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A NOTE ON PROJECTION OF FUZZY SETS ON HYPERPLANES *

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Abstract

The aim of this paper is to realize a comparative study between the concepts of projection and shadow of fuzzy sets on a closed hyperplane in a Hilbert space $\mathcal{X}$, this last one introduced by Zadeh in [8] on finite dimensional spaces and recently studied by Takahashi [1,7] in a real Hilbert space $\mathcal{X}$

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1. Introduction

Zadeh [8] introduce the concept of shadow of fuzzy sets on finite dimensional spaces and proved the following result: Let u, v be two convex fuzzy sets on $\mathbf{R}^n$. If $S_H[u] = S_H[v]$ for every hyperplane H in $\mathbf{R}^n$, then $u = v$.

However, Takahashi [7] shows a counterexample for this theorem and then proves that, in order to ensure this last equality, it is necessary to assume that u and v are closed-convex fuzzy sets.

In the first part of this work we introduce the concept of projection $P_H[u]$ of a fuzzy set u on a closed hyperplane H, analyzing when $P_H[u]$ is well defined. Hereafter, we realize a comparative study between shadow and projection of fuzzy sets on a closed hyperplane and we prove that this concepts are coincidents when they have compact levels.

This paper is organized as follows. In Section 2, we give the basic material that will be used in the article and, in Section 3, we realize a comparative study between shadow and projection of fuzzy sets, $S_H[u]$ and $P_H[u]$ respectively, on a closed hyperplane H. In this direction we present an example of a closed and convex fuzzy set with $S_H[u] \neq P_H[u]$, and then we prove that $S_H[u] = P_H[u]$ when u is a compact and convex fuzzy set. Finally, we prove that if u, v are compact and convex fuzzy sets and $P_H[u] = P_H[v]$ for every closed hyperplane, then $u = v$.

2. Preliminaries

In the sequel, $\mathcal{X}$ will denote a real separable Hilbert space with norm $\| \ \|$ and with topological dual $\mathcal{X}^*$. It is well known that if C is a closed and convex subset of X then, for every $x \in \mathcal{X}$, there exists a unique point $P_C(x) \in C$ such that $\|x - P_C(x)\| \leq \|x - c\|$ for all $c \in C$ (P_C is called the metric projection of $\mathcal{X}$ on C). Also, the application $x \in \mathcal{X} \mapsto P_C(x) \in C$ is uniformly continuous and, moreover

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \ , \ \forall x, y \in \mathcal{X}.$$

On the other hand, if f is a linear functional on $\mathcal{X}$ and $\alpha \in \mathbf{R}$ then $[f = \alpha] = \{x / f(x) = \alpha\}$ is called a hyperplane in $\mathcal{X}$. It is well known that $H = [f = \alpha]$ is closed if and only if $f \in \mathcal{X}^*$.

Theorem 2.1. (Hahn-Banach Theorem, [2]). *Let A, B be two non-empty disjoint closed and convex subsets of $\mathcal{X}$. If A is a compact set then there exists a closed hyperplane $H = [f = \alpha]$ strictly separating A and B. That is to say, $\exists\, f \in \mathcal{X}^*$, $\alpha \in \mathbf{R}$ and $\epsilon > 0$ such that $f(x) \leq \alpha - \epsilon$, $\forall x \in A$, and $f(x) \geq \alpha + \epsilon$, $\forall x \in B$.*

As a direct consequence of this result we obtain:

Corollary 2.2. *If A, B are closed-convex subsets of $\mathcal{X}$ then*

$$P_H(A) = P_H(B)\ ,\ \text{for all closed hyperplane } H \Rightarrow A = B.$$

Proof. Suppose that $A \neq B$. Then, there exists $a_0 \in A$ such that $a_0 \notin B$. But then, B and $\{a_0\}$ are closed-convex sets with $\{a_0\} \cap B = \emptyset$. So, due compactness of $\{a_0\}$, there exist a closed hyperplane F, we say $F = [f = \alpha]$, separating (strictly) $\{a_0\}$ and B.

