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SEPARATION PROBLEM FOR STURM-LIOUVILLE EQUATION WITH OPERATOR COEFFICIENT

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Abstract

Let H be a separable Hilbert Space. Denote by $H_1 = L_2(a,b;H)$ the set of function defned on the interval $a < x < b$ $(-\infty \leq a < x < b \leq \infty)$ whose values belong to H strongly measurable [12] and satisfying the condition

$$\int_a^b ||f(x)||_H^2 dx < \infty$$

If the inner product of function $f(x)$ and $g(x)$ belonging to H_1 is defined by

$$(f,g)_1 = \int_a^b (f(x), g(x))_H dx$$

then H_1 forms a separable Hilbert space . We study separation problem for the operator formed by $-y'' + Q(x)y$ Sturm-Liouville differential expression in $L_2(-\infty,\infty;H)$ space has been proved where $Q(x)$ is an operator which transforms at H in value of x,self-adjoint,lower bounded and its inverse is complete continous.

1. Introduction

Let us consider

$$-y^{''} + Q(x)y \tag{1}$$

differential expression in $H_1 = L_2(-\infty, \infty; H)$. Let us assume that $Q(x)$, satisfies the following conditions as a self-adjoint operator making transformation at H in each value obtained $(-\infty, \infty)$ interval of x

1) $Q(x)$ operator family have the same definition set indepent of x $(-\infty < x < \infty)$ Let us show this set by $\mathcal{D}$.

2) Let $Q(x) \geq I$ and for $\forall f \in \mathcal{D}$ $Q(x)f$ be a strong countinous function in $(-\infty, \infty)$ and $Q^{-1}(x)$ is completely continous in H for $\forall f \in (-\infty, \infty)$

3) When $|x - y| \leq 2$ let

$$\| \left(Q(x) - Q(y)\right) Q^{-1}(y) \| < \delta$$

$\delta > 0$ is any number.

Let us form L_0 operator by (2) expression.Let the definition set $\mathcal{D}(L_0)$ of L_0 be by the functions $y(x)$ satisfying the following conditions:

1) Let $y(x)$ having compact support in $(-\infty, \infty)$,$Q(x)y(x)$,$y^{''}(x)$ be continious.

2) Let $-y''(x) + Q(x)y(x) \in L_2(-\infty, \infty; H)$.

We have formed

$$L_0 y = -y'' + Q(x)y \quad , \quad y \in \mathcal{D}(L_0)$$

Since H which is the definition set of $Q(x)$ is dense almost everywhere and since $Q(x)f$ is continuous at $(-\infty, \infty)$ $\forall f \in \mathcal{D}$ definition set $\mathcal{D}(L_0)$ of L_0 operator forms a dense linear monifold almost everywhere of the space $L_2(-\infty, \infty; H)$. L_0 is a symmetric operator bounded from below in $L_2(-\infty, \infty; H)$ Let us assume that L which is a closure of L_0 is a self-adjoint operator.

In this work we study the seperability of the operator L. According to the definition of seperability when $y(x)$ is any function belonging to $\mathcal{D}(L)$ we will show that $y''(x)$ and $Q(x)y(x)$ functions also belong to $L_2(-\infty,\infty;H)$ space.

Let us show the resolvent of L operator in reguler λ value (λ is complex number) by $R_\lambda = (L-\lambda I)^{-1}$. According to the definition of Resolvent, R_λ operator is bounded operator in H_1 space.

Lemma 1 : [9] If $Q(x)R_\lambda$ operator is bounded in H_1, then L operator is seperable in H_1.

Proof: Let $y(x)$ be an arbitrary element belonging to $\mathcal{D}(L)$ then $(L-\lambda I)y = f \quad f \in H_1$. We can write $y = R_\lambda f$ and $Q(x)y = Q(x)R_\lambda f$ Since $Q(x)R_\lambda$ is a bounded operator, $Q(x)R_\lambda f \in H_1$ i.e $Q(x)y \in H_1$•

Many books, by B. M. Levitan and I. S. Sargsyan [8], E. C. Titcmarch [11], M. Otelbayev [10], and papers by T.C.Fulton and S.A.Pruess [6] belonging to singular Sturm-Liouville problem have been written. English mathematicians W. N. Everitt and M. A. Giertz [3], [4], [5] have proved that they introduced separation definition for operator L consisting of expression (1) being real valued function $Q(x)$ with series papers and studies, and separation theorem of $Q(x)$ for operator L in various conditions. For the separability problem,there are works by M. Bayramoglu and A. Abudov [1], K. Boymatov [2], A. Izmaylov and M. Otelbayev [13], M. Otelbayev [14] and many mathematicians; it has taken big place in the book [9] by the K. Minbayev and M. Otelbayev and given many references in the book.

