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A NONRESONANCE BETWEEN NON-CONSECUTIVE EIGENVALUES OF SEMILINEAR ELLIPTIC EQUATIONS : VARIATIONAL METHODS

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Abstract

We study the solvability of the problem

$$-\Delta u = f(x,u) + h \quad in\ \Omega\ ;\ u = 0 \qquad on\ \ \partial\Omega$$

when the nonlinearity f is assumed to lie asymptotically between two non- consecutive eigenvalues of $-\Delta$. We show that this problem is nonresonant.

Key words *: Eigenvalue, resonance, nonresonance, variational method.*

1. Introduction

In this paper, we will examine the existence of a solution of the problem:

$$(1.1)\textit{where} \qquad \begin{cases} -\Delta u &= f(x,u)+h \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Ω is a smooth bounded domain of ^{N}, $N \geq 2$, Δ denotes the laplacian $\Delta u = div(\nabla u)$, $f : \Omega \times \ \to$ is assumed to be a Carathéodory function such that

$$(f_0)\textit{and} \qquad m_R(x) = \sup_{|s|\leq R} |f(x,s)| \in L^2(\Omega) \qquad \text{for each } R > 0$$

h $\in L^2(\Omega)$.

We are interested in the conditions to be imposed on f and on the primitive F, $F(x,s) = \int_0^s f(x,t)\,dt$ in order to have the nonresonance i.e. the solvability of (1.1) for every h in $L^2(\Omega)$.

First we introduce some notations, the inequality

$\alpha(x) \preceq \beta(x)$ means that $\alpha(x) \leq \beta(x)$ for a.e. $x \in \Omega$ with a strict inequality $\alpha(x) < \beta(x)$ holding on subset of Ω of positive measure. $\lambda_i < \lambda_{i+1} < \lambda_{i+2}$ are the consecutive eigenvalues of the problem

$$-\Delta u = \lambda u \quad \text{in}\,\Omega\,;\, u = 0 \quad \text{on}\,\partial\Omega$$

$E(\lambda_i)$ is the subspace of $H_0^1(\Omega)$ spanned by the eigenfunctions corresponding to λ_i. $\|.\|$ denotes the norm in $H_0^1(\Omega)$ induced by the inner product $< u, v >= \int_\Omega \nabla u \nabla v$; $u, v \in H_0^1(\Omega)$.

Theorem 1.1. *Assume* (f_0) *and*

(H_1) $\quad \lambda_i \preceq l(x) = \liminf_{|s|\to\infty} \dfrac{f(x,s)}{s} \leq k(x) = \limsup_{|s|\to+\infty} \dfrac{f(x,s)}{s} \preceq \lambda_{i+2}$ *uniformly for*

a.e. $x \in \Omega$

(H_2) *for some* j *such that* $2 \leq j \leq i$, λ_j *is not simple i.e.* $dim E(\lambda_j) \geq 2$

(H_3) $\quad L(x) = \liminf_{|s|\to+\infty} \dfrac{2F(x,s)}{s^2} \succeq \lambda_{i+1}$ *uniformly for a.e.* $x \in \Omega$

then for any given $h \in L^2(\Omega)$, *there exists a weak solution of* (1.1).

Remark: We can replace (H_3), by one of the following conditions

1) $\lim\limits_{\|v\|\to+\infty, v\in\oplus_{1\le p\le i+1}E(\lambda_p)} \int_\Omega F(x,v(x)) - \frac{1}{2}\lambda_{i+1}v^2 = +\infty.$

2) $K(x) = \limsup\limits_{|s|\to+\infty} \frac{2F(x,s)}{s^2} \preceq \lambda_{i+1}.$

3) $\int_{v>0}(L_+(x)-\lambda_{i+1})v^2 + \int_{v<0}(L_-(x)-\lambda_{i+1})v^2 > 0$; $v \in E(\lambda_{i+1})$ and $v \neq 0$ and $L_\pm(x) = \liminf\limits_{s\to\pm\infty} \frac{2F(x,s)}{s^2} \geq \lambda_{i+1}.$

