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A MULTIPLIER GLIDING HUMP PROPERTY FOR SEQUENCE SPACES

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Abstract

We consider the Banach-Mackey property for pairs of vector spaces E and E' which are in duality. Let $\mathcal{A}$ be an algebra of sets and assume that P is an additive map from $\mathcal{A}$ into the projection operators on E. We define a continuous gliding hump property for the map P and show that pairs with this gliding hump property and another measure theoretic property are Banach-Mackey pairs,i.e., weakly bounded subsets of E are strongly bounded. Examples of vector valued function spaces, such as the space of Pettis integrable functions, which satisfy these conditions are given.

1. INTRODUCTION

H. Lebesgue introduced the gliding hump technique of proof to establish several uniform boundedness results for concrete function spaces such as L[0,1] ([L]). Subsequently, Schur and Hellinger/Toeplitz also used the gliding hump method to establish similar uniform boundedness principles for concrete function spaces ([Sc],[HT]). The early proofs of abstact uniform boundedness principles by Banach, Hahn and Hilldebrandt all employed gliding techniques ([B],[Ha],[Hi]). Absract gliding hump assumptions have been used to treat a number of topics in sequence spaces;for example, Noll used a "strong gliding hump" property to establish the weak sequential completeness of the beta dual of a sequence space ([N] ; see [BF] for a list of various gliding hump properties for sequence spaces). In this paper we introduce a gliding hump assumption involving multipliers from a scalar sequence space which is particularly useful in establishing uniform boundedness results for a vector-valued sequence space and its beta dual; in particular, our results establish Banach-Mackey properties for sequence spaces.

2. DEFINITIONS AND EXAMPLES

We begin with the notations and assumptions which will be used. Let X be a Hausdorff locally convex space and let E be a vector space of X-valued sequences containing $c_{00}(X)$, the space of all X-valued sequences which are eventually 0. We assume that E has a Hausdorff locally convex topology under which E is a K-space, i.e., the coordinate maps $x = \{x_k\} \to x_k$ from E into X are continuous for every k. An interval in $\mathbf{N}$ is a set of the form $[m, n] = \{k \in \mathbf{N} : \mathbf{m} \le \mathbf{k} \le \mathbf{n}\}$, where $m \le n$; a sequence of intervals $\{I_k\}$ is increasing if $\max I_k < \min I_{k+1}$ for every k. If I is an interval in $\mathbf{N}$ the characteristic function of I is denoted by χ_I , and if $x = \{x_k\}$ is an X-valued sequence, $\chi_I\, x$ denotes the coordinatewise product of χ_I and x.

Let λ be a vector space of scalar valued sequences which contains c_{00} the space of sequences which are eventually 0. The β-dual of λ, λ^β, is defined to be $\{t = \{t_k\} : \sum t_k s_k \ converges\ for\ every\ s = \{s_k\} \in \lambda\}$. If $s \in \lambda$ and $t \in \lambda^\beta$, we set $t \cdot s = \sum t_k s_k$; λ and λ^β are in duality with respect to the bilinear pairing $(s, t) \to s \cdot t$.

Definition 1. E has the strong λ gliding hump property (strong λ-GHP) if whenever $\{I_k\}$ is an increasing sequence of intervals and $\{x^k\}$ is a bounded sequence in E, then for every $t = \{t_k\} \in \lambda$ the coordinate sum of the series $\sum t_k \chi_{I_k} x^k$ belongs to E.

Definition 2. E has the weak λ gliding hump property (weak λ-GHP) if whenever $\{I_k\}$ is an increasing sequence of intervals and $\{x^k\}$ is a bounded sequence in E, there is a subsequence $\{n_k\}$ such that the coordinate sum $\sum t_k \chi_{I_{n_k}} x^k$ belongs to E for every $t \in \lambda$.