Localization method for seperability of L operator. This method was firstly used by R.Ismagilov [7], and were developed by M.Otelbayev [10]. We use that developed method.

Let us show the operator L_j ,formed by the following differential expression

$$-y'' + Q(x)y \quad , \quad j-1 < x < j+1 \tag{2}$$

and

$$y\,(j-1) = y\,(j+1) = 0 \qquad j = 0, \pm 1, \pm 2, ... \textbf{ are integers} \tag{3}$$

boundary conditions in space $L_2(j-1, j+1; H)$. Each L_j operator is a positive defined self-adjoint operator.

Let $w(x)$ be a function satisfying the following properties and differentiable defined in $(-\infty, \infty)$

Let

$$w(x) = \begin{cases} 1 & |x| \le 1 \\ \\ 0 & |x| \ge 1.5 \end{cases}$$

and $\varphi_j(x) = w(x-j)$.

Let us show

$$\psi_j(x) = \begin{cases} \frac{1}{2} & |x-j| \le 1 \\ \\ 0 & |x-j| > 1 \end{cases}$$

It is seen easily that

$$\sum_{j=-\infty}^{\infty} \psi_j(x) = 1 \tag{4}$$

Let $f \in H_1$. $\lambda > 0$ let us show

$$M_\lambda f = \sum_{j=-\infty}^{\infty} \varphi_j (L_j + \lambda I)^{-1} \psi_j f$$

operator by M_λ.

$$(L + \lambda I) M_\lambda f = \sum_{j=-\infty}^{\infty} (L + \lambda I) \varphi_j (L_j + \lambda I)^{-1} \psi_j f \tag{5}$$

Since $L + \lambda I$ and $L_j + \lambda I$ operators coincide in the interval $(j - 1.5, j + 1.5)$ and $\varphi_j(x)$ has compact support in the interval $(j - 1.5, j + 1.5)$, we can write

$$\begin{aligned} &(L + \lambda I)\, M_\lambda f = \\ &- \textstyle\sum_{j=-\infty}^{\infty} \left[\varphi"_j \,(L_j + \lambda I)^{-1} \,\psi_j f + 2\varphi'_j \tfrac{d}{dx} \,(L_j + \lambda I)^{-1} \,\psi_j f \right] \\ &+ \textstyle\sum_{j=-\infty}^{\infty} \varphi_j \,(L_j + \lambda I)\,(L_j + \lambda I)^{-1} \,\psi_j\ f \\ &= B_\lambda f + \textstyle\sum_{j=-\infty}^{\infty} \varphi_j\ \psi_j\ f \end{aligned} \tag{6}$$

where

$$B_\lambda f = \sum_{j=-\infty}^{\infty} \left[\varphi_j''(L_j+\lambda I)^{-1}\psi_j f + 2\varphi_j' \frac{d}{dx}(L_j+\lambda I)^{-1}\psi_j f \right]$$

If we consider $\varphi_j\psi_j = \psi_j$ and condition (4), the expression (6) can be written as

$$(L+\lambda I)M_\lambda f = (I+B_\lambda)f \tag{7}$$

If we apply $(L+\lambda I)^{-1}$ operator to both side of (7) we can write

$$M_\lambda f = (L+\lambda I)^{-1}(I+B_\lambda)f \tag{8}$$

Let $(I+B_\lambda) = g$ or $f = (I+B_\lambda)^{-1}g$. Then (8) equality becomes

$$M_\lambda (I+B_\lambda)^{-1} g = (L+\lambda I)^{-1} g \tag{9}$$

Let us evalute the norm of B_λ operator transforming in H_1. Let $f \in H_1$

$$\begin{aligned} \|B_\lambda f\|^2 &= \left\| \sum_{j=-\infty}^{\infty} \left[\varphi_j''(L_j+\lambda I)^{-1}\psi_j f + 2\varphi_j' \tfrac{d}{dx}(L_j+\lambda I)^{-1}\psi_j f \right] \right\|^2 \\ &\leq 8 \sum_{j=-\infty}^{\infty} \left\| \varphi_j''(L_j+\lambda I)^{-1}\psi_j f \right\|^2 \\ &+ 8 \sum_{j=-\infty}^{\infty} \left\| \varphi_j'(L_j+\lambda I)^{-1}\psi_j f \right\|^2 \end{aligned} \tag{10}$$

