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PRESERVING FUZZY SG-CLOSED SETS
Conservando conjuntos Fuzzy sg-cerrados

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Abstract

In this paper we consider new weak and stronger forms of fuzzy irresolute and fuzzy semi-closure via the concept Fsg-closed sets which we call Fap-irresolute maps, Fap-semi-closed maps and contra-fuzzy irresolute and we use it to obtain several results in the literature concerning the preservation of fuzzy sg-closed sets and to obtain also a characterization of fuzzy semi-$T_{1/2}$ spaces.

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1. Introduction.

The concept of a fuzzy semi-generalized closed set (written as Fsg-closed set) of a fuzzy topological space was introduced by H.Maki, T.Fukutake, M.Kojima and H.Harada [3]. These sets were also considered and investigated by R.K.Saraf, M.Caldas and M.Khanna [4].

In this paper we shall introduce the concept of fuzzy irresoluteness called Fap-irresolute maps and Fap-semi-closed maps by using Fsg-closed sets and study some of their basic properties. This definition enables us to obtain conditions under which maps and inverse maps preserve Fsg-closed sets. Also in this paper we present a new generalization of fuzzy irresoluteness called contra-fuzzy irresolute. We define this last class of map by the requeriment that the inverse image of each fuzzy semi-open set in the codomain is fuzzy semi-closed in the domain . This notion is a stronger form of Fap-irresoluteness. Finally, we also characterize the class of fuzzy Semi-$T_{1/2}$ spaces in terms of Fap-irresolute and Fap-semi-closed maps.

Throughout this paper we adopt the notations and terminology of [5],[6],[1] and [2] and the following conventions: $(X,\tau),(Y,\sigma)$ and (Z,γ) (or simply X,Y and Z) represent non-empty fuzzy topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a fuzzy subset A of a space X, $Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A respectively.

2. Preliminares.

Since we shall require the following known definitions, notations and some properties, we recall them in this section .

Definition 1. *A fuzzy subset A of a fuzzy topological space X is said to be fuzzy semi-open (in short Fs-open) [1] if, there exists a fuzzy open O in X such that $O \leq A \leq Cl(O)$. The fuzzy semi-interior [9] of A denoted by $sInt(A)$, is defined by the union of all fuzzy semi-open sets of X contained in A.*

Remark 2.1. *A fuzzy subset A is Fs-open [9] if and only if $sInt(A) = A$.*

By $FSO(X)$ we mean the collection of all fuzzy semi-open sets in X.

Definition 2. *A fuzzy subset B of a fuzzy topological space X is said to be fuzzy semi-closed (in short Fs-closed) [1] if, its complement B^c is fuzzy semi-open in X. The fuzzy semi-closure [9] of a fuzzy set B of X denoted by $sCl_X(B)$ briefly $sCl(B)$, is defined to be the intersection of all fuzzy semi-closed sets of X containing B.*

Remark 2.2. *A fuzzy subset B is fuzzy semi-closed [9] if and only if $sCl(B) = B$.*

Definition 3. *Let $f : X \to Y$ be a map from a fuzzy topological space X to a fuzzy topological space Y.*
(i) f is called fuzzy rresolute [8] if, $f^{-1}(O)$ is fuzzy semi-open in X for every fuzzy semi-open set O of Y.
(ii) f is called fuzzy pre-semi-closed (resp. fuzzy pre-semi-open) [7] if, for every fuzzy semi-closed (resp. fuzzy semi-open) set B of $X, f(B)$ is fuzzy semi-closed (resp. fuzzy semi-open) in Y.

Definition 4. *A fuzzy subset F of a fuzzy topological space X is said to be fuzzy semi-generalized closed (written in short as Fsg-closed) in X [3] if, $sCl(F) \leq O$ whenever $F \leq O$ and O is Fs-open in X. A subset B is said to be fuzzy semi-generalized open (written as Fsg-open) in X [3] if, its complement B^c is Fsg-closed in X.*

Definition 5. *The intersection of all Fsg-closed sets containing a set A is called fuzzy semi-generalized closure of A [4], and is denoted by $sgCl(A)$.*

Remark 2.3. *(i) Every fuzzy semi-closed set is Fsg-closed but the converse may not be true in general [3].*
(ii) If A is Fsg-closed set, then $sgCl(A) = A$.
(iii) The fuzzy topological space X is said fuzzy Semi -$T_{1/2}$ [4], if every Fsg-closed set is fuzzy semi-closed.