Now, if p is a nonzero vector such that $f(p) = 0$ then $H = \{y \in \mathcal{X} /\ \langle y, p\rangle = \langle a_0, p\rangle\}$ is a closed hyperplane orthogonal to F and containing a_0. Moreover, it is clear that $a_0 \in P_H(A)$ and $a_0 \notin P_H(B)$. Thus, $P_H(A) \neq P_H(B)$. □

Denote $\mathcal{K}(\mathcal{X}) = \{A \subseteq \mathcal{X} /\ A \text{ nonempty, compact and convex}\}$. A linear structure of convex cone in $\mathcal{K}(\mathcal{X})$ is defined by

$$\begin{aligned} A + B &= \{a + b / a \in A, b \in B\} \\ \lambda A &= \{\lambda a / a \in A\} \end{aligned}$$

for all $A, B \in \mathcal{K}(\mathcal{X})$, $\lambda \in \mathbf{R}$. Also, the Hausdorff metric h on $\mathcal{K}(\mathcal{X})$ is defined by

$$h(A, B) = inf\,\{\epsilon > 0 /\ A \subseteq N(B, \epsilon) \text{ and } B \subseteq N(A, \epsilon)\}$$

where $N(A, \epsilon) = \{x /\ d(x, A) < \epsilon\}$ and $d(x, A) = \inf_{a \in A} \| x - a \|$.
It is well known that $(\mathcal{K}(\mathcal{X}), h)$ is a complete and separable metric space (see [3]).

On the other hand, if (A_p) is a sequence in $\mathcal{K}(\mathcal{X})$ we define the lower and upper Kuratowski limits of (A_p) by mean

$$\begin{aligned} lim\ inf\ A_p &= \{x \in \mathcal{X} /\ x = \lim_{p\to\infty} x_p,\ x_p \in A_p\} \\ lim\ sup\ A_p &= \{x \in \mathcal{X} /\ x = \lim_{k\to\infty} x_{p_k},\ x_{p_k} \in A_{p_k}\}. \end{aligned}$$

If $lim\ inf\ A_p = lim\ sup\ A_p = A$, we say that (A_p) converges to A in the Kuratowski sense and, in this case, we write $A = lim\ A_p$ or $A_p \xrightarrow{K} A$, and we say that A_p K-converges to A.
It is well known that

Proposition 2.3. If (A_p) is a sequence in $\mathcal{K}(\mathcal{X})$ then

$$\begin{aligned} lim\ sup\ A_p &= \bigcap_{p=1}^{\infty} \left(\overline{\bigcup_{m\geq p} A_m} \right) \\ lim\ inf\ A_p &= \bigcap_{F} \left(\overline{\bigcup_{m\in F} A_m} \right), \end{aligned}$$

respectively, where the last intersection is over all sets F cofinal in $\mathbf{N}$ (we recall that F is a cofinal subset of $\mathbf{N}$ if for all $n \in \mathbf{N}$ there is $m \in F$ such that $m > n$).
The following result is very useful.

Proposition 2.4. *A sequence* (A_p) in $\mathcal{K}(\mathcal{X})$ *converges to a compact set* A *respect to the Hausdorff metric if and only if there is* K;*compact in* $\mathcal{X}$ *such that* $A_p \subseteq K$ *for all* p *and*

$$lim\ inf\ A_p = lim\ sup\ A_p = A.$$

For details on h and K-convergence and its relationships see [3].
As an extension of $\mathcal{K}(\mathcal{X})$, we consider the space of compact-convex fuzzy sets on $\mathcal{X}$ defined by

$$\mathcal{F}(\mathcal{X}) = \{u : \mathcal{X} \to [0,1]/\ L_\alpha u \in \mathcal{K}(\mathcal{X}),\ \forall \alpha \in [0,1]\},$$

where $L_\alpha u = \{x/\ u(x) \geq \alpha\}$ is the α-level of u, for $\alpha > 0$, and $L_0 u = \mathbf{cl}\{x/\ u(x) > 0\}$ is the support of u.

Remark 2.5. The family $\{L_\alpha u/\alpha \in [0,1]\}$ satisfies the following properties:
a) $L_0 u \supseteq L_\alpha u \supseteq L_\beta u$ for all $0 \leq \alpha \leq \beta$.
b) If $\alpha_n \nearrow \alpha$ then $L_\alpha u = \bigcap\limits_{n=1}^{\infty} L_{\alpha_n} u$ (i.e. the level-application is left-continuous)
c) $u = v \Leftrightarrow L_\alpha u = L_\alpha v$ for all $\alpha \in [0,1]$.
d) $L_\alpha u \neq \emptyset$ for all $\alpha \in [0,1]$ is equivalent to $u(x) = 1$ for some $x \in \mathcal{X}$.