We refer to the elements of λ in Definitions 1 and 2 as multipliers since their coordinates multiply the blocks $\{\chi_{I_k}\}$ determined by $\{I_k\}$ and $\{x^k\}$. The weak $\lambda - GHP$ is like the strong gliding humps property introduced by Noll ([N]) where the multipliers consist only of the constant sequence $\{1\}$. After giving examples of spaces with λ-GHP we will make remarks comparing λ-GHP with other gliding hump properties.

We proceed to give an extensive list of examples of spaces with λ-GHP. The reader may want to skip ahead to section 3 where the main results are established and then refer back to the examples. For our first example we need a definition.

Definition 3. E satisfies the boundedness property (B) if for every increasing sequence of intervals $\{I_k\}$ and every bounded set $A \subset E$, the set $\{\chi_{I_k} x : k \in \mathbf{N}, \mathbf{x} \in \mathbf{A}\}$ is bounded in E.

For example, if $\mathcal{I}$ is the family of all intervals in $\mathbf{N}$ and the maps $\chi_I : E \to E, x \to \chi_I x, I \in \mathcal{I}$ are equicontinuous, then (B) holds. This is the case if $p(\chi_I x) \leq p(x)$ holds for every $I \in \mathcal{I}, \S \in \mathcal{E}$ and continuous seminorm p on E.

Proposition 4. If E is a locally complete space with property (B), then E has strong $l^1 - GHP$.

Proof: Let $\{I_k\}$ be an increasing sequence of intervals and $\{x^k\} \subset E$ be bounded. By (B) $\{\chi_{I_k} x^k : k\}$ is bounded so if $t = \{t_k\} \in l^1$, the series $\sum_{k=1}^{\infty} t_k \chi_{I_k} x^k$ is absolutely convergent in E and, therefore, converges to an element $x \in E$ by local completeness. Since X is a $K - space$, x is also the coordinate sum of the series.

Proposition 4 gives a large supply of spaces with $l^1 - GHP$. We also have

Example 5. l^∞ and c_0 have strong $c_0 - GHP$; l^p has strong $l^p - GHP$ for $0 < p \leq \infty$.

We now give examples of non-complete scalar sequence spaces with weak $l^p - GHP$.

Example 6. Let $1 \leq p < \infty$. Let $\mathbf{P}$ be the power set of $\mathbf{N}$ and let $\mu : \mathbf{P} \to [\mathbf{0}, \infty)$ be a finitely additive set function with $\mu(\{j\}) > 0$ for every j. Put $l^p(\mu) = L^p(\mu)$, the space of all pth power μ-integrable functions with the norm $\|f\|_p = (\int_N \mid f \mid^p d\mu)^{1/p}$ [see [RR] for details on the integration with repect to finitely additive set functions; the assumption $\mu(\{j\}) > 0$ for every j makes $l^p(\mu)$ a $K - space$]. We show that $l^p(\mu)$ has weak $l^p - GHP$. Let $\{I_k\}$ be an increasing sequence and $\{f_k\} \subset l^p(\mu)$ be bounded with $\|f_k\|_p \leq 1$. By Drewnowski's Lemma ([Dr],[Sw2]2.2.3) , there is a subsequence $\{n_k\}$ such that μ is countably additive on the σ-algebra generated by $\{I_{n_k}\}$. Suppose that $t \in l^p$. Put $f = \sum_{k=1}^{\infty} t_k \chi_{I_{n_k}} f_{n_k}$ [coordinatewise]. We claim that $f \in l^p(\mu)$ and the series converges to f in $l^p(\mu)$ by using Theorem 4.6.10 of [RR]. Put $s_n = \sum_{k=1}^{n} t_k \chi_{I_{n_k}} f_{n_k}$ and note that $s_n \to f$ μ-hazily [μ-measure] since if $\epsilon > 0$,

$$\mu(\{j :\mid s_n(j) - f(j) \mid \geq \epsilon\}) \leq \mu(\cup_{j=n+1}^{\infty} I_{n_j}) = \sum_{j=n+1}^{\infty} \mu(I_{n_j}) \to 0$$

by countable additivity. Next, $\{s_n\}$ is Cauchy in $l^p(\mu)$ since

$$\|s_n - s_m\|_p^p = \| \sum_{j=m}^{n} t_j \chi_{I_{n_j}} f_{n_j} \|_p^p \leq \sum_{j=m}^{n} \mid t_j \mid^p \to 0.$$