3. Fuzzy ap-Irresolute, Fuzzy ap-Semi-closed and Contra-Fuzzy Irresolute Maps.

Let $f : X \to Y$ be a map from a fuzzy topological space X into a fuzzy topological space Y.

Definition 6. *A map $f : X \to Y$ is said to be fuzzy approximately irresolute (or Fap-irresolute) if, $sCl(F) \leq f^{-1}(O)$ whenever O is a fuzzy semi-open subset of Y, F is a Fsg-closed subset of X, and $F \leq f^{-1}(O)$.*

Definition 7. *A map $f : X \to Y$ is said to be fuzzy approximately semi-closed (or Fap-semi-closed) if, $f(B) \leq sInt(A)$ whenever A is a Fsg-open subset of Y, B is a fuzzy semi-closed subset of X, and $f(B) \leq A$.*

Clearly fuzzy irresolute maps are Fap-irresolute and fuzzy pre-semi-closed maps are Fap-semi-closed, but not conversely.

The proof follows from Definitions.

The following example shows the converse implications do not hold.

Example 3.1. *Let $X = \{a, b\}$ and $Y = \{x, y\}$. Define A and B as:*
$A(a) = 0.3$, $A(b) = 0.4$, $B(x) = 0.7$, $B(y) = 0.8$.
Let $\tau = \{0, A, 1\}$ and $\sigma = \{0, B, 1\}$. Then the map $f : (X, \tau) \to (Y, \sigma)$ defined by $f(a) = x$, $f(b) = y$ is Fap-irresolute but not fuzzy irresolute.

Example 3.2. *Let $X = \{x, y, z\}$ and $Y = \{a, b, c\}$. Define A, B and H as:*
$A(x) = 0$, $A(y) = 0.3$, $A(z) = 0.2$; $B(a) = 0$, $B(b) = 0.3$, $B(c) = 0.2$; $H(a) = 0.9$ $H(b) = 0.6$, $H(c) = 0.7$.
Let $\tau = \{0, A, 1\}$ and, $\sigma = \{0, B, H, 1\}$. Then the map $f : (X, \tau) \to (Y, \sigma)$ defined by $f(x) = a$, $f(y) = b$ and $f(z) = c$ is Fap-semi-closed but not fuzzy pre-semi-closed.

Theorem 3.3. *(i) $f : X \to Y$ is Fap-irresolute if, $f^{-1}(O)$ is fuzzy semi-closed in X for every $O \epsilon FSO(Y)$.*
(ii) $f : X \to Y$ is Fap-semi-closed if, $f(B) \epsilon FSO(Y)$ for every fuzzy semi-closed subset B of X.

Proof. (i) Let $F \leq f^{-1}(O)$ where $O \epsilon FSO(Y)$ and F is a Fsg-closed subset of X. Therefore $sCl(F) \leq sCl(f^{-1}(O)) = f^{-1}(O)$ since $f^{-1}(O)$ is fuzzy semi-closed. Thus f is Fap-semi-continuous.
(ii) Let $f(B) \leq A$,where B is a fuzzy semi-closed subset of X and A is a Fsg-open subset of Y. Therefore $sInt(f(B)) \leq sInt(A)$. Then $f(B) \leq sInt(A)$ since $f(B)$ is Fs-open. Thus f is Fap-semi-closed. □

The converse of Theorem 3.3 need not be true (see Remark 3.9).

In the following theorem, we get under certain conditions that the converse of Theorem 3.3 is true.

Theorem 3.4. *Let $f : X \to Y$ be a map from a fuzzy topological space X in a fuzzy topological space Y.*
(i) If the fuzzy semi-open and fuzzy semi-closed sets of X coincide, then f is Fap-irresolute if and only if, $f^{-1}(O)$ is fuzzy semi-closed in X for every $O \epsilon FSO(Y)$.
(ii) If the fuzzy semi-open and fuzzy semi-closed sets of Y coincide, then f is Fap-semi-closed if and only if, $f(B) \epsilon FSO(Y)$ for every fuzzy semi-closed subset B of X.

