Proyecciones
Vol. 20, N^o 3, pp. 305-321, December 2001.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172001000300004

EXISTENCE OF SOLUTIONS FOR A SYSTEM OF ELASTIC WAVE EQUATIONS

LUIS A. CORTÉS–VEGA *
Universidad de Bío - Bío - Chile

Abstract

A simple and short proof of the existence of solutions for the direct scattering problem associated with the system of elastic wave equations is shown.

Subjclass : Primary 35B40, 76Q05, Secondary 73D35, 73C02
Keywords : Elastic waves, exterior domain, resonant frequencies

*This research was done with partial support by Diprode 000406–3 UBB-Chile and CNPq–Brazil

1. Introduction

The propagation of time-harmonic elastic waves

$$\mathbf{v}^{in}(x) = -\sigma^{-2}\text{grad div } [\mathbf{p}_0 \exp(i\ \sigma_L\ d \cdot x)] \\ +\sigma^{-2} \text{ rot rot } [\mathbf{p}_0 \exp(i\ \sigma_T\ d \cdot x)]$$

by impenetrable bounded obstacle $D \subset \mathbf{R}^3$ with $\Omega = \mathbf{R}^3 \backslash \overline{D}$ its complement in $\mathbf{R}^3$ and polarization $\mathbf{p}_0 \in \mathbf{R}^3$ leads to exterior boundary value problems for the system

$$b^2\, \boldsymbol{\Delta} \mathbf{v} + (a^2 - b^2)\, \text{grad (div } \mathbf{v}) + \sigma^2\, \mathbf{v} = 0,$$

here $\sigma \in \mathbf{C}$ is the frequency of the incident wave $\mathbf{v}^{in}$ and $\mathbf{v} = \mathbf{v}^{in} + \mathbf{v}^{sc}$, where $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{v}^{sc} = (v_1^{sc}, v_2^{sc}, v_3^{sc})$, denote the displacement of the refracted and scattered wave, respectively.

The total wave $\mathbf{v}$ it is required to fulfill the Kupradze-Sommerfeld radiation conditions

$$\text{(1.1)} \qquad \begin{cases} \mathbf{v}^L(x) = o(1), & \text{as } |x| \to \infty, \\ \frac{\partial}{\partial\, |x|}\mathbf{v}^L(x) - i\ \sigma_L\ \mathbf{v}^L(x) = o(\frac{1}{|x|}), & \text{as } |x| \to \infty, \end{cases}$$

and

$$\text{(1.2)} \qquad \begin{cases} \mathbf{v}^T(x) = o(1), & \text{as } |x| \to \infty, \\ \frac{\partial}{\partial\, |x|}\mathbf{v}^T(x) - i\ \sigma_T\ \mathbf{v}^T(x) = o(\frac{1}{|x|}), & \text{as } |x| \to \infty, \end{cases}$$

uniformly for all directions $\widehat{x} = (\frac{1}{|x|})\ x$, where $\mathbf{v} = \mathbf{v}^L + \mathbf{v}^T$ it is a sum of an irrotational (lamellar) vector $\mathbf{v}^T$ and a solenoidal vector $\mathbf{v}^L$. Here $b^2 = \mu$ and $a^2 = \lambda + 2\ \mu$, where λ, μ are the Lamé coeficientes of the Elastycity Theory, $\sigma_L \in \mathbf{C}$ is the longitudinal (dilational) wave number, $\sigma_L = \frac{\sigma}{b}$ and transverse (shear) wave number $\sigma_T = \frac{\sigma}{a} \in \mathbf{C}$, with $a^2 > (\frac{4}{3})\ b^2 > 0$.