It follows that $\{\int \mid f_{n_j} \mid^p d\mu : j\}$ is uniformly μ-continuous. The claim is thus justified, and it follows that $l^p(\mu)$ has weak $l^p - GHP$.

Problem. Does $l^p(\mu)$ have strong l^p?

We next give examples of vector-valued sequence spaces with $\lambda - GHP$. Let $\mathcal{X}$ be a family of semi-norms which generate the topology of X . Let μ be a normal (scalar) K-space whose topology is generated by the family of semi-norms $\mathcal{M}$. If $t = \{t_k\} \in \mu$, we set $\mid t \mid = \{\mid t_k \mid\}$. We make the following assumptions on μ:

(*) If $A \subset \mu$ is bounded, then $\mid A \mid = \{\mid t \mid : t \in A\}$ is bounded in μ.

(**) If $s, t \in \mu$ with $\mid s \mid \leq \mid t \mid$ and if $q \in \mathcal{M}$, then $q(s) \leq q(t)$.

These assumptions are satisfied by many of the classical sequence spaces.

We define $\mu\{X\}$ to be the space of all $X - valued$ sequences $x = \{x_k\}$ such that $\{p(x_k)\} \in \mu$ for every $p \in \mathcal{X}$. Since μ is normal, $\mu\{X\}$ is a vector space. We assume that $\mu\{X\}$ has the locally convex topology generated by the semi-norms

$$\pi_{q,p}(\{x_k\}) = q(\{p(x_k)\}), p \in \mathcal{X}, \text{II} \in \mathcal{M}. \tag{1}$$

Spaces of this type were considered in [FP] and [F].

The spaces $l^p\{X\}$ and $c_0\{X\}$ are the usual spaces of pth power convergent and null sequences, respectively. As in Example 5 it is easily seen that $l^\infty\{X\}$ and $c_0\{X\}$ have strong $c_0 - GHP$ and $l^p\{X\}$ has strong $l^p - GHP$. More generally. we have

Proposition 7. If μ has strong $\lambda - GHP$, then $\mu\{X\}$ has strong $\lambda - GHP$.

Proof: Let $\{I_k\}$ be an increasing sequence of intervals and $\{x^k\} \subset \mu\{X\}$ be bounded. Let $t \in \lambda$ and put $x = \sum_{k=1}^{\infty} t_k \chi_{I_k} x^k$ {coordinatewise} . Let $p \in \mathcal{X}$ and note $p(x(\cdot)) = \sum_{k=1}^{\infty} \mid t_k \mid \chi_{I_k} p(x^k(\cdot))$,where $x(\cdot)$ is the function $j \to x_j$. Now $\{\{p(x_j^k)\}_{j=1}^{\infty} : k\}$ is bounded in μ by the definition in (1). By strong $\lambda - GHP$, $\{p(x_j)\} \in \mu$,i.e., $x \in \mu\{X\}$.

Proposition 8. If μ has weak $\lambda - GHP$ and X is normed, then $\mu\{X\}$ has weak $\lambda - GHP$.

Proof: Continue the notation from Proposition 7 and let $\| \;\; \|$ be the norm on X. For every k $\{\|x_j^k\|\}_{j=1}^{\infty} \in \mu$ and $\{\{\|x_j^k\|\}_j : k\}$

is bounded in μ so by weak $\lambda - GHP$ there is a subsequence $\{n_k\}$ such that $\sum_{k=1}^{\infty} t_k \chi_{I_{n_k}} \|x^{n_k}(\cdot)\| = s \in \mu$ for every $t \in \lambda$. Therefore, $x = \sum_{k=1}^{\infty} t_k \chi_{I_{n_k}} x^{n_k} \in \mu\{X\}$.