In (10) equality it is considered that support of φ_j' and φ_{j+k} $(k \geq 2)$ functions are not intersected. Let us evaluate the terms of the first sum on the right side of (10)

$$\begin{aligned} \left\| \varphi_j''(L_j+\lambda I)^{-1}\psi_j f \right\|^2 \leq \tfrac{c^2}{\lambda^2} \|\psi_j f\|^2 &= \tfrac{c^2}{\lambda^2} \int_{-\infty}^{\infty} \|\psi_j f\|_H^2 \, dx \\ &= \tfrac{c^2}{\lambda^2} \int_{j-1}^{j+1} \|f(x)\|_H^2 \, dx \end{aligned} \tag{11}$$

Thus,

$$\left\| \varphi_j''(L_j+\lambda I)^{-1}\psi_j f \right\|^2 \leq \frac{c^2}{\lambda^2} \int_{j-1}^{j+1} \|f(x)\|_H^2 dx$$

Second sum is

$$\left\|\varphi_j'(L_j+\lambda I)^{-1}\psi_j f\right\|^2 \le c^2\|\frac{d}{dx}(L_j+\lambda I)^{-1}\|^2\|\psi_j f\|^2$$
$$\le c^2\|\frac{d}{dx}(L_j+\lambda I)^{-1}\|^2\int_{j-1}^{j+1}\|f(x)\|^2dx$$

Let us prove the following lemma.

Lemma 2: $\|\frac{d}{dx}(L_j+\lambda I)^{-1}\| \le c\frac{1}{\sqrt{\lambda}}$ $(\lambda > 0)$ inequality holds.

Proof: Let us consider $y(j-1)=y(j+1)=0$ by multiply with $y(x)$ both sides of equation $-y''+Q(x)y=f$. Then,

$$\int_{j-1}^{j+1}(-y''+Q(x)y,y)dx = \int_{j-1}^{j+1}(f,y)dx$$

$$\int_{j-1}^{j+1}\left[\|y'\|+(Q(x)y,y)\right]dx = \int_{j-1}^{j+1}(f,y)dx$$

If we consider in this inequality that $Q(x)=Q^*(x)\ge I$ and use the Schwartz inequality, we obtain

$$\int_{j-1}^{j+1}\|y'\|^2 \le \left(\int_{j-1}^{j+1}\|f(x)\|^2\right)^{1/2}\left(\int_{j-1}^{j+1}\|y(x)\|^2\right)^{1/2}$$

Since

$$y(x)=(L_j+\lambda I)^{-1}f(x)$$

then

$$\|y(x)\|^2_{L_2^j} \le \frac{1}{\lambda^2}\|f\|^2_{L_2^j} = \frac{1}{\lambda^2}\int_{j-1}^{j+1}\|f(x)\|^2dx$$

Here $L_2^j = L_2(j-1,j+1)$. If we consider

$$\int_{j-1}^{j+1}\|y'\|^2dx \le \frac{1}{\lambda}\int_{j-1}^{j+1}\|f(x)\|^2dx$$

$$y' = \frac{d}{dx}(L_j+\lambda I)^{-1}f$$

$$\|\frac{d}{dx}(L_j+\lambda I)^{-1}f\|_{L_2^j} \le \frac{1}{\sqrt{\lambda}}\|f\|_{L_2^j}$$

is obtain from the last inequality. Since f is the arbitary element of L_2^j we find

$$\|\frac{d}{dx}(L_j+\lambda I)^{-1}\| < \frac{1}{\sqrt{\lambda}}$$

Thus, the terms belonging to the second sum in (10) are

$$\|\varphi_j'\frac{d}{dx}(L_j+\lambda I)^{-1}\psi_j f\|^2 \le \frac{c}{\lambda}\int_{j-1}^{j+1}\|f(x)\|^2dx \qquad \lambda\ge 1 \tag{12}$$

