain

$$\begin{aligned} & \|G(x, y, t) - e^{tQ(y)} G_1(x-y, y, t)\|_1 \leq \\ & \leq A_1 \int_0^t (t-\tau)^{-\frac{n}{2}} d\tau \int_{\mathbb{R}^n} \exp\left[-c \frac{|x-\xi|^2}{t-\tau}\right] \\ & \quad \| \exp[-(t-\tau)Q(\xi)] \varphi(\xi, y, \tau) \|_1 d\xi \\ & = A_1 \int_0^t (t-\tau)^{-\frac{n}{2}} d\tau \int_{|y-\xi| \leq 1} \exp\left[-c \frac{|x-\xi|^2}{t-\tau}\right] \| \\ & \quad \exp[-(t-\tau)Q(\xi)] \varphi(\xi, y, \tau) \|_1 d\xi \\ & + A_1 \int_0^t (t-\tau)^{-\frac{n}{2}} d\tau \int_{|y-\xi| > 1} \exp\left[-c \frac{|x-\xi|^2}{t-\tau}\right] \| \\ & \quad \exp[-(t-\tau)Q(\xi)] \varphi(\xi, y, \tau) \|_1 d\xi \\ (3.12) \quad & = B_1 + B_2 \end{aligned}$$

where A_1 is a positive constant. By using the inequalities (3.9) and (3.11) we shall estimate the integral B_1 :

$$\begin{aligned} B_1 & = A_1 \int_0^t (t-\tau)^{-\frac{n}{2}} d\tau \int_{|y-\xi| \leq 1} \exp\left[-c \frac{|x-\xi|^2}{t-\tau}\right] \cdot \| \\ & \quad \exp[-(t-\tau)Q(\xi)] \varphi(\xi, y, \tau) \|_1 d\xi \\ & \leq \text{const.} \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{n}{2}} \tau^{-b} d\tau \int_{|y-\xi| \leq 1} \exp\left[-c \frac{|x-\xi|^2}{t-\tau}\right] \| \exp[-(t-\tau)Q(\xi)] \|_1 \cdot \end{aligned}$$

$$\begin{aligned}
 & \exp[-c \frac{|\xi - y|^2}{\tau}] d\xi + \text{sabit} \int_{\frac{t}{2}}^t (t - \tau)^{-\frac{n}{2}} d\tau \int_{|y - \xi| \leq 1} \exp[-c \frac{|x - \xi|^2}{t - \tau}]. \\
 & \exp[-(t - \tau)\alpha_1(\xi)] \exp[-c \frac{|\xi - y|^2}{\tau}] \{\tau^{-b} \|\exp[-c\tau Q(y)]\|_1 + \|Q^{-l}(y)\|_1\} d\xi \\
 (3.13) \quad & = B_1^{(1)} + B_1^{(2)}.
 \end{aligned}$$

Since $Q(x)$ satisfies the condition 6) and from the inequality (3.8) and the formula

$$\int_0^t \tau^\alpha (t - \tau)^\beta d\tau = t^{\alpha+\beta+1} B(\alpha + 1, \beta + 1) \quad (\alpha > -1, \beta > -1),$$

where

$$B(m, k) = \frac{\Gamma(m)\Gamma(k)}{\Gamma(m+k)}, \quad \Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx$$

for $B_1^{(1)}$ we find

$$\begin{aligned}
 B_1^{(1)} &= \text{const.} \int_0^{\frac{t}{2}} (t - \tau)^{-\frac{n}{2}} \tau^{-b} d\tau \int_{|y - \xi| \leq 1} \exp[-c \frac{|x - \xi|^2}{t - \tau}]. \\
 & \|\exp[-(t - \tau)Q(\xi)]\|_1 \exp[-c \frac{|\xi - y|^2}{\tau}] d\xi \\
 & \leq \text{const.} \|\exp[-c_1 t Q(y)]\|_1 \int_0^{\frac{t}{2}} \tau^{\frac{n}{2}-b} d\tau. \\
 & \int_{\mathbb{R}^n} \exp[-c \frac{|x - \xi|^2}{t - \tau} - c \frac{|\xi - y|^2}{\tau}] [\tau(t - \tau)]^{-\frac{n}{2}} d\xi \\
 (3.14) \quad & \leq \text{const.} t^{1-b} \|\exp[-c_1 t Q(y)]\|_1 \exp[-c_1 \frac{|x - y|^2}{t}].
 \end{aligned}$$