Proof. (i) Assume f is Fap-irresolute. Let A be an arbitrary fuzzy subset of X such that $A \leq Q$ where $Q \epsilon FSO(X)$. Then by hypothesis $sCl(A) \leq sCl(Q) = Q$. Therefore all fuzzy subsets of X are Fsg-closed (and hence all are Fsg-open). So for any $O \epsilon FSO(Y)$, $f^{-1}(O)$ is Fsg-closed in X. Since f is Fap-irresolute $sCl(f^{-1}(O)) \leq f^{-1}(O)$.Therefore $sCl(f^{-1}(O)) = f^{-1}(O)$, i.e., $f^{-1}(O)$ is fuzzy semi-closed in X (Remark 2.2). The converse is clear by Theorem 3.3(i).
(ii) Assume f is Fap-semi-closed. Reasoning as in (i), we obtain that all fuzzy subsets of Y are Fsg-open. Therefore for any fuzzy semi-closed fuzzy subset B of X, $f(B)$ is Fsg-open in Y. Since f is Fap-semi-closed $f(B) \leq sInt(f(B))$. Therefore $f(B) = sInt(f(B))$, i.e, $f(B)$ is fuzzy semi-open (Remark 2.1). The converse is clear by Theorem 3.3(ii). □

As immediate consequence of Theorem 3.4, we have the following.

Corollary 3.5. *Let $f : X \to Y$ be a map from a fuzzy topological space X in a fuzzy topological space Y.*

(i) If the fuzzy semi-open and fuzzy semi-closed sets of X coincide, then f is Fap-irresolute if and only if, f is fuzzy irresolute.
(ii) If the fuzzy semi-open and fuzzy semi-closed sets of Y coincide, then f is Fap-semi-closed if and only if, f is fuzzy pre-semi-closed.

Definition 8. *A map $f : X \to Y$ is called contra-fuzzy irresolute if $f^{-1}(O)$ is fuzzy semi-closed in X for each $O \,\epsilon FSO(Y)$, and contra-fuzzy pre-semi-closed if $f(B)\epsilon FSO(Y)$ for each fuzzy semi-closed set B of X.*

Remark 3.6. *The concepts of contra-fuzzy irresoluteness and fuzzy irresoluteness are independent. For*

Example 3.7. *A fuzzy irresolute map need not be contra-fuzzy irresolute.*
Let $X = \{x, y, z\}$ and $Y = \{a, b\}$. Define A and B as:
$A(x) = 0.3$, $A(y) = 0.5$, $A(z) = 0.4$; $B(a) = 0.4$, $B(b) = 0.6$. Let $\tau = \{0, A, 1\}$ and $\sigma = \{0, B, 1\}$. Then the map $f : (X, \tau) \to (Y, \sigma)$ defined by $f(x) = a$, $f(y) = f(z) = b$ is fuzzy irresolute but not contra-.fuzzy irresolute.

In the same manner one can prove that, contra-fuzzy irresoluteness does not imply fuzzy irresoluteness and that, contra-fuzzy pre-semi-closed maps and fuzzy pre-semi-closed are independent notions.

The following result can be easily verified. Its proof is straightfoward.

Theorem 3.8. *Let $f : X \to Y$ be a map. Then the following conditions are equivalent:*
(i) f is contra-fuzzy irresolute ,
(ii) The inverse image of each fuzzy semi-closed set in Y is fuzzy semi-open in X.

Remark 3.9. *By Theorem 3.3 , we have that every contra-fuzzy irresolute map is Fap–irresolute and every contra-fuzzy semi-closed is Fap-semi-closed, the converse implication do not hold, as the map defined in Example 3.1 is Fap–irresolute but not contra-fuzzy irresolute and the map defined in Example 3.2 is Fap–semi-closed but not contra fuzzy semi-closed.*

A map $f : X \to Y$ is called perfectly contra -fuzzy irresolute if the inverse of every fuzzy semi-open set in Y is fuzzy semi-clopen in X. Hence, every perfectly contra-fuzzy irresolute map is contra-fuzzy irresolute and fuzzy irresolute.