In connection with the above properties, we recall the following representation theorem (Negoita&Ralescu Theorem) for fuzzy sets

Theorem 2.6.([4]). *Let $(N_\alpha)_{\alpha\in[0,1]}$ be a family of subsets of $\mathcal{X}$ such that:*

i) $\alpha \leq \beta \Rightarrow N_\beta \subseteq N_\alpha$

ii) $\alpha_1 \leq \alpha_2 \leq ... \nearrow \alpha \;\; \Rightarrow \;\; N_\alpha = \bigcap_{p=1}^{\infty} N_{\alpha_p}$.

Then there exists a unique fuzzy sets $u : \mathcal{X} \to [0,1]$ such that $L_\alpha u = N_\alpha$, $\forall \alpha \in (0,1]$ and $L_0 u \subseteq N_0$. Moreover, u is given by

$$u(x) = \begin{cases} 0 & if \quad x \notin N_0 \\ sup\{\alpha/\ x \in N_\alpha\} & if \quad x \in N_0. \end{cases}$$

A linear structure of convex cone in $\mathcal{F}(\mathcal{X})$ is defined via α-level sets by mean

$$\begin{aligned} L_\alpha(u+v) &= L_\alpha(u) + L_\alpha(v) \\ L_\alpha \lambda u &= \lambda L_\alpha u, \end{aligned}$$

for all $u, v \in \mathcal{F}(\mathcal{X})$ and $\alpha \in [0,1]$. On the other hand, as an extension of the Hausdorff metric h, we define

$$D(u,v) = \sup_{\alpha\in[0,1]} H(L_\alpha u, L_\alpha v),\ \forall u, v \in \mathcal{F}(\mathcal{X}),$$

and it is well known that $(\mathcal{F}(\mathcal{X}), D)$ is a complete but nonseparable metric space (see [4,5,6]).

3. Projection and shadow of fuzzy sets

In this section we will recall the concept of shadow of fuzzy sets introduced by Zadeh in [8] on $\mathbf{R}^n$ and recently studied by Takahashi [1,7] in a real Hilbert space $\mathcal{X}$. On the other hand, we want introduce the concept of projection of fuzzy sets on a closed hyperplane H in $\mathcal{X}$ and to make a comparative study between shadow and projection of fuzzy sets.

Definition 3.1 ([8],[1,7]). *Let u be a fuzzy set in $\mathcal{X}$. The shadow of u on a closed hyperplane H in $\mathcal{X}$ is a fuzzy set in H defined by*

$$S_H[u](z) = sup\{u(x)/\ x \in P_H^{-1}(z)\},$$

for all $z \in H$, where $P_H(\cdot)$ is the projection mapping from $\mathcal{X}$ onto H and

$$P_H^{-1}(z) = \{x \in X/\ P_H(x) = z\}.$$

With this definition, in [7] the authors shows that:

Theorem 3.2. *Let u, v be two convex-closed fuzzy sets in $\mathcal{X}$ such that $S_H[u] = S_H[v]$ for any closed hyperplane H, then we have $u = v$.*

Now, if $u \in \mathcal{F}(\mathcal{X})$ and H is a hyperplane in $\mathcal{X}$ then we wish to define $P_H[u]$, the projection of u on the hyperplane H.

For this, we will use the Negoita&Ralescu Theorem and we will define $P_H[u]$ by mean its α-level sets.