In the similar form, we can estimate $B_1^{(2)}$:

$$B_1^{(2)} = \text{const.} \int_{\frac{t}{2}}^t (t - \tau)^{-\frac{n}{2}} d\tau \int_{|y - \xi| \leq 1} \exp[-c \frac{|x - \xi|^2}{t - \tau}] \exp[-(t - \tau)\alpha_1(\xi)].$$

$$\begin{aligned}
& \exp\left[-c\frac{|x-\xi|^2}{t-\tau} - c\frac{|\xi-y|^2}{\tau}\right]\{\tau^{-b}\|\exp[-c\tau Q(y)]\|_1 + \|Q^{-l}(y)\|_1\}d\xi \\
& \leq \text{const.}\{t^{-b}\|\exp[-c_1tQ(y)]\|_1 + \|Q^{-l}(y)\|_1\} \\
& \quad \int_0^t \tau^{\frac{n}{2}}d\tau \int_{\mathbb{R}^n} \exp\left[-c\frac{|x-\xi|^2}{t-\tau} - c\frac{|\xi-y|^2}{\tau}\right][\tau(t-\tau)]^{-\frac{n}{2}}d\xi \\
& \leq \text{const.}t\{t^{-b}\|\exp[-c_1tQ(y)]\|_1 + \|Q^{-l}(y)\|_1\}\exp\left[-c_1\frac{|x-y|^2}{t}\right].
\end{aligned}$$

From (3.13), (3.14) and the last relation we find

$$\begin{aligned}
B_1 & \leq \text{const.}t\{t^{-b}\|\exp[-c_1tQ(y)]\|_1 + \|Q^{-l}(y)\|_1\}\exp\left[-c_1\frac{|x-y|^2}{t}\right] \leq \\
(3.15) \quad & \leq \text{const.}t^{1-b}\exp\left[-c_1\frac{|x-y|^2}{t}\right]\{\|\exp[-c_1tQ(y)]\|_1 + \|Q^{-l}(y)\|_1\}.
\end{aligned}$$

This time, let us estimate the expression B_2 in (3.12) using the inequalities (3.8) and (3.10).

$$\begin{aligned}
B_2 & = A_1 \int_0^t (t-\tau)^{-\frac{n}{2}}d\tau \int_{|y-\xi|>1} \exp\left[-c\frac{|x-\xi|^2}{t-\tau}\right] \\
& \quad \|\exp[-(t-\tau)Q(\xi)]\varphi(\xi, y, \tau)\|_1d\xi \\
& \leq A_1 \int_0^t (t-\tau)^{-\frac{n}{2}}d\tau \int_{|y-\xi|>1} \|\exp[-(t-\tau)Q(\xi)]\| \\
& \quad \exp\left[-c\frac{|x-\xi|^2}{t-\tau}\right]\|\varphi(\xi, y, \tau)\|_1d\xi \\
& \leq \text{const.}\|Q^{-l}(y)\|_1 \int_0^t \tau^{\frac{n}{2}}d\tau \int_{\mathbb{R}^n} \\
& \quad \exp\left[-c\frac{|x-\xi|^2}{t-\tau} - c\frac{|\xi-y|^2}{\tau}\right][\tau(t-\tau)]^{-\frac{n}{2}}d\xi \\
(3.16) \quad & \leq \text{const.}t\|Q^{-l}(y)\|_1\exp\left[-c_1\frac{|x-y|^2}{t}\right].
\end{aligned}$$

From (3.12), (3.15) and (3.16) we obtain

$$\begin{aligned} & \|G(x, y, t) - e^{-tQ(y)}G_1(x - y, y, t)\|_1 \leq \\ & \leq \text{const.} t^{1-b} \exp[-c_1 \frac{|x - y|^2}{t}] \{ \|\exp[-c_1 t Q(y)]\|_1 + \|Q^{-l}(y)\|_1 \} \end{aligned}$$

or

$$\begin{aligned} G(x, y, t) &= G_1(x - y, y, t) e^{-tQ(y)} + t^{1-b} \exp[-c_1 \frac{|x - y|^2}{t}] \cdot \\ & \{ \|\exp[-c_1 t Q(y)]\|_1 + \|Q^{-l}(y)\|_1 \} O(1). \end{aligned}$$

This completes the proof.