Propositions 7 and 8 give a large supply of spaces with $\lambda - GHP$ many of which are not sequentially complete [e.g., $l^p\{X\}$or $c_0\{X\}$].

We now give other examples of (non-monotone) vector-valued sequence spaces.

Example 9. Let $CS(X)$ be the space of all X-valued sequences $\{x_k\}$ such that the series $\sum x_k$ is Cauchy in X. If X is the scalar field, $CS(X)$ is the space cs of convergent series. We define a topology on $CS(X)$ induced by the semi-norms $p'(\{x_k\}) = \sup\{p(\sum_{j\in I} x_j) : I \in \mathcal{I}\}$, $p \in \mathcal{X}$.

We claim that $CS(X)$ has strong $l^1 - GHP$. Suppose $\{I_k\}$ is increasing and $\{x^k\} \subset CS(X)$ is bounded. If $t \in l^1$, put $x = \sum_{k=1}^{\infty} t_k \chi_{I_k} x^k$ [coordinatewise]. Let $\varepsilon > 0, p \in X$ and set $M = \sup\{p(\sum_{j\in I} x_j^k) : I \in \mathcal{I}, \|\}$. Pick N such that $\sum_{k=N}^{\infty} \mid t_k \mid < \varepsilon$. Suppose $I \in \mathcal{I}$ and $\min I > N$. Then

$$p(\sum_{j\in I} x_j) \leq \sum_{k=N}^{\infty} \mid t_k \mid M \leq M\varepsilon$$

so $x \in CS(X)$.

Example 10. Let $BS(X)$ be all X-valued sequences $\{x_k\}$ such that the partial sums $\{\sum_{k=1}^{n} x_k\}$ are bounded. If X is the scalar field, $BS(X)$ is the space of bounded series bs. As above define a topology on $BS(X)$ by the semi-norms $p'(\{x_k\}) = \sup\{p(\sum_{j\in I} x_j : I \in \mathcal{I}\}$, $p \in \mathcal{X}$. It is easily checked that $BS(X)$ has strong $l^1 - GHP$.

Example 11. Let $BV(X)$ be all $X-$valued sequences $\{x_k\}$ such that the series $\sum_{i=1}^{\infty}(x_{i+1} - x_i)$ is absolutely convergent in X, i.e., $\{x_{i+1} - x_i\} \in l^1\{X\}$. If X is the scalar field $BV(X)$ is the space bv of sequences of bounded variation. If $p \in \mathcal{X}$, we define a semi-norm $p'(\{x_k\}) = \sum_{i=1}^{\infty} p(x_{i+1} - x_i) + \lim p(x_i)$ and topologize $BV(X)$ by the semi-norms $\{p' : p \in \mathcal{X}\}$.

We show that $BV(X)$ has strong $l^1 - GHP$. First note that if $x \in BV(X)$, then $\sup\{p(x_i) : i\} \leq p'(x)$ for $p \in X$ [for $n > m, x_m =$

$\sum_{k=m}^{n}(x_k - x_{k+1}) + x_{n+1}]$, so if $I \in \mathcal{I}, p'(\chi_I x) \leq p'(x) + 2\sup_i p(x_i) \leq 3p'(x)$. If $\{I_k\}$ is increasing, $\{x^k\} \subset BV(X)$ is bounded, $t \in l^1$ and we set $x = \sum_{k=1}^{\infty} t_k \chi_{I_k} x^k$, we have $\sum_{k=1}^{\infty} p(x_{k+1} - x_k) \leq \sum_{k=1}^{\infty} \mid t_k \mid 3p'(x^k) < \infty$ so $x \in BV(X)$.