Definition 3.3. *If $u \in \mathcal{F}(\mathcal{X})$ and H is a hyperplane in $\mathcal{X}$, then we define the projection of u on the hyperplane H as the fuzzy sets $P_H[u]$ defined on H by mean*

$$P_H[u] : H \longrightarrow [0,1],\ L_\alpha P_H[u] = P_H(L_\alpha u),\ \ \forall \alpha \in (0,1].$$

Now, in order to checking that $P_H[u]$ is well-defined (i. e., $(P_H(L_\alpha u))_{\alpha\in[0,1]}$ is an α-level family), we must to verify the hypotheses of Theorem 2.6. For this, firstly we observe that, due continuity of P_H

and compactness of $L_\alpha u$, we have $L_\alpha P_H[u] = P_H(L_\alpha u) \in \mathcal{K}(\mathcal{X}),\ \forall \alpha \in [0,1]$.
On the other hand,
i) $\alpha \le \beta \Rightarrow L_\beta\, u \ \subseteq L_\alpha\, u \Rightarrow P_H(L_\beta\, u) \subseteq P_H(L_\alpha\, u)$.

ii) Let $\alpha_n \nearrow \alpha$ be. Then, $L_{\alpha_n} u \ \supseteq L_\alpha\, u,\ \forall_n$, which implies $P_H(L_{\alpha_n} u) \supseteq P_H(L_\alpha\, u),\ \forall_n$. Therefore,

$$\bigcap_{n=1}^{\infty} P_H(L_{\alpha_n} u) \supseteq P_H(L_\alpha\, u). \tag{3.1}$$

Conversely, if $x \in \bigcap_{n=1}^{\infty} P_H(L_{\alpha_n} u)$ then $x \in P_H(L_{\alpha_n} u),\ \forall_n$.
Therefore $\forall n \in \mathbf{N},\ \exists\ y_n \ \in L_{\alpha_n} u$ such that $P_H(y_n) = x$. So, (y_n) is a sequence in $L_0 u$ and, due compactness of $L_0 u$, there exists a subsequence (y_{n_k}) such that $y_{n_k} \to y \in L_0 u$ as $k \to \infty$.
Now, because $y_{n_k} \in L_{\alpha_{n_k}} u \ ,\forall k$; then $y \in lim\ sup\, L_{\alpha_{n_k}} u$. Thus, due $(L_{\alpha_n} u)$ is a decreasing sequence, by Proposition 2.3 and Remark 2.5 b) we obtain

$$y \in \bigcap_{n=1}^{\infty} \left(\overline{\bigcup_{m \ge n} L_{\alpha_m} u} \right) = \bigcap_{n=1}^{\infty} L_{\alpha_n} u = L_\alpha u.$$

On the other hand, due continuity of the projection P_H , we have

$$P_H(y) = \lim_{k \to \infty} P_H(y_{n_k}) = x.$$

Therefore, $x \in P_H(L_\alpha\, u)$ and, consequently, we obtain

$$\bigcap_{n=1}^{\infty} P_H(L_{\alpha_n} u) \subseteq P_H(L_\alpha u)\,. \tag{3.2}$$

Thus, from (1) and (2) we conclude that

$$P_H(L_\alpha\, u) = \bigcap_{n=1}^{\infty} P_H(L_{\alpha_n}\ u). \tag{3.3}$$

This shows that $P_H[u]$ is well defined.Finally, by using Theorem 2.6, if $z \in H$ then an explicit formula for $P_H[u](z)$ is given by

$$P_H\,[u]\,(z) = sup\ \{\alpha\,/\,z \in\ P_H(L_\alpha u)\}\,.$$

The following example shows that, in general, projection and shadow of fuzzy sets are not equivalent concepts.

Example 3.4. Let $\mathcal{X} = \mathbf{R}^2$ be and consider the set (see Figure:1)

$$A = \left\{(x, y) \in (0, +\infty) \times (0, +\infty)/\ y \geq \frac{1}{x}\right\}. \tag{3.4}$$

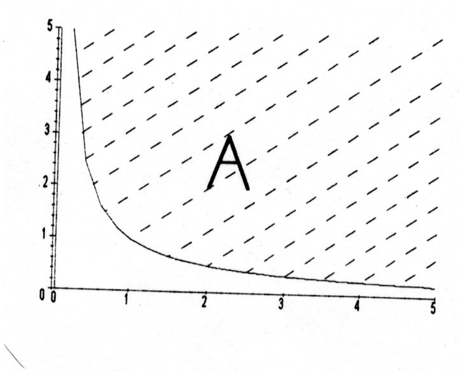

Figure 1

Then A is a closed and convex subset of $\mathcal{X}$. Now, define $u : \mathbf{R}^2 \to [0, 1]$ by mean

$$u(\mathbf{x}) = \begin{cases} 0 & if \quad \mathbf{x} \notin [0, +\infty) \times [0, +\infty) \\ 1 - \frac{d(\mathbf{x},A)}{\sqrt{2}} & if \quad \mathbf{x} \in [0, +\infty) \times [0, +\infty). \end{cases}$$