Now, we can prove that the spectrum of L is pure discrete and find an asymptotic formula for the number of its eigenvalues.

Theorem 3.3. If the coefficients of differential expression (1.1) satisfy the conditions 1) -7) then the spectrum of L is pure discrete.

Proof. The operator function $G_0(x - y, y, t)$ is in the form

$$G_0(x - y, y, t) = \frac{I}{(2\pi)^n} \int_{\mathbb{R}^n} \exp[-t \sum_{i,j=1}^n a_{ij}(y) s_i s_j + i(s, x - y)] ds,$$

where $I : H \rightarrow H$ is an identity operator. By translating $s_i = t^{-\frac{1}{2}} \xi_i$ we have

$$G_0(x - y, y, t) = \frac{I}{(2\pi\sqrt{t})^n} \int_{\mathbb{R}^n} \exp[-\sum_{i,j=1}^n a_{ij}(y) \xi_i \xi_j + it^{-\frac{1}{2}}(\xi, x - y)] d\xi.$$

From here, we find

(3.17)

$$G_0(0, x, t) = \frac{I}{(2\pi\sqrt{t})^n} \int_{\mathbb{R}^n} \exp[-\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j] d\xi = \frac{\Phi(x)}{(2\pi\sqrt{t})^n} I.$$

From the formulas (1.4), (3.17) and theorem 3.2 we obtain

(3.18)

$$\text{tr} G(x, x, t) = \frac{\Phi(x)}{(2\pi\sqrt{t})^n} \text{tr} e^{-tQ(x)} + O(1) t^{1-b} [\|e^{-ctQ(x)}\|_1 + \|Q^{-l}(x)\|_1].$$

By hypothesis, for every $x \in \mathbb{R}^n$, $Q(x) \geq I$ and hence

$$\|e^{-ctQ(x)}\|_1 = tre^{-ctQ(x)}.$$

On the other hand, since $Q(x)$ satisfies the conditions 5) -7) and from (3.18) we find

$$(3.19) \quad \int_{\mathbb{R}^n} trG(x, x, t)dx = \frac{1}{(2\pi\sqrt{t})^n} \int_{\mathbb{R}^n} \Phi(x)tre^{-tQ(x)}dx + O(1)t^{1-b}.$$

Since the expression $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j$ satisfies the condition 2) there exist positive constants M_1 and M_2 such that

$$(3.20) \quad M_1 < \Phi(x) < M_2.$$

Thus, from (3.19) we obtain

$$\int_{\mathbb{R}^n} trG(x, x, t)dx < \infty.$$

Since $u = e^{-tL}\psi(x)$ and $u = \int_{\mathbb{R}^n} G(x, y, t)\psi(y)dy$, $G(x, y, t)$ is the kernel of positive bounded operator e^{-tL} ($t > 0$). In this case, for every $t > 0$, e^{-tL} is the kernel operator, [8]. For the spectrums of operators e^{-tL} and L

$$s\{e^{-tL}\} = e^{-ts\{L\}}.$$

Since for every $t > 0$, e^{-tL} is a complete continuous operator and from the last formula we obtain that the spectrum of L is pure discrete.

Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$, be eigenvalues of L and let

$$N(\lambda) = \sum_{\lambda_n < \lambda} 1,$$

where $\lambda > 0$.

Theorem 3.4. If the coefficients of differential expression (1.1) satisfy the conditions 1)-8) then while $\lambda \rightarrow \infty$ the asymptotic formula

$$N(\lambda) \sim \frac{1}{(2\pi)^n \Gamma(\frac{n+2}{2})} \sum_i \int_{\alpha_i(x) < \lambda} \Phi(x)(\lambda - \alpha_i(x))^{\frac{n}{2}} dx$$

is satisfied.

Proof. The following equality holds:

$$(3.21) \quad \operatorname{tr} e^{-tL} = \int_0^\infty e^{-\lambda t} dN(\lambda).$$

From (3.19) and (3.20) while $t \rightarrow +0$ we obtain the asymptotic formula

$$(3.22) \quad \int_{\mathbb{R}^n} \operatorname{tr} G(x, x, t) dx = F(t) + o(1)F(t),$$

where

$$(3.23) \quad F(t) = \frac{1}{(2\pi\sqrt{t})^n} \int_{\mathbb{R}^n} \Phi(x) \operatorname{tr} e^{-tQ(x)} dx$$

and $o(1)$ is a function of t satisfying the condition $\lim_{t \rightarrow +0} o(1) = 0$. On the other hand, in [8] it is given

$$\operatorname{tr} e^{-tL} = \int_{\mathbb{R}^n} \operatorname{tr} G(x, x, t) dx.$$

From this last, (3.21) and (3.22) we find

$$(3.24) \quad \int_0^\infty e^{-\lambda t} dN(\lambda) = \int_{\mathbb{R}^n} \operatorname{tr} G(x, x, t) dx = F(t) + o(1)F(t).$$

Moreover,

$$\begin{aligned} F(t) &= \frac{1}{(2\pi\sqrt{t})^n} \int_{\mathbb{R}^n} \Phi(x) [\sum_{i=1}^\infty e^{-t\alpha_i(x)}] dx \\ &= \frac{1}{(2\pi\sqrt{t})^n} \sum_{i=1}^\infty \int_{\mathbb{R}^n} \Phi(x) e^{-t\alpha_i(x)} dx. \end{aligned}$$

On the other hand, in [11] it is shown that

$$\frac{1}{(2\pi\sqrt{t})^n} \int_{\mathbb{R}^n} \Phi(x) e^{-t\alpha_i(x)} dx = \int_0^\infty e^{-\lambda t} d\rho_i(\lambda),$$

where

$$\rho_i(\lambda) = \frac{1}{(2\pi)^n \Gamma(\frac{n+2}{2})} \int_{\alpha_i(x) < \lambda} \Phi(x) (\lambda - \alpha_i(x))^{\frac{n}{2}} dx.$$

Hence, from the last three equalities we obtain

$$(3.25) \quad F(t) = \int_0^\infty e^{-\lambda t} d\rho(\lambda),$$

where

$$\rho(\lambda) = \frac{1}{(2\pi)^n \Gamma(\frac{n+2}{2})} \sum_i \int_{\alpha_i(x) < \lambda} \Phi(x) (\lambda - \alpha_i(x))^{\frac{n}{2}} dx.$$

From (3.24) and (3.25), while $t \rightarrow +0$ we find

$$(3.26) \quad \int_0^\infty e^{-\lambda t} dN(\lambda) \sim \int_0^\infty e^{-\lambda t} d\rho(\lambda).$$

By hypothesis, we have

$$\lambda \rho'(\lambda) < a_0 \rho(\lambda)$$

Thus, from (3.26) and according to [12], while $\lambda \rightarrow \infty$ we obtain

$$N(\lambda) \sim \rho(\lambda)$$

or

$$N(\lambda) \sim \frac{1}{(2\pi)^n \Gamma(\frac{n+2}{2})} \sum_i \int_{\alpha_i(x) < \lambda} \Phi(x) (\lambda - \alpha_i(x))^{\frac{n}{2}} dx.$$

Example 3.1. Here we will give an example of scalar functions $a_{ij}(x)$ ($i, j = 1, 2, \dots, n$) and an operator function $Q(x)$ satisfying the conditions 1) – 8). Now, let

$$a_{ij}(x) = \begin{cases} 1 & , & i = j \\ 0 & , & i \neq j \end{cases} \quad (i, j = 1, 2, 3, \dots, n)$$

These functions obviously satisfy the conditions 1) – 2).

For an orthonormal basis $\{e_i\}_{i=1}^\infty$ of separable Hilbert space H , we define the operator function in the form

$$Q(x)f = \sum_{i=1}^\infty (f, e_i) i^4 (1 + |x|^{5/4}) e_i \quad , \quad (f \in D(Q(x)))$$

It can be easily shown that the operator function $Q(x)$ satisfies the conditions 3) – 8).

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