For a soft obstacle the unknown scattered wave $\mathbf{v}^{sc} = \mathbf{v} - \mathbf{v}^{in}$ has to satisfy the Dirichlet boundary condition $\mathbf{v}^{sc} = -\ \mathbf{v}^{in}$ on $\partial\Omega$ whereas for hard obstacle $\mathbf{v}^{sc}$ has to satisfy a Neumann boundary condition $\mathbf{T_n v}^{sc} = -\mathbf{T_n v}^{in}$ on $\partial\Omega$, where $\mathbf{n} = (n_1, n_2, n_3)$ denotes the unit outward normal to the boundary $\partial\Omega$ and $\mathbf{T_n}$ is the stress vector calculated on the surface element

$$\mathbf{T_n}\, \mathbf{v} = 2\, b^2\ \frac{\partial\, \mathbf{v}}{\partial\, \mathbf{n}} + (a^2 - 2\, b^2)\ \mathbf{n}\ \ div\ \ \mathbf{v} + b^2\ \mathbf{n} \times rot\ \ \mathbf{v},$$

where $\frac{\partial \mathbf{v}}{\partial \mathbf{n}} = (\frac{\partial v_1}{\partial \mathbf{n}}, \frac{\partial v_2}{\partial \mathbf{n}}, \frac{\partial v_3}{\partial \mathbf{n}})$ is the derivative with respect to the outer normal $\mathbf{n}$ on $\partial\Omega$.

Problems involving the propagation of time-harmonic elastic waves as above arise naturally in many situations, particularly those involving fluid–structure interaction (see for instance [7, 14] and [16]) and in the existence (localization) of the scattering frequencies (see for instance [3, 4] and [17]) which are problems of significant interest.

The existence of solutions for the exterior boundary value problems above based on boundary integral equations, appear, for example in [1, 2, 11, 16] and by other methods in [7, 8], and references therein. Here, the basic results needed to devolp efficient tools for inverse scattering problems are provided. This is of great practical interest (see for instance [9] and [10]).

In this work we consider the scattering of time-harmonic plane elastic waves in a homogeneus isotropic medium at an obstacle D. We hence present a simple and short proof of the existence of solutions for the exterior boundary value problem associated with the reduced system of elastic wave equations

$$b^2\,\mathbf{\Delta v}(x) + (a^2 - b^2)\,\mathrm{grad}\,(\mathrm{div}\,\mathbf{v}(x)) + \sigma^2\,\mathbf{v}(x) = \mathbf{h}(x), \quad x \in \Omega \tag{1.3}$$

together with the Dirichlet boundary condition

$$\mathbf{v}(x) = 0, \quad x \in \partial\Omega \tag{1.4}$$

and the Kupradze–Sommerfeld radiation condition (1.1) and (1.2), respectively. In (1.3) $\mathbf{h} = (h_1, h_2, h_3)$ is a given function and $\sigma \in \mathbf{C}$. To this end, we use a tecnique similar to the one discussed in ([18], p.p, 35-36) and [5, 6], in this sense our approach is new.

Outline of the work: In section §2 we present the formulation of the main result. In section §3 we give the proof the main theorem. Finally, in the section §4 we present the meromorphic extension of the solution for every $\sigma \in \mathbf{C}$ with $\Im(\sigma) \leq 0$.

We shall use the standart notation: Here and throughout this work we assume that $\Omega = \mathbf{R}^3 \backslash \overline{D}$ is the exterior of $\overline{D}$ with smooth boundary $\partial\Omega$. Also, we denote by grad the gradient, by rot the rotational vector, $\mathbf{\Delta v} = (\triangle v_1, \triangle v_2, \triangle v_3)$, where $\triangle$ is the usual Laplacian operator and div the divergence. For any positive integer p and $1 \leq s \leq \infty$ we consider the Sobolev space $W^{p,s}(\Omega)$ of (classes of) functions in $L^s(\Omega)$ which together with their derivatives up to order p belong to $L^s(\Omega)$. The norm of $W^{p,s}(\Omega)$ will denoted by $\|\cdot\|_{p,s}$ in the case $s = 2$ we write $H^p(\Omega)$ instead of $W^{p,2}(\Omega)$. If E is a vector space then we denoted $[E]^3 = \oplus_{i=1}^3 E$ and the norm of a vector $\mathbf{v}$ wich belong to $[E]^3$ will be denoted by $\|\cdot\|_{[E]^3}$. $C_0^\infty(\mathbf{R}^3)$ denotes the space of all C^∞ functions defined on $\mathbf{R}^3$ with compact support. If E is a Banach space, we consider the space $B(E,E)$ of linear bounded operators in E. If $\mathbf{h} :\mathbf{R}^3 \to \mathbf{R}^3$, $\mathbf{h} = (h_1, h_2, h_3)$ then we denoted by supp $\mathbf{h} = \cap_{i=1}^3$ supp h_i the support of $\mathbf{h}$ and