Then, it is clear that $L_\alpha u = \{\mathbf{x} \in [0, +\infty) \times [0, +\infty)/\ d(\mathbf{x}, A) \leq \sqrt{2}(1 - \alpha)\}$ is a closed and convex set in $\mathcal{X}$, for all $\alpha > 0$. In particular, $L_1 u = A$. Now, if $\alpha_p \nearrow 1$ and H is the closed hyperplane $y = 0$, then $P_H(L_{\alpha_p} u) = [0, +\infty)$ for every p and, consequently, $\bigcap\limits_{n=1}^{\infty} P_H(L_{\alpha_p} u) = [0, +\infty)$, whereas $P_H(L_1 u) = (0, +\infty)$. Thus, in this case is not possible to define $P_H[u]$. On the other hand, $S_H[u]$ is well defined and, in particular, $S_H[u](\mathbf{0}) = sup\{u\mathbf{x})/\ \mathbf{x} \in P_H^{-1}(\mathbf{0})\} = 1$ and $L_1 S_H[u] = [0, +\infty)$.

Theorem 3.5. *If $u \in \mathcal{F}(\mathcal{X})$, then $S_H[u] = P_H[u]$ for all closed hyperplane H in $\mathcal{X}$.*
Proof. We know that, in this case, $P_H[u]$ is well defined and if $z \in H$ then

$$\begin{aligned} P_H\,[u]\,(z) &= sup\,\{\alpha \,/\, z \in\, P_H(L_\alpha u)\} \quad (by\ (4)) \\ &= sup\,\{\alpha \,/\, z =\, P_H(x),\ x \in L_\alpha u\} \\ &= sup\,\left\{\alpha \,/\, x \in (P_H^{-1}(z) \cap L_\alpha u)\right\} \\ &\leq sup\,\left\{u(x) / \, x \in (P_H^{-1}(z) \cap L_\alpha u)\right\} \\ &\leq sup\,\left\{u(x) / \, x \in P_H^{-1}(z)\right\} \\ &= S_H[u](z). \end{aligned}$$

Conversely, suppose that $S_H[u](z) = \alpha_0$, then $\alpha_0 = sup\{u(x)/\ x \in P_H^{-1}(z)\}$. Thus, $\forall n \in \mathbf{N}$, $\exists x_n \in P_H^{-1}(z)$ such that $u(x_n) > \alpha_0 - \dfrac{1}{n}$. This implies that $x_n \in L_{\alpha_0 - \frac{1}{n}} u$ and $z = P_H(x_n) \in P_H(L_{\alpha_0 - \frac{1}{n}} u)$, for every $n \in \mathbf{N}$. Therefore, due (3), $z \in \bigcap_{n=1}^{\infty} P_H(L_{\alpha_0 - \frac{1}{n}} u) = P_H(L_{\alpha_0} u)$. Consequently,

$$S_H[u](z) = \alpha_0 \leq sup\,\{\alpha \,/\, z \in\, P_H(L_\alpha u)\} = P_H\,[u]\,(z)$$

and the proof is completed. □

Theorem 3.6. *Let $u, v \in \mathcal{F}(\mathcal{X})$ be. If $P_H[u] = P_H[v]$ for any closed hyperplane H, then we have $u = v$.*

Proof.

$$\begin{aligned} P_H[u] = P_H[v],\ \forall H \ &\Rightarrow\ L_\alpha P_H[u] = L_\alpha P_H[v],\ \forall H,\ \forall\alpha \\ &\Rightarrow\ P_H[L_\alpha u] = P_H[L_\alpha v],\ \forall H,\ \forall\alpha \\ &\Rightarrow\ L_\alpha u = L_\alpha v,\ \forall\alpha \quad (by\ Prop.2.2). \end{aligned}$$

Consequently, due Remark 2.5 c), $u = v$. □

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