Clearly the following diagram holds and none of its implications is reversible:

$$
\begin{array}{ccccc}
 & \not\swarrow & \text{contra-fuzzy irresolute} & \not\nwarrow & \\
 & \nearrow & & \searrow & \\
\text{Perfectly contra-fuzzy irresolute} & & \not\uparrow \ \not\downarrow & & \text{Fap-irresolute} \\
 & \searrow & & \nearrow & \\
 & \not\nwarrow & \text{fuzzy irresolute} & \not\swarrow &
\end{array}
$$

The next two theorem are used to the preservation of Fsg-closed sets.

R.K.Saraf, M.Caldas and M.Khanna in ([4], Theorem 2.5) showed that the fuzzy irresolute pre-semi-closed inverse image of a Fsg-closed set is Fsg-closed. We strengthen this result slightly by replacing the fuzzy pre-semi-closed requirement with Fap-semi-closed.

Theorem 3.10. *If a map $f : X \to Y$ is fuzzy irresolute and Fap-semi-closed, then $f^{-1}(A)$ is Fsg-closed (resp.Fsg-open) whenever A is Fsg-closed (resp. Fsg-open) subset of Y.*

Proof. Let A be a Fsg-closed subset of Y. Suppose that $f^{-1}(A) \leq O$ where $O \epsilon FSO(X)$. Taking complements we obtain $O^c \leq f^{-1}(A^c)$ or $f(O^c) \leq A^c$. Since f is an Fap-semi-closed $f(O^c) \leq sInt(A^c) = (sCl(A))^c$. It follows that $O^c \leq (f^{-1}(sCl(A)))^c$ and hence $f^{-1}(sCl(A)) \leq O$. Since f is fuzzy irresolute $f^{-1}(sCl(A))$ is fuzzy semi-closed. Thus we have $sCl(f^{-1}(A)) \leq sCl(f^{-1}(sCl(A)) = f^{-1}(sCl(A)) \leq O$. This implies that $f^{-1}(A)$ is Fsg-closed in X.

A similar argument shows that inverse images of sg-open are sg-open . □

The following theorem show that the fuzzy ap-irresolute pre-semi-closed image of a Fsg-closed set is Fsg-closed

Theorem 3.11. *If a map $f : X \to Y$ is Fap-irresolute and fuzzy pre-semi-closed, then for every Fsg-closed F of $X, f(F)$ is Fsg-closed set of Y.*

Proof. Let F be a Fsg-closed subset of X. Let $f(F) \leq O$ where $O \epsilon FSO(Y)$. Then $F \leq f^{-1}(O)$ holds. Since f is Fap-irresolute $sCl(F) \leq f^{-1}(O)$ and hence $f(sCl(F)) \leq O$. Therefore, we have :

$sCl(f(F)) \leq sCl(f(sCl(F))) = f(sCl(F)) \leq O$. Hence $f(F)$ is Fsg-closed in Y. □

However the following theorem holds. The proof is easy and hence omitted.

Theorem 3.12. *Let $f : X \to Y$ and $g : Y \to Z$ be two maps such that $gof : X \to Z$. Then,*
(i) gof is contra-fuzzy irresolute, if g is fuzzy irresolute and f is contra-fuzzy irresolute.
(ii) gof is contra-fuzzy irresolute, if g is contra-fuzzy irresolute and f is fuzzy irresolute.

In an analogous way, we have the following.

Theorem 3.13. *Let $f : X \to Y$, $g : Y \to Z$ be two maps such that $g \circ f : X \to Z$. Then*
(i) $g \circ f$ is Fap-semi-closed, if f is fuzzy pre-semi-closed and g is Fap-semi-closed..
(ii) $g \circ f$ is Fap-semi-closed, if f is Fap-semi-closed and g is fuzzy pre-semi-open and g^{-1} preserves Fsg-open sets.
(iii) $g \circ f$ is Fap-irresolute If f is Fap-irresolute and g is fuzzy irresolute.