As noted earlier the weak $\lambda - GHP$ resembles the strong gliding hump property introduced by Noll where the mutipliers consist of the single constant sequence $\{1\}$ ([N]). A weaker gliding hump property is the $zero{-}GHP$; E has $zero{-}GHP$ if $x^k \to 0$ in E and $\{I_k\}$ increasing implies there exists a subsequence $\{n_k\}$ such that $x = \sum_{k=1}^{\infty} \chi_{I_{n_k}} x^{n_k} \in E$ ([Sw3] 12.5). We give an example of a space with $l^1 - GHP$ but without $zero - GHP$.

Example 12. Let E be l^2 with the weak topology. Since E is sequentially complete, E has strong $l^1 - GHP$ by Proposition 4. However, E fails to have $zero - GHP$ [consider $\{k\}$ and $\{e^k\}$].

Problem. Does $zero - GHP$ imply $l^1 - GHP$?

3. MAIN RESULTS

We now prove several uniform boundedness results for spaces with weak $\lambda - GHP$. The (scalar) $\beta - dual$ of E is defined to be $E^\beta = \{\{y_k\} : y_k \in X', \sum_{k=1}^{\infty} \langle y_k, x_k \rangle \textit{ converges for every } x = \{x_k\} \in E\}$. If $x = \{x_k\} \in E$ and $y = \{y_k\} \in E^\beta$, we write $y \cdot x = \sum_{k=1}^{\infty} \langle y_k, x_k \rangle$; E and E^β are then in duality with respect to the bilinear pairing $(x, y) \to y \cdot x$.

If Z and Z' are two vector spaces in duality, we denote the weak (strong) topology of Z with resect to this duality by $\sigma(Z, Z')(\beta(Z, Z'))$. Recall that the pair Z, Z' is a Banach-Mackey pair if $\sigma(Z, Z')$ bounded sets in Z are $\beta(Z, Z')$ bounded, and X is a Banach-Mackey space if X, X' is a Banach-Mackey pair ([Wi] 10.4).

We begin with a basic lemma. If $A \subset E$ and $B \subset E^\beta$, we write $\mid B \cdot A \mid = \sup\{\mid y \cdot x \mid : y \in B, x \in A\}$.

Lemma 1. Let X be a Banach-Mackey space. Suppose $A \subset E$ is coordinatewise bounded and $B \subset E^\beta$ has coordinates which are $\sigma(X', X)$ bounded. If $\mid B \cdot A \mid = \infty$, then there exists an increasing sequence of intervals $\{I_k\}, \{x^k\} \subset A$ and $\{y^k\} \subset B$ such that $\mid y^k \cdot \chi_{I_k} x^k \mid > k^2$.

Proof: There exist $y^k \in B, x^k \in A$ such that $\mid y^k \cdot x^k \mid > k^2 + 1$. Set $k_1 = 1$. There exists n_1 such that $\mid \sum_{j=1}^{n_1} \langle y_j^{k_1}, x_j^{k_1} \rangle \mid > k_1^2 + 1$. For every j $\{x_j^k : k\}$ is bounded in X by hypothesis and $\{y_j^k : k\}$ is $\sigma(X', X)$ bounded since B has $\sigma(X', X)$ bounded coordinates. Since X is Banach-Mackey, $\{\langle y_j^k, x_j^k \rangle : k\}$ is bounded for every j so $\lim_k \frac{1}{k} \langle y_j^k, x_j^k \rangle = 0$. Hence, there exists $k_2 > k_1$ such that $\sum_{j=1}^{n_1} \mid \langle y_j^{k_2}, x_j^{k_2} \rangle \mid < 1$. Then $\mid \sum_{j=n_1+1}^{\infty} \langle y_j^{k_2}, x_j^{k_2} \rangle \mid > k_2^2$. Pick $n_2 > n_1$ such that $\mid \sum_{j=n_1+1}^{n_2} \langle y_j^{k_2}, x_j^{k_2} \rangle \mid > k_2^2$ and set $I_2 = [n_1+1, n_2]$ so $\mid y^{k_2} \cdot \chi_{I_{k_2}} x^{k_2} \mid > k_2^2$. Now just continue this construction and relabel.