$$\int_{\mathbf{R}^3} \mathbf{h}\; dx = \left(\int_{\mathbf{R}^3} h_1\; dx, \int_{\mathbf{R}^3} h_2\; dx \int_{\mathbf{R}^3} h_3\; dx \right).$$

If $R > 0$ then $B(R)$ is the ball centered at zero and radius R. Also, we denoted by

$$\partial\overline{B}(R) = \left\{ x \in \mathbf{R}^3 : |x| = R \right\}$$

and by $[L_R^2(\mathbf{R}^3)]^3$ the space

$$[L_R^2(\mathbf{R}^3)]^3 = \left\{ \mathbf{v} \in [L^2(\mathbf{R}^3)]^3 :\ \mathbf{v} = 0,\ \text{if}\ |x| \geq R \right\}.$$

With all these notations we stablish now our main theorem

2. Formulation of result

In this section we shall establish the existence of solutions for the systems of elastic waves

$$\begin{cases} b^2\, \boldsymbol{\Delta}\, \mathbf{v}(x) + (a^2 - b^2)\, \mathrm{grad}(\mathrm{div}\ \mathbf{v}(x)) + \sigma^2\, \mathbf{v}(x) = \mathbf{h}(x), & x \in \Omega, \\ \mathbf{v}(x) = 0, & x \in \partial\Omega, \\ \text{Radiation condition.} & \end{cases}$$

This will be done based on [5, 6]. Our starting point is the following lemma whose proof appears in [6, 11].

Lemma 1. *Let $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$ and take $v \in [H^2(\mathbf{R}^3)]^3$ solution of system*

$$b^2\, \boldsymbol{\Delta}\, \mathbf{v}(x) + (a^2 - b^2)\, grad\ (div\ \mathbf{v}(x)) + \sigma^2\, \mathbf{v}(x) = 0,\ x \in \mathbf{R}^3,$$

satisfying the Kupradze–Sommerfeld radiation condition for $a^2 > (\frac{4}{3})$ b^2*. Then we have*

$$\int_{|x|=R} \overline{\mathbf{v}} \cdot \mathbf{T}_n v \, ds = 0, \quad \text{as } R \to \infty.$$

Lemma 2. *Let* $\sigma \in \mathbf{C}$ *with* $\Im(\sigma) > 0$*. Then, for any* $\mathbf{g} \in [L^2_R(\mathbf{R}^3)]^3$*, the system*

$$\begin{aligned} b^2 \, \boldsymbol{\Delta} \, \mathbf{v}(x) + (a^2 - b^2) \, grad \, (\text{div } \mathbf{v}(x)) \\ + \sigma^2 \, \mathbf{v}(x) = \mathbf{g}(x), \; x \in \mathbf{R}^3, \end{aligned} \tag{2.1}$$

admits a solution $\mathbf{v} \in [H^2(\mathbf{R}^3)]^3$ *and* $\mathbf{v} = \mathbf{A}(\sigma) \, \mathbf{g}$*, where*

$$\mathbf{A}(\sigma) : [L^2_R(\mathbf{R}^3)]^3 \to [H^2(\mathbf{R}^3)]^3$$

is a linear continuous operator. In particular, if $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *solves (2.1) and satisfies the Kupradze–Sommerfeld radiation condition, then* $\mathbf{v}_1(x) = \mathbf{v}_2(x)$ *for all* $x \in \mathbf{R}^3$*. See [5] for the proof.*