Proof. In order to prove the statement (i), suppose B is an arbitrary fuzzy semi-closed subset in X and A a Fsg-open subset of Z for which $g \circ f(B) \leq A$. Then $f(B)$ is fuzzy semi-closed in Y because f is fuzzy pre-semi-closed. Since g is Fap-semi-closed, $g(f(B)) \leq sInt(A)$.

This implies that $g \circ f$ is Fap-semi-closed.
In order to prove the statement (ii), suppose B is an arbitrary fuzzy semi-closed subset of X and A a Fsg-open subset of Z for which $g \circ f(B)) \leq A$. Hence $f(B) \leq g^{-1}(A)$. Then $f(B) \leq sInt(g^{-1}(A))$ because $g^{-1}(A)$ is Fsg-open and f is Fap-semi-closed. Thus ,
$(g \circ f)(B) = g(f(B)) \leq g(sInt(g^{-1}(A))) \leq sInt(gg^{-1}(A)) \leq sInt(A)$.
This implies that $g \circ f$ is Fap-semi-closed.
In order to prove the statement(iii), suppose F is an arbitrary Fsg-closed subset of X and $O \epsilon FSO(Z)$ for which $F \leq (g \circ f)^{-1}(O)$. Then $g^{-1}(O) \epsilon FSO(Y)$ because g is fuzzy irresolute. Since f is Fap-irresolute, $sCl(F) \leq f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O)$. This proved that $g \circ f$ is Fap-irresolute □

As a consequence of Theorem 3.13, we have:

Corollary 3.14. *Let $f_i : X \to Y_i$ be a map for each $i \epsilon \Omega$ and $f : X \to \prod Y_i$ the product map given by $f(x) = (f_i(x))$. If f is Fap-irresolute, then f_i is Fap-irresolute for each i.*

Proof. For each j let $P_j : \prod Y_i \to Y_j$ be the projection map. Then $f_j = P_j \circ f$ where P_j is fuzzy irresolute ([9], Theorem 3.8). By Theorem 3.13(iii), f_j is Fap-irresolute □

4. A Characterization of Fuzzy Semi-T$_{1/2}$ Spaces.

In the following theorem we give a characterization of a class of topological space called fuzzy Semi-T$_{1/2}$ space by using the concepts of Fap-irresolute maps and Fap-semi-closed maps

We recall, that a topological space X is said to be fuzzy Semi-T$_{1/2}$ space (written in short as F-Semi T$_{1/2}$) [4], if every Fsg-closed set is fuzzy semi-closed.

Theorem 4.1. *Let X be a fuzzy topological space. Then the following statements are equivalent.*
(i) X is a F-Semi-T$_{1/2}$ space,
(ii) For every space Y and every map $f : X \to Y, f$ is Fap-irresolute.

Proof. $(i) \to (ii)$: Let F be a Fsg-closed subset of X and suppose that $F \leq f^{-1}(O)$ where $O \epsilon FSO(Y)$. Since X is a F-semi-$T_{1/2}$ space, F is fuzzy semi-closed (i.e., $F = sCl(F)$). Therefore $sCl(F) \leq f^{-1}(O)$. Then f is Fap-irresolute.
$(ii) \to (i)$: Let B be a Fsg-closed subset of X and let Y be the fuzzy set X with the topology $\sigma = \{0, B, 1\}$. Finally let $f : X \to Y$ be the identity map. By assumption f is Fap-irresolute. Since B is Fsg-closed in X and fuzzy semi-open in (Y, σ) and $B \leq f^{-1}(B)$, it follows that $sCl(B) \leq f^{-1}(B) = B$. Hence B is fuzzy semi-closed in X and therefore is F-Semi-$T_{1/2}$. □

Theorem 4.2. *Let Y be a fuzzy topological space. Then the following statements are equivalent.*
(i) Y is a F-semi-$T_{1/2}$ space,
(ii) For every space X and every map $f : X \to Y$, f is Fap-semi-closed.

Proof. Analogous to Theorem 4.1 making the obvious changes. □

We refer the reader to [4] for others results on F-Semi-$T_{1/2}$ spaces.

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