We now establish our first uniform boundedness result for E and its β-dual. In what follows e^k is the canonical vector with a 1 in the kth coordinate and 0 in the other coordinates.

Theorem 2. Let X be a Banach-Mackey space and suppose that E has weak $\lambda - GHP$. Assume

$$\{e^k : k\} \textit{ is } \beta(\lambda, \lambda^\beta) \textit{ bounded in } \lambda. \tag{2}$$

If $A \subset E$ is bounded and $B \subset E^\beta$ is $\sigma(E^\beta, E)$ bounded, then $\mid B \cdot A \mid < \infty$.

Proof: If the conclusion fails, Lemma 1 applies. Let the notation be as in Lemma 1 and let $\{n_j\}$ be the subsequence in the definition of the weak $\lambda - GHP$. Define a linear operator $T : \lambda \to E$ by $Tt = \sum_{j=1}^{\infty} t_j \chi_{I_{n_j}} x^{n_j}$ [coordinatewise sum].

We claim that T is $\sigma(\lambda, \lambda^\beta) - \sigma(E, E^\beta)$ continuous. For this let $t \in \lambda, y \in E^\beta$. Then $y \cdot Tt = \sum_{j=1}^{\infty} t_j (y \cdot \chi_{I_{n_j}} x^{n_j})$ and since this series converges for every $t \in \lambda$, $\{y \cdot \chi_{I_{n_j}} x^{n_j}\}$ belongs to λ^β and $y \cdot Tt = \{y \cdot \chi_{I_{n_j}} x^{n_j}\} \cdot t$ which implies that T is $\sigma(\lambda, \lambda^\beta) - \sigma(E, E^\beta)$ continuous. Hence, T is also $\beta(\lambda, \lambda^\beta) - \beta(E, E^\beta)$ continuous ([Wi] 11.2.6,[Sw1] 26.15). Thus, by hypothesis, $\{Te^k\} = \{\chi_{I_{n_k}} x^{n_k}\}$ is $\beta(E, E^\beta)$ bounded. But this contradicts the conclusion of Lemma 1.

A similar uniform boundedness result for spaces with $zero-GHP$ is given in [Sw3] 12.5.7.

Corollary 3. Under the hypothesis of Theorem 2 if $E' \subset E^{\beta}$, then E is a Banach-Mackey space.

We have a general criterion for the hypothesis in Corollary 3 to hold. If $z \in X$, we define $e^k \otimes z$ to be the sequence with z in the kth coordinate and 0 in the other coordinates. We say that E is an AK-space if the series $\sum_{k=1}^{\infty} e^k \otimes x_k$ converges to $x = \{x_k\} \in E$ in the topology of E for all x.

Proposition 4. Assume that the map $z \to e^k \otimes z$ from X into E is continuous for every k. If E is an AK-space, then $E' \subset E^{\beta}$.

Proof: Let $f \in E'$. For every k define $y_k : X \to \mathbf{R}$ by $\langle y_k, z \rangle = \left\langle f, e^k \otimes z \right\rangle$. Then $y_k \in X'$ by hypothesis, and if $x \in E$, $\langle f, x \rangle = \langle f, \sum_{k=1}^{\infty} e^k \otimes z \rangle = \sum_{k=1}^{\infty} \langle y_k, x_k \rangle$ so $y \in E^{\beta}$ and $\langle f, x \rangle = y \cdot x$.

Example 5. $CS(X)$ is an AK-space so it follows from Proposition 4, Corollary 3 and Example 2.9 that $CS(X)$ is a Banach-Mackey space when X is a Banach-Mackey space.