Let $\mathbf{f} \in [L^2(\boldsymbol{\Omega})]^3$ and take $\mathbf{f}_0$ given by

$$\mathbf{f}_0(x) = \begin{cases} \psi(x) \, \mathbf{f}(x), & \text{if } \; x \in \Omega, \\ 0, & \text{if } \; x \in \overline{D}, \end{cases} \tag{2.2}$$

where ψ is the function

$$\psi(x) = \begin{cases} 1, & \text{if } \; x \in \Omega_R, \\ 0, & \text{if } \; x \notin \Omega_R, \end{cases} \tag{2.3}$$

and $\Omega_R = \{x \in \boldsymbol{\Omega} : \, |x| < R\}$.

Let $\sigma \in \mathbf{C}$ be such that $\Im(\sigma) > 0$ and $\mathbf{v}_0 \in [H^2(\mathbf{R}^3)]^3$ a solution of the system

$$b^2 \, \boldsymbol{\Delta} \mathbf{v}_0(x) + (a^2 - b^2) \, \text{grad}(\text{div } \mathbf{v}_0(x)) + \sigma^2 \, \mathbf{v}_0(x) = \mathbf{f}_0(x), \, x \in \mathbf{R}^3.$$

Lemma 3. *The system of elastic waves*

$$\begin{cases} b^2\,\mathbf{\Delta}\,\mathbf{w}(x) + (a^2-b^2)\,grad(div\ \ \mathbf{w}(x)) = 0, & x \in \Omega_R, \\ \mathbf{w}(x) = -\mathbf{v}_0(x), & x \in \partial\Omega, \\ \mathbf{w}(x) = 0, & x \in \partial\overline{B}(R) \end{cases}$$

has a (unique) solution on $[H^2(\mathbf{\Omega}_R)]^3$*. See [6] for the proof.*

Next we summarize the well known result given, for example in [15].

Lemma 4. *Let* $\mathbf{w} \in [H^2(B(R))]^3$ *be a solution of the system*

$$b^2\,\mathbf{\Delta}\,\mathbf{w}(x) + (a^2-b^2)\,grad\ (div\ \mathbf{w}(x)) = 0,\ x \in B(R)$$

and

$$\mathbf{w}(x) = 0,\ x \in \partial\overline{B}(R).$$

Then $\mathbf{w}(x) = 0$ for every $x \in B(R)$. See [15] or [6] for the proof.

At this point, we derive from the above lemmas the proof the main theorem.

Theorem 1. *Let* $\sigma \in \mathbf{C}$ *with* $\Im(\sigma) > 0$*. Then, for any* $\mathbf{h} \in [L^2(\mathbf{\Omega})]^3$ *with supp* $\mathbf{h} \subset \mathbf{\Omega}_{\ R}$ *the system of elastic waves*

$$\begin{cases} b^2\,\mathbf{\Delta}\,\mathbf{v}(x) + (a^2-b^2)\,grad(div\ \ \mathbf{v}(x)) + \sigma^2\,\mathbf{v}(x) = \ \mathbf{h}(x), & x \in \Omega, \\ \mathbf{v}(x) = 0, & x \in \partial\Omega, \\ \textit{Radiation condition} & \end{cases}$$

(2.4)

has a unique solution $\mathbf{v} \in [H^2(\mathbf{\Omega})]^3$*. Furthermore,* $\mathbf{v}$ *can be extended in a meromorphic way to* $\sigma \in \mathbf{C}$ *with* $\Im(\sigma) \le 0$ *except, for some countable number of poles in* $\Xi = \{\sigma \in \mathbf{C} : \Im(\sigma) \le 0\}$*.*

3. Proof of theorem 1

The proof of Theorem 1 is divided into two parts.