If we consider (10), (11), (12) inequalities we find

$$\begin{aligned}
\|B_\lambda f\|^2 &\le \frac{c^2}{\lambda^2}\sum_{j=-\infty}^{\infty}\int_{j-1}^{j+1}\|f(x)\|^2dx + \frac{c}{\lambda}\sum_{j=-\infty}^{\infty}\int_{j-1}^{j+1}\|f(x)\|^2dx\\
&\le \frac{c_1}{\lambda}\sum_{j=-\infty}^{\infty}\int_{j-1}^{j+1}\|f(x)\|^2dx\\
&= \frac{c_1}{\lambda}\sum_{j=-\infty}^{\infty}\left(\int_{j-1}^{j}\|f(x)\|^2dx + \int_{j}^{j+1}\|f(x)\|^2dx\right)\\
&= \frac{c_1}{\lambda}\sum_{j=-\infty}^{\infty}\int_{j-1}^{j}\|f(x)\|^2dx + \frac{c_1}{\lambda}\sum_{j=-\infty}^{\infty}\int_{j}^{j+1}\|f(x)\|^2dx\\
&= \frac{2c_1}{\lambda}\int_{-\infty}^{\infty}\|f(x)\|^2dx = \frac{2c_1}{\lambda}\|f\|^2
\end{aligned}$$

Thus,

$$\|B_\lambda\| \le 2\frac{c_1}{\lambda}$$

where in big pozitive values of λ, it is found that $\|B_\lambda\|$ is as small as desired.Therefore, we can write

$$M_\lambda = (L+\lambda I)^{-1}(I+B_\lambda)$$

formula as

$$(L+\lambda I)^{-1} = M_\lambda(I+B_\lambda)^{-1}$$

Lemma 3: If $Q(x)M_\lambda$ is bounded, then L operator is separable.

Proof: According to Lemma 1, if $Q(x)(L+\lambda I)^{-1}$ is bounded, then L is separable. By the equation

$$(L+\lambda I)^{-1} = M_\lambda(I+B_\lambda)^{-1}$$

the proof of Lemma 3 is obtained.

Lemma 4: If $\sup_j \|Q(x)(L_j+\lambda I)^{-1}\| < \infty$, then $Q(x)M_\lambda$ operator is bounded.

Proof:

$$\begin{aligned}
\|Q(x)M_\lambda f\|^2 &= \left\|\sum_{j=-\infty}^{\infty} \varphi_j(x)(L+\lambda I)^{-1}\psi_j f\right\|^2 \\
&\leq 4\sum_{j=-\infty}^{\infty} \left\|\varphi_j Q(x)(L+\lambda I)^{-1}\psi_j f\right\|^2 \\
&\leq \sum_{j=-\infty}^{\infty} \left\|\varphi_j Q(x)(L+\lambda I)^{-1}\right\|^2 \|\psi_j f\|^2 \\
&\leq \sup_j \left(\left\|\varphi_j Q(x)(L+\lambda I)^{-1}\right\|^2\right) \sum_{j=-\infty}^{\infty} \|\psi_j f\|^2 \\
&= 2A\|f\|_{H_1}^2
\end{aligned}$$

Thus, $\|Q(x)M_\lambda f\|^2 \leq 2A\|f\|^2$. From this we find

$$\|Q(x)M_\lambda\| \leq \sqrt{2A}$$

and Lemma is proved. Let us show that $\|Q(x)(L_j+\lambda I)^{-1}\|$ is finite. When $|x-y| \leq 2$ let us assume that

$$\|(Q(x)-Q(y))Q^{-1}(y)\| \leq \delta \tag{13}$$

($\delta =$ const. > 0) Let us consider the following boundary value problem

$$-y'' + Q(x)y = f$$

$$y(j-1) = y(j+1) = 0$$

Let us write this problem as

$$-y'' + (-Q(j)+Q(x))y + Q(j)y = f \tag{14}$$

$$y(j-1) = y(j+1) = 0$$

Let

$$-y'' + Q(j)y = v$$

and

$$A_j y = -y'' + Q(j)y$$

Then $v = A_j y$. If we consider these, we can write (14) as

$$\begin{aligned} f = L_j y &= A_j y + (Q(x) - Q(j))y \quad (15) \\ v + (Q(x) - Q(j))A_j^{-1}v &= (I + (Q(x) - Q(j))A_j^{-1})v \quad (1.2) \end{aligned}$$

Let us evaluate the norm of operator $(Q(x) - Q(j))A_j^{-1}$. For this let us write the expression as

$$(Q(x) - Q(j))A_j^{-1} = (Q(x) - Q(j))Q(j)^{-1}(Q(j)A_j^{-1})$$

If we use the condition (13) we can write

$$\|(Q(x) - Q(j))A_j^{-1}\| \leq \delta \|Q(j)A_j^{-1}\| \tag{16}$$

Now let us prove the following Lemma.