For the vector-valued sequence spaces $\mu\{X\}$, we have

Example 6. It is easily checked that $\mu\{X\}$ is an AK-space when μ is an AK-space. If

(3) X is a Banach-Mackey space and either μhas strong $\lambda - GHP$ or μ has weak $\lambda - GHP$ and X is normed,

(2) holds and μ is anAK-space, then $\mu\{X\}$ is a Banach-Mackey space [Proposition 4, Corollary 3 and Propositions 2.7 or 2.8].

In particular, $c_0\{X\}$ is a Banach-Mackey space when X is a Banach-Mackey space; this was established by Mendoza ([M]). It also follows that $l^p\{X\}$ is a Banach-Mackey space for $1 \leq p < \infty$; Fourie has given a general criterion for spaces of the type $\mu\{X\}$ to be Banach-Mackey spaces ([F] 3.7) but his result does not cover $l^1\{X\}$.

We also have a general uniform boundedness result for the spaces $\mu\{X\}$ and their β-duals.

Corollary 7. Assume (3). If $A \subset \mu\{X\}$ is bounded and $B \subset \mu\{X\}^\beta$ is $\sigma(\mu\{X\}^\beta, \mu\{X\})$ bounded, then $| B \cdot A |< \infty$.

We consider conditions which guarantee that E, E^β form a Banach-Mackey pair and then consider specific examples. From Theorem 2, we obtain

Corollary 8. Assume that X is a Banach-Mackey space, E has weak $\lambda - GHP$ and (2) holds. If E is such that $\sigma(E, E^\beta)$ bounded sets are bounded in the topology of E, then E, E^β is a Banach-Mackey pair.

Example 9. The space $l^\infty\{X\}$ satisfies the boundedness criterion in Corollary 8. For suppose $A \subset l^\infty\{X\}$ is $\sigma(l^\infty\{X\}, l^\infty\{X\}^\beta)$ bounded. For $t \in l^1, x' \in X'$ define $t \otimes x' \in l^\infty\{X\}^\beta$ by $t \otimes x' \cdot x = \sum_{k=1}^\infty t_k \langle x', x_k \rangle$. Then $\sup\{|t \otimes x' \cdot x| : x \in A\} < \infty$. Thus, $\{\{\langle x', x_k \rangle : x \in A, k\} \subset l^\infty$ is $\sigma(l^\infty, l^1)$ bounded and, therefore, norm bounded in l^∞. Hence, $\sup\{| \langle x', x_k \rangle |: x \in A, k\} < \infty$ and $\{x_k : x \in A, k\}$ is bounded in X or A is bounded in $l^\infty\{X\}$.From Corollary 8 and Proposition 7, it follows that $l^\infty\{X\}, l^\infty\{X\}^\beta$ is a Banach-Mackey pair when X is a Banach-Mackey space [the β-dual of $l^\infty\{X\}$ is described in [GKR] 2.6].

Similarly, $c_0\{X\}, c_0\{X\}^\beta$ is a Banach-Mackey pair.

When E is a monotone space [or more generally when E has the signed weak GHP] and X' is weak* sequentially complete, then $(E^\beta, \sigma(E^\beta, E))$ is sequentially complete so E, E^β form a Banach-Mackey pair ([Sw3] 12.4.1,[Wi] 10.4). This result applies to $l^\infty\{X\}$ and $c_0\{X\}$ when X' is weak* sequentially complete; however, our assumption on X being a Banach-Mackey space is weaker.

We show that the (non-monotone) space $BS(X)$ satisfies the boundedness criterion of Corollary 8. For this we require a description of the β-dual of $BS(X)$. Let X'_b be the dual of X equipped with the strong topology and let $BV_0(X)$ be the subspace of $BV(X)$ consisting of the null sequences.

Proposition 10. $BS(X)^\beta = BV_0(X'_b)$.