Proof. Uniqueness: Let $\mathbf{v}$ be the difference of two solutions $\mathbf{v}_1$ and $\mathbf{v}_2$ of (2.4), then $\mathbf{v}$ satisfies (2.4) with $\mathbf{h} = 0$. Now, let $R > 0$ be such that $\partial\overline{B}(R)$ is contained in Ω and denoted by $\overline{\Omega}_R = \{x \in \mathbf{\Omega} : |x| \leq R\}$, the Bettis-Green formula (see for instance [11] or [13]) yields to

$$\int_{\Omega_R} \overline{\mathbf{v}} \cdot \widetilde{\mathbf{\Delta}}\mathbf{v} dx + \int_{\Omega_R} e(\overline{\mathbf{v}}, \mathbf{v}) dx = \int_{\partial\Omega_R} \overline{v} \cdot T_n v \, ds,$$

where

$$e(\overline{\mathbf{v}}, \mathbf{v}) = \frac{3a^2 - 4b^2}{3} |divv|^2 + \frac{b^2}{2} \sum_{p \neq q} \left| \frac{\partial v_p}{\partial x_q} + \frac{\partial v_q}{\partial xp} \right|^2 + \frac{b^2}{3} \sum_{p,q=1}^{3} \left| \frac{\partial v_p}{\partial x_p} - \frac{\partial v_q}{\partial xq} \right|^2$$

and

$$\widetilde{\mathbf{\Delta}} v = b^2 \, \Delta \, v + (a^2 - b^2) \, grad(div \;\; v).$$

Recall that $\widetilde{\mathbf{\Delta}}\mathbf{v} = -\sigma^2 \; v$ on $\Omega_R \subset \Omega$ and $\partial\overline{\Omega}_R = \partial\overline{B}(R) \cup \partial\Omega$. A direct calculation shows that

$$\begin{aligned} -\sigma^2 \textstyle\int_{\Omega_R} ||v||^2 dx + \int_{\Omega_R} e(\overline{v}, v) dx = \int_{\partial\overline{B}(R)} \overline{v} \cdot T_n v ds \\ - \textstyle\int_{\partial\Omega} \overline{v} \cdot T_n v ds. \end{aligned} \tag{3.1}$$

Now, using (3.1) toghether with Lemma 1, the homogeneus boundary condition and passing to the limit as $R \to \infty$ we get

$$\int_{\Omega} e(\overline{v}, v) \; dx = \sigma^2 \int_{\Omega} ||v||^2 dx,$$

so

$$\int_{\Omega} e(\overline{v}, v) dx = [(\Re(\sigma)^2 - \Im(\sigma)^2) - 2i\Re(\sigma)\Im(\sigma)] \int_{\Omega} ||v||^2 dx. \tag{3.2}$$

From (3.2) we obtain

$$0 = 2\Re(\sigma)\Im(\sigma) \int_{\Omega} ||v||^2 dx \tag{3.3}$$

and

$$\int_{\Omega} e(\overline{v}, v) dx = [\Re(\sigma)^2 - \Im(\sigma)^2] \int_{\Omega} ||v||^2 dx. \tag{3.4}$$

Thus, we have two possibilities

(a) If $\Re(\sigma) = 0$, from (3.4) we get

$$\int_\Omega e(\overline{v}, v)dx = [\Re(\sigma)^2 - \Im\ (\sigma)^2] \int_\Omega ||v||^2 dx.$$

With the formula above, $\Im(\sigma) > 0$ and

$$\int_\Omega e(\overline{v}, v)dx \geq 0,$$

it is easy to see that $v = 0$ on Ω. Similarly, (b) If $\Re(\sigma) \neq 0$, taking into account the fact that $\Im(\sigma) > 0$, from (3.3) we obtain

$$\int_\Omega ||v||^2 dx = 0.$$

Hence, $v = 0$ on Ω. Therefore, (a) and (b) implies $v_1 = v_2$. And the uniqueness is proved for all $\sigma \in C$ with $\Im(\sigma) > 0$.

Existence: Now we study the existence of solutions for the system (2.4), to this end, we assume that $\partial\Omega$ is sufficiently regular for the use of the Betti-Green formula. Let $R > 0$ and $R_0 > 0$ be such that $B(R_0) \subset D$, $\partial\Omega \subset B(R)$. We start with an arbitrary function $\zeta \in C_0^\infty(\mathbf{R}^3)$ satisfying

(ζ1) supp $\zeta \subset B(R)/B(R_0)$,

(ζ2) $\zeta = 1$, in a neighborhood of $\partial\Omega$

and

(ζ3) $\zeta = 0$, if $|x| = R$.

In order to analyze our existence problem we introduce here the following function

$$\mathbf{v}(x) = \mathbf{v}_0(x) + \zeta(x)\ \tilde{\mathbf{u}}(x), x \in \mathbf{R}^3, \tag{3.5}$$

where $\tilde{\mathbf{u}}$ is the Calderón extension to $\mathbf{R}^3$ of a (see for instance [12], Theorem 5.3.1) solution $\mathbf{w} \in [H^2(\mathbf{\Omega}_R)]^3$ of the system (see Lemma 3)

(w1) $b^2\, \mathbf{\Delta}\, \mathbf{w}(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\ \mathbf{w}(x)) = 0, \quad x \in \Omega_R,$

(w2) $\mathbf{w}(x) = -\mathbf{v}_0(x), \quad x \in \partial\Omega$

and

(w3) $\mathbf{w}(x) = 0, \quad x \in \partial\overline{B}(R).$

Here, $\mathbf{v}_0$ satisfies (see Lemma 2) the differential equations

$$b^2\,\boldsymbol{\Delta}\mathbf{v}_0(x) + (a^2-b^2)\,\mathrm{grad}(\mathrm{div}\ \mathbf{v}_0(x)) + \sigma^2\,\mathbf{v}_0(x) = \mathbf{f}_0(x), \quad x\in\mathbf{R}^3$$

and the Kupradze–Sommerfeld radiation condition. From (3.5) and (w2) we obtain

$$\mathbf{v}(x) = \mathbf{v}_0(x) + \mathbf{w}(x) = 0, \ \ x\in\partial\boldsymbol{\Omega}.$$

Furthermore, it is easy to see from (ζ1) and (3.5) that $\mathbf{v}(x) = \mathbf{v}_0(x)$, for every $x\in\mathbf{R}^3/\overline{B}(R)$. Now, the function $\mathbf{v}_0$ satisfies the Kupradze–Sommerfeld radiation condition (1.1) and (1.2). In view of this, the function $\mathbf{v}$ has this property. Thus, for any $\mathbf{h}\in[L^2(\boldsymbol{\Omega})]^3$ with supp $\mathbf{h}\subset\Omega_R$ and $\sigma\in\mathbf{C}$ with $\Im(\sigma)>0$ the function $\mathbf{v}(x) = \mathbf{v}_0(x) + \zeta(x)\ \tilde{\mathbf{u}}(x)$, $x\in\mathbf{R}^3$, will be a solution of the system (2.4) if only if, for every $x\in\Omega$, we obtain

$$\mathbf{h}(x) = b^2\,\boldsymbol{\Delta}\mathbf{v}(x) + (a^2-b^2)\,\mathrm{grad}(\mathrm{div}\ \mathbf{v}(x)) + \sigma^2\,\mathbf{v}(x) =$$

$$= \mathbf{f}_0(x) + b^2\,\boldsymbol{\Delta}\ \zeta(\mathbf{x})\tilde{\mathbf{u}}(\mathbf{x}) + (a^2-b^2)\,\mathrm{grad}(\mathrm{div}\ \zeta(x)\tilde{\mathbf{u}}(x)) + \sigma^2\,\zeta(x)\tilde{\mathbf{u}}(x). \tag{3.6}$$

It is simple to see from (ζ1), (ζ3) and (2.3) that (3.6) is valid on the set $\Omega^R = \{x\in\boldsymbol{\Omega} : |x|\geq R\}$, since supp $\mathbf{h}\subset\boldsymbol{\Omega}_R$. Thus,

$$\mathbf{v}(x) = \mathbf{v}_0(x) + \zeta(x)\tilde{\mathbf{u}}(x),\ x\in\mathbf{R}^3,$$

well be solution of the system (3.6) if only if, for every $x\in\Omega_R$, we have

$$\mathbf{h}(x) = \mathbf{f}(x) + b^2\,\boldsymbol{\Delta}\ \zeta(x)\mathbf{w}(x) + (a^2-b^2)\ \ \mathrm{grad}\ (\mathrm{div}\ \zeta(x)\ \mathbf{w}(x)) + \tag{3.7}$$
$$+\,\zeta(x)\ \mathbf{w}(x).$$

Applying to div $\mathbf{w}$ the operator grad, on Ω_R we find

$$\mathrm{rot}\ (\mathrm{rot}\ \mathbf{w}(x)) = -\boldsymbol{\Delta}\ \mathbf{w}(x) + \ \mathrm{grad}\ (\mathrm{div}\ \mathbf{w}(x)).$$

Now, $\mathbf{w}$ on Ω_R solves

$$b^2\,\boldsymbol{\Delta}\ \mathbf{w}(x) + (a^2-b^2)\,\mathrm{grad}\ (\mathrm{div}\ \mathbf{w}(x)) = 0.$$

Therefore, the ansatz (3.7) takes the form

$$\mathbf{h} = \mathbf{f} + \mathbf{G}_\zeta(\sigma)\mathbf{w}, \tag{3.8}$$

where $\mathbf{G}_\zeta(\sigma)$ is a continuous linear operator

$$\mathbf{G}_\zeta(\sigma) : [H^2(\mathbf{\Omega}_R)]^3 \to [H^1(\mathbf{\Omega}_R)]^3 \tag{3.9}$$

given by the formula

$$\begin{aligned} \mathbf{G}_\zeta(\sigma)\mathbf{w} &= (a^2+b^2)[(\text{grad } \zeta \cdot \text{grad) } \mathbf{w}] + [b^2 \triangle \zeta + \sigma^2 \zeta] \ \mathbf{w} + \\ &+ (a^2-b^2)[(\mathbf{w} \cdot \text{grad) grad } \zeta + \text{grad } \zeta \times \text{rot } \mathbf{w} + \text{grad } \zeta \text{ div } \mathbf{w}]. \end{aligned} \tag{3.10}$$

On the other hand, the solution operator $P(\sigma)$ associated with the system (w1-w3) above, that is, $P(\sigma)\ \mathbf{g} = \mathbf{w}$, where $\mathbf{g} = -\mathbf{v}_0 \in [H^{1/2}(\partial\mathbf{\Omega})]^3$, is well defined, of course, $P(\sigma)$ is a continuous linear operator

$$P(\sigma) : [H^{1/2}(\partial\mathbf{\Omega})]^3 \to [H^2(\mathbf{\Omega}_R)]^3, \tag{3.11}$$

and depends analytically from $\sigma \in \mathbf{C}$, because $\mathbf{v}_0$ has this property. In a similar fashion, the trace

$$\Lambda_n : [H^2(\mathbf{\Omega}_R)]^3 \to [H^{1/2}(\partial\mathbf{\Omega})]^3, \tag{3.12}$$

is a continuous linear operator. Thus, with this operators and taking into account the fact that $\mathbf{v}_0 = \mathbf{v}_{0|\mathbf{\Omega}_R}$ on $\mathbf{\Omega}_R$, (3.8) can be written in the form

$$\mathbf{h} = \mathbf{f} - \mathbf{G}_\zeta(\sigma)P(\sigma)\Lambda_n F_R(\sigma)\widetilde{A}(\sigma)\mathbf{f}, \tag{3.13}$$

where $F_R(\sigma)\mathbf{v}_0 = \mathbf{v}_{0|\mathbf{\Omega}_R}$,

$$F_R(\sigma) : [H^2(\mathbf{R}^3)]^3 \to [H^2(\mathbf{\Omega}_R)]^3 \tag{3.14}$$

is a restriction, continuous linear operator. Also,

$$\widetilde{A}(\sigma) : [L^2(\mathbf{\Omega}_R)]^3 \to [H^2(\mathbf{R}^3)]^3, \tag{3.15}$$

is a continuous linear operator given by the compotition

$\widetilde{A}(\sigma)\mathbf{r} = \mathbf{A}(\sigma)M_\psi\mathbf{r}$, where $A(\sigma)$ it is the solution operator of the system $b^2\,\mathbf{\Delta v}_0(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\ \mathbf{v}_0(x)) + \sigma^2\,\mathbf{v}_0(x) = \mathbf{f}_0(x),\ \ x \in \mathbf{R}^3$, (see Lemma 2) and M_ψ it is the multiplication operator

$$(M_\psi\mathbf{r})(x) = \begin{cases} \mathbf{r}(x), & \text{if } \ x \in \Omega_R, \\ 0, & \text{if } \ x \notin \Omega_R. \end{cases}$$

Note that

$$\|M_\psi\mathbf{r}\|^2_{[L^2(\mathbf{R}^3)]^3} = \|\psi\mathbf{r}\|^2_{[L^2(\mathbf{R}^3)]^3} = \int_{\mathbf{R}^3} |\psi|^2 ||\mathbf{r}||^2 dx = \int_{\mathbf{\Omega}_R} |\psi|\ ||\mathbf{r}||^2 dx < \infty.$$

Therefore, M_ψ is a continuos linear operator $M_\psi : [L^2(\mathbf{\Omega}_R)]^3 \to [L^2_R(\mathbf{R}^3)]^3$, since, $M_\psi\mathbf{r} = 0$, if $|x| \geq R$. Thus, $M_\psi\mathbf{r} \in [L^2_R(\mathbf{R}^3)]^3$ for every $\mathbf{r} \in [L^2(\mathbf{\Omega}_R)]^3$. Let $B_\zeta(\sigma)$be the operator defined by

(3.16) $$B_\zeta(\sigma)\mathbf{f} = -\mathbf{G}_\zeta(\sigma)P(\sigma)\Lambda_n F_R(\sigma)\widetilde{A}(\sigma)\mathbf{f}.$$

Thus, (3.13) can be written as

(3.17) $$\mathbf{h} = \mathbf{f} + B_\zeta(\sigma)\mathbf{f}.$$

From these considerations we see that the theorem will be proved if:

(I) The set of operators $\{B_\zeta(\sigma)\}$, $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$ given in (3.16) is a family of compact operators of $[L^2(\mathbf{\Omega}_R)]^3$onto itself, and

(II) the homogeneous equation $\mathbf{f} + B_\zeta(\sigma)\mathbf{f} = 0$, has only the trivial solution.

Proof.

(I): We denote by $S_g \subset [H^2(\mathbf{\Omega}_R)]^3$ the space ($g = -\mathbf{v}_0 \in [H^{1/2}(\partial\mathbf{\Omega})]^3$) of solutions of the system

$$\begin{cases} b^2\,\mathbf{\Delta}\,\mathbf{w}(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\ \ \mathbf{w}(x)) = 0, & x \in \Omega_R, \\ \mathbf{w}(x) = -\mathbf{v}_0\,(x)\,, & x \in \partial\Omega \\ \mathbf{w}(x) = 0, & x \in \partial\overline{B}\,(R). \end{cases}$$

Now, the definitions for $\mathbf{G}_\zeta(\sigma)$, $P(\sigma)$, Λ_n, $F_R(\sigma)$, $\widetilde{A}(\sigma)$, given