Lemma 5: The set of $Q(j)A_j^{-1}$ operator is smooth bounded with 1.

$$\|Q(j)A_j^{-1}\| \leq 1$$

Proof: We will prove the Lemma by using opening formula according to eigenvector of $Q(j)$ operator.

Let us find A_j^{-1} operator in order to show that $\|Q(j)A_j^{-1}\| \leq 1$

Let us consider

$$A_j y = -y'' + Q(j)y = f(x)f(x) \in L_2(j-1, j+1; H) \tag{17}$$

$$y(j-1) = 0 \qquad y(j+1) = 0 \tag{18}$$

where $Q^{-1}(j)$ complete continous and therefore its spectrum is pure-disjoint.

Let eigenvalues of $Q(j)$ be

$$\alpha_1(j) \leq \alpha_2(j) \leq ...$$

and the corresponding eigenvectors to these eigenvalues be

$$g_1(j), g_2(j), ...$$

These form a base in H.

$$
\begin{array}{ll}
 & (x) \\
 & = \\
 & \sum_{k=1}^{\infty} h_k(x) g_k(j) \qquad h_k(x) = (f(x), g_k(j)) \\
(19) & y_j(x) \\
 & = \\
 & \sum_{k=1}^{\infty} y_{k,j}(x) g_k(j) \qquad y_{k,j}(x) = (y(x), g_k(j))
\end{array}
$$

If we put these equations in (17) we find

$$(20) - \textstyle\sum_{k=1}^{\infty} y_k''(x)\, g_k(j) + \sum\limits_{k=1}^{\infty} \alpha_k(j)\, y_k(x)\, g_k(j) = \sum\limits_{k=1}^{\infty} h_k(x)\, g_k(j)$$

$$
\begin{array}{l}
- \quad y_k''(x) + \alpha_k(j) y_k = h_k(x) \qquad k = 1, 2, ... \\
\sum\limits_{k=1}^{\infty} y_k(x) g_k(j)\Big|_{x=j-1} = 0 \quad , \quad \sum\limits_{k=1}^{\infty} y_k(x) g_k(j)\Big|_{x=j+1} = 0 \\
\sum\limits_{k=1}^{\infty} y_k(j-1) g_k(j) = 0 \quad , \quad \sum\limits_{k=1}^{\infty} y_k(j+1) g_k(j) = 0
\end{array}
$$

Thus, (17),(18) problems are transformed

$$(21) \qquad -y_k''(x) + \alpha_k(j) y_k = h_k(x)$$

$$(22) \qquad y_k(j-1) = 0 \quad , \quad y_k(j+1) = 0$$

The eigenvalues of these problems are

$$\lambda = \frac{k^2\pi^2}{4}$$

The corresponding normalized eigenvectors are the following:
if j is odd;

$$y = \sin\frac{k\pi}{2}x$$

if j even and k is odd;

$$y_{k,j} = \cos\frac{k\pi}{2}x$$

if k is even;

$$y_{k,j} = \sin\frac{k\pi}{2}x$$

$$y_{k,j} = \sum_{m=0}^{\infty}\frac{1}{\lambda_m+\alpha_k(j)}(h_k,\varphi_{m,j})\varphi_{m,j} \tag{23}$$

$$(h,\varphi_m) = \int_{j-1}^{j+1} h(x)\varphi_m(x)dx$$

Let us consider (23) in (19)

$$y_j(x) = \sum_{k=1}^{\infty}\left(\sum_{m=0}^{\infty}\frac{1}{\lambda_m+\alpha_k(j)}(h_k,\varphi_{m,j})\varphi_{m,j}\right)g_k(j)$$