Proof: Let $y \in BS(X)^\beta$. To show that $y_k \to 0$ strongly, it suffices to show that $\langle y_k, x_k \rangle \to 0$ for every bounded sequence $\{x_k\} \subset X$. If $x_0 = 0$, then $\{x_k - x_{k-1}\} \in BS(X)$ so $\sum_{k=1}^\infty \langle y_k, x_k - x_{k-1} \rangle$ converges and we have that $\lim_k \langle y_k, x_k - x_{k-1} \rangle = 0$ for every bounded sequence $\{x_k\}$. This implies that $\lim_k \langle y_k, x_k \rangle = 0$ for every bounded sequence [Define a bounded sequence $\{z_j\}$ by $0, x_1, 0, x_3, 0 ...$; then the sequence $\{\langle y_j, z_{j+1} - z_j \rangle\}$ contains the sequence $\{\langle y_{2j+1}, x_{2j+1} \rangle\}$ as a subsequence so $\lim_j \langle y_{2j+1}, x_{2j+1} \rangle = 0$. Similarly, $\lim_j \langle y_{2j}, x_{2j} \rangle = 0$ so $\lim_j \langle y_j, x_j \rangle = 0$.]. Thus, $y \in c_0\{X'_b\}$.

Put $w_k = x_{k+1} - x_k$ so $\{w_k\} \in BS(X)$ and $\sum_{k=1}^\infty \langle y_k, w_k \rangle$ converges. Now

$$\sum_{i=1}^{n} \langle y_i, w_i \rangle = \sum_{i=1}^{n} \langle y_i, x_{i+1} - x_i \rangle = \sum_{i=1}^{n-1} \langle y_i - y_{i+1}, x_i \rangle - \langle y_n, x_n \rangle. \tag{4}$$

By the above $\langle y_n, x_n \rangle \to 0$ so $\sum_{i=1}^\infty \langle y_i - y_{i+1}, x_i \rangle$ converges for every bounded $\{x_k\}$ by (4). Hence, $\sum_{i=1}^\infty (y_i - y_{i+1})$ is absolutely convergent in X'_b, i.e.,$y \in BV_0(X'_b)$.

Next, let $y \in BV_0(X'_b)$ and $x \in BS(X)$. $\{s_i = \sum_{j=1}^i x_j\}$ is bounded so $\sum_{i=1}^\infty \langle y_{i+1} - y_i, s_i \rangle$ converges absolutely. Now

$$\sum_{i=1}^{n} \langle y_i, x_i \rangle = \sum_{i=1}^{n-1} \langle y_i - y_{i+1}, s_i \rangle + \langle y_n, s_n \rangle. \tag{5}$$

$\langle y_n, s_n \rangle \to 0$ since $y_n \to 0$ strongly so (5) implies that $\sum_{i=1}^\infty \langle y_i, x_i \rangle$ coverges. That is, $y \in BS(X)$.

Proposition 11. If $A \subset BS(X)$ is $\sigma(BS(X), BS(X)^\beta)$ bounded, then A is bounded in $BS(X)$.

Proof: For $t \in bv_0$ and $x' \in X'$ define $tx' \in BV_0(X'_b)$ by $(tx')_k = t_k x'$. If $x \in A$,

$$tx' \cdot x = \sum_{j=1}^{\infty} t_j \langle x', x_j \rangle. \tag{6}$$

Since $\{\langle x', x_j \rangle\} \in bs$, (6) implies $\{\{\langle x', x_j \rangle\} : x \in A\}$ is $\sigma(bs, bv_o)$ bounded and, therefore, bounded in bs ([KG] p.69). Therefore, $\{\sum_{j=1}^{n} \langle x', x_j \rangle : x \in A, n\}$ is bounded. Hence, $\{\sum_{j=1}^{n} x_j : x \in A, n\}$ is $\sigma(X, X')$ bounded and, therefore, bounded in X. That is, A is bounded in $BS(X)$.

From Corollary 8 and Example 10, we have

Example 12. If X is a Banach-Mackey space, then $BS(X), BS(X)^\beta$ is a Banach-Mackey pair.

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