4) $\int_{v>0}(\lambda_{i+1}-K_+(x))v^2 + \int_{v<0}(\lambda_{i+1}-K_-(x))v^2 > 0$; $v \in E(\lambda_{i+1})$ and $v \neq 0$ and $K_\pm(x) = \limsup\limits_{s\to\pm\infty} \frac{2F(x,s)}{s^2} \leq \lambda_{i+1}$

these limits are taken uniformly for a.e. $x \in \Omega$

Corollary 1.1. *Assume* (f_0), (H_1), (H_2) *and*
(H_4) $F(x,s) \leq 0$ *for* $|s| \leq \delta$ $(\delta > 0)$
(H_5) $F(x,s) \leq As^2 + B$ $A,\ B \in$
then if $h = 0$ *the problem* (1.1) *possesses a nontrivial solution.*

Corollary 1.2. *Assume* (f_0), (H_1), (H_2) *and*
(H_6) $F(x,t+s) \geq F(x,t) + F(x,s) + B(x); B(.) \in L^1(\Omega)$
(H_7) $\lim\limits_{\|v\|\to+\infty; v\in E(\lambda_{i+1})} \int_\Omega F(x,v(x)) - \frac{1}{2}\lambda_{i+1}v^2\,dx = +\infty$
then the problem (1.1) *possesses a solution for any given h in* $E(\lambda_{i+1})^\perp$.

Where $E(\lambda_{i+1})^\perp = \{h \in L^2(\Omega) : \int_\Omega h\varphi = 0 \quad \forall \varphi \in E(\lambda_{i+1})\}$

This generalizes many of the existing results for doubly resonant problems. see e.g. [3]; [7]; [4]; ... Our approach to the problem (1.1) is variational and uses the well-known saddle point theorem of P. Rabinowitz.

2. Preliminary Lemmas

From the conditions (f_0) and (H_1), it follows that there exists constants $a,\ A > 0$ and functions $b(.) \in L^2(\Omega), B(.) \in L^1(\Omega)$ such that

$$|f(x,s)| \leq a|s| + b(x) \tag{1}$$

and

$$|F(x,s)| \leq A|s|^2 + B(x) \tag{2}$$

hence, the functional

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) - \int_\Omega hu$$

is well defined, lower semi-continuous and of class [1] on the Sobolev space $H_0^1(\Omega)$ with derivative $\Phi'(u)$ given by

$$\Phi'(u)w = \int_\Omega \nabla u \nabla w - \int_\Omega f(x,u)w - \int_\Omega hw.$$

for all $u, w \in H_0^1(\Omega)$, thus the critical points of Φ are precisely the weak solutions of (1.1).

Let $(u_n) \subset H_0^1(\Omega)$ be an unbounded sequence, such that

$$\Phi(u_n) \text{ is bounded and } \Phi'(u_n) \to 0. \tag{3}$$

Defining (v_n) by $v_n = \dfrac{u_n}{\|u_n\|}$, we have $\|v_n\| = 1$ and, passing to a subsequence (still denoted by (v_n)), we may assume

$$v_n \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega)$$

$$v_n \to v \quad \text{strongly in } L^2(\Omega)$$

$$v_n(x) \to v(x) \ a.e.\ x \in \Omega$$

$$|v_n| \leq z(x) \quad \text{where } z(.) \in L^2(\Omega).$$

Assuming (f_0) and (H_1), we obtain that the sequence $(\dfrac{f(x,u_n)}{\|u_n\|})$ is bounded in $L^2(\Omega)$, so we may assume that

$$\frac{f(x,u_n)}{\|u_n\|} \rightharpoonup \tilde{f} \quad \text{weakly in } L^2(\Omega). \tag{4}$$

An easy calculation (see [4]) shows that

$$l(x) \leq \frac{\tilde{f}(x)}{v(x)} \leq k(x) \quad \text{if } v(x) \neq 0 \tag{5}$$

and

$$\tilde{f}(x) = 0 \quad \text{if } v(x) = 0. \tag{6}$$

Let us define

$$m(x) = \begin{cases} \dfrac{\tilde{f}}{v(x)} & \text{if} \quad v(x) \neq 0 \\ \dfrac{1}{2}(l(x) + k(x)) & \text{if} \quad v(x) = 0. \end{cases}$$

Then by (5) and (6), we have

$$\tilde{f}(x) = m(x)v(x) \quad \text{and} \quad l(x) \leq m(x) \leq k(x) \tag{7}$$

Lemma 2.1. *Assume* (f_0) *and* (H_1), *then* v *is a nontrivial solution of the following problem*

$$-\Delta u = m(.)u \quad \textit{in } \Omega \ ; \ u = 0 \quad \textit{on } \partial\Omega \tag{1.2}$$

Proof. Using (3) we have $|\Phi'(u_n)w| \leq \varepsilon_n \|w\|$ for all $w \in H_0^1(\Omega)$, where $\varepsilon_n \to 0$, therefore

$$\frac{|\Phi'(u_n)u_n|}{\|u_n\|^2} = \left| 1 - \int_\Omega \frac{f(x,u_n)}{\|u_n\|} v_n - \frac{1}{\|u_n\|} \int_\Omega h v_n \right| \leq \frac{\varepsilon_n}{\|u_n\|}$$

hence, by (4) and the fact that $v_n \to v$ in $L^2(\Omega)$, we obtain $\int_\Omega \tilde{f} v = 1$, so that $v \neq 0$.
On the other hand, for any $w \in H_0^1(\Omega)$ we have

$$\frac{|\Phi'(u_n)w|}{\|u_n\|} = \left| \int_\Omega \nabla v_n \nabla w - \int_\Omega \frac{f(x,u_n)}{\|u_n\|} w - \frac{1}{\|u_n\|} \int_\Omega h w \right| \leq \frac{\varepsilon_n}{\|u_n\|} \|w\|$$

passing to the limit, we conclude

$$\int_\Omega \nabla v \nabla w - \int_\Omega \tilde{f} w = 0 \quad \forall w \in H_0^1(\Omega)$$

that is
$$\int_\Omega \nabla v \nabla w - \int_\Omega m(x) v w = 0 \quad \forall w \in H_0^1(\Omega)$$
in other words v is a weak solution of (1.2), moreover $v \neq 0$. So the proof of lemma 2.1 is complete.

Lemma 2.2. *Assume (f_0), (H_1) and (H_2) then the functional Φ satisfies the Palais-Smale condition (PS), that is whenever $(u_n) \subset H_0^1(\Omega)$ is a sequence such that $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$ then (u_n) possesses a convergent subsequence*

Proof : Let $(u_n) \subset H_0^1(\Omega)$ be such that $|\Phi(u_n)| \leq c$, $\Phi'(u_n) \to 0$. Since $\Phi'(u) = u - T(u)$ where T is a compact operator from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ $(T(u)w = \int_\Omega f(x,u)w - \int_\Omega hw)$, in order to show that (u_n) has a convergent subsequence it suffices to show that (u_n) is bounded. Suppose by contradiction that

$$\|u_n\| \to +\infty \tag{8}$$

Let $v_n = \dfrac{u_n}{\|u_n\|}$, then, as we observed in (7) and lemma 2.1 there exists a subsequence of (v_n) (still denoted by (v_n)) such that $v_n \rightharpoonup v$ in $H_0^1(\Omega)$, $v_n \to v$ strongly in $L^2(\Omega)$ and v is a nontrivial solution of the problem
$$-\Delta u = m(.)u \quad \text{in } \Omega \; ; \; u = 0 \quad \text{on } \partial\Omega$$
where
$$l(x) \leq m(x) \leq k(x)$$
so we conclude that

$$1 \in \sigma(-\Delta, m(.)) \tag{9}$$

and

$$\lambda_i \preceq m(.) \preceq \lambda_{i+2}. \tag{10}$$

On the other hand, by (10) and the strict monotonicity of $\lambda_i(m)$ we deduce

$$\lambda_i(\lambda_i(1)) > \lambda_i(m(.)) \text{ and } \lambda_{i+2}(m(.)) > \lambda_{i+2}(\lambda_{i+2}(1))$$

hence

$$1 > \lambda_i(m(.)) \text{ and } \lambda_{i+2}(m(.)) > 1$$

that is

$$\lambda_i(m(.)) < 1 < \lambda_{i+2}(m(.)). \tag{11}$$

it follows from (9) and (11) that

$$1 = \lambda_{i+1}(m(.)). \tag{12}$$

In view of the variational characterization of λ_{i+1} we have

$$1 = \sup_{F_{i+1}} \inf \left\{ \int_\Omega m(.)u^2 \ : \ \|u\| = 1 \ , u \in F_{i+1} \right\} \tag{13}$$

where F_{i+1} varies over all $i+1$- dimensional subspace of $H_0^1(\Omega)$.
On the other hand, we claim that there exists $\varepsilon > 0$ such that

$$\inf \left\{ \int_\Omega mu^2 \ : \ \|u\| = 1, \ u \in \oplus_{1\leq p\leq i} E(\lambda_p) \right\} \ \geq 1 + \varepsilon \tag{14}$$

Indeed, suppose (14) is false, then there exists a sequence (u_n) in $\oplus_{1\leq p\leq i} E(\lambda_p), \ \|u_n\| = 1$ and a sequence (ε_n) in $^{*+}$ such that

$$\varepsilon_n \to 0 \text{ and } \int_\Omega m(.)u_n^2 \ \leq 1 + \varepsilon_n. \tag{15}$$

Since $\|u_n\| = 1$ and $dim \oplus_{1\leq p\leq i} E(\lambda_p) < \infty$, we deduce

$$u_n \to u, \ \|u\| = 1 \text{ and } \|u\|^2 \leq \lambda_i \int_\Omega u^2. \tag{16}$$

Passing to the limit in (15) we conclude

$$\int_\Omega m(x)u^2 \, dx \leq 1 \tag{17}$$

combining (10), (16) and (17) we obtain

$$1 = \|u\|^2 = \lambda_i \int_\Omega u^2 = \int_\Omega mu^2$$

hence

$$u \in E(\lambda_i) \text{ and } \int_\Omega (m-\lambda_i)u^2 = 0. \tag{18}$$

Since $m(.) \geq \lambda_i$ and $u \in E(\lambda_i)$ satisfies the unique continuation principle, (18) implies that $m(.) = \lambda_i$ a.e. $x \in \Omega$, which contradicts (H_1) and shows that (15) can not occur. To complete the proof of lemma 2.2, let $F \subset \oplus_{1\leq p\leq i} E(\lambda_p)$ such that $dimF = i+1$ (it is possible by hypothesis (H_2)), it follows from (14) that

$$\inf\left\{\int_\Omega mu^2,\ \|u\| = 1,\ u \in F\right\} \geq 1+\varepsilon$$

which contradicts (13), so (8) can not occur and the proof of lemma 2.2 is complete.

Let us take the decomposition $H_0^1(\Omega) = V \oplus W$ where V is the subspace spanned by the eigenfunctions corresponding to $\lambda_j, j = 1, \ldots, i+1$, and $W = V^\perp$. It is easy to see that (H_1) implies

$$K(x) = \limsup_{|s|\to+\infty} \frac{2F(x,s)}{s^2} \preceq \lambda_{i+2} \tag{19}$$

Lemma 2.3. *Assume* $(f_0)\,,(H_1)$ *and* (H_3), *then we have*
i) $\displaystyle\lim_{\|v\|\to+\infty; v\in V} \Phi(v) = -\infty$
ii) $\displaystyle\lim_{\|w\|\to+\infty; w\in W} \Phi(w) = +\infty$

Proof. Combining (H_3) and (19) the above results follows.

3. Proof Of The Main Results

Proof of theorem 1.1. We can easily see that the functional Φ :

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u) - hu\ dx$$

is weakly lower semicontinuous. Therefore, since $\Phi_{|W}$ is coercive, (lemma2.3 ii)) the infinimum $\beta = \inf \Phi_{|W} > -\infty$ is attained. Taking

$\alpha < \beta$ by $i)$ of lemma 2.3 there exists $R > 0$ such that $\Phi(v) \leq \alpha$ for all $v \in V$ with $\|v\| \geq R$. Finally since Φ satisfies the Palais-Smale condition, (lemma 2.2) we can apply the saddle point theorem of P. Rabinowitz to conclude the existence of a critical point $u_0 \in H_0^1$ of Φ, so the proof is complete. ■

Proof of corollary 1.1. To show the corollary 1.1, it suffices to show :

$i)$ there exists $\rho > 0$ such that

$$\Phi(u) \geq \alpha > 0 \text{ if } \|u\| = \rho, u \in H_0^1(\Omega) \tag{20}$$

and $ii)$ By hypotheses (H_1) and (H_2) we deduce

$$\lim_{|t| \to +\infty} \Phi(t\varphi_1) = -\infty \tag{21}$$

φ_1 is a λ_1 normalized eigenfunction.

Combining (20), (21) and lemma 2.2 we can apply the Mountain-Pass theorem to conclude that Φ has a critical value $\Phi(u_0)$ with $\Phi(u_0) \geq \alpha > 0$.

Proof of corollary 1.2. It is easy to show from (H_1), (H_6) and (H_7) that

$$\lim_{\|v\| \to +\infty; v \in V} \Phi(v) = -\infty \tag{20}$$

On the other hand we have

$$\lim_{\|w\| \to +\infty,\, w \in W} \Phi(w) = +\infty \tag{21}$$

so by lemma 2.2 ,(20) and (21) we deduce the result.

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