$$Q(j)A_j^{-1}f = Q(j)y_j(x) \tag{24}$$

$$\begin{aligned}
&= Q(j)\sum_{k=1}^{\infty}\left(\sum_{m=0}^{\infty}\frac{1}{\lambda_m+\alpha_k(j)}(h_k,\varphi_{m,j})\varphi_{m,j}\right)g_k(j)\\
&= \sum_{k=1}^{\infty}\left(\sum_{m=0}^{\infty}\frac{1}{\lambda_m+\alpha_k(j)}(h_k,\varphi_{m,j})\varphi_{m,j}\right)Q(j)g_k(j)\\
&= \sum_{k=1}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\alpha_k(j)}{\lambda_m+\alpha_k(j)}(h_k,\varphi_{m,j})\varphi_{m,j}\right]g_k(j)
\end{aligned}$$

Thus,

$$Q(j)f = A_j^{-1}f = \sum_{k=1}^{\infty}\left[\sum_{m=0}^{\infty}\frac{\alpha_k(j)}{\lambda_m+\alpha_k(j)}(h_k,\varphi_{m,j})\varphi_{m,j}\right]g_k(j)$$

Since $\{\varphi_{m,j}(x)\}_{m=0}^{\infty}$ functions form a complete ortonormal system in $L_2(j-1,j+1)$,and the element $\{g_k(j)\}_{j=1}^{\infty}$ form a complete ortonormal system in H $\{\varphi_{m,j}(x)g_k(j)\}_{m=0,k=1}^{\infty}$ forms a complete ortonormal system in Hilbert space.

Since the system $\{\varphi_{m,j}(x)g_k(j)\}_{m=0,k=1}^{\infty}$ forms a complete ortonormal system in $L_{2,j}=L_2(j-1,j+1;H)$ if we use Parseval equation we can write from (24)

$$\|Q(j)A_j^{-1}f\|^2 = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\left(\frac{\alpha_k(j)}{\lambda_m+\alpha_k(j)}\right)^2|(h_k,\varphi_{m,j})|^2 \tag{25}$$

Since $\alpha_k(j)\geq 1$, $\frac{\alpha_k(j)}{\lambda_m+\alpha_k(j)}<1$ and

$$\|Q(j)A_j^{-1}f\|^2 \leq \sum_{k=1}^{\infty}\sum_{m=0}^{\infty}|(h_k,\varphi_{m,j})|^2 = \|f\|^2$$

or

$$\|Q(j)A_j^{-1}f\| \leq \|f\|$$

that is $\|Q(j)A_j^{-1}\|\leq 1$. Thus, we have proved $Q(j)A_j^{-1}$ operators are straight bounded.

According to Lemma, (16) becomes

$$\|\left(Q(x)-Q(j)\right)A_j^{-1}\| \leq \delta$$

Let us assume that $\delta<1$. Then from (15) we find

$$v = (I+(Q(x)-Q(j))A_j^{-1})^{-1}f$$

Let

$$T_j = (I+(Q(x)-Q(j))A_j^{-1})^{-1}$$

The last expression becomes

$$v = T_j f$$

and we obtain

$$y(x) = A_j^{-1} T_j f$$

Thus

$$\begin{array}{rcl} Q(x)y(x)\| & = & \|(Q(x)Q^{-1}(j))(Q(j)A_j^{-1}T_j f)\| \\ & \leq & \|Q(x)Q^{-1}(j)\| c \|T_j f\| \\ & \leq & \|Q(x)Q^{-1}(j)\| c_1 \|f\|_{L_{2,j}} \end{array} \qquad (26)$$

If we consider here condition

$$\|(Q(x) - Q(j))Q^{-1}(j)\| \leq \delta$$

then we find

$$\|Q(x)Q^{-1}(j) - I\| \leq \delta$$

Therefore, inequality (26) becomes

$$\|Q(x)y(x)\| \leq c_2 \|f\|_{L_{2,j}}$$

Here, if we consider $y(x) = L_j^{-1} f$, the last inequality takes the form of

$$\|Q(x)L_j^{-1} f\| \leq c_2 \|f\|_{L_{2,j}}$$

If we consider, then we find

$$\|Q(x)(L_j + \lambda I)^{-1} f\| \leq c_2 \|f\|_{L_{2,j}}$$

or

$$\|Q(x)(L_j + \lambda I)^{-1}\| \leq c$$

Thus, we have proved the theorem of separability.

Theorem: When the conditions 1) - 3) are satisfied, L operator is separable in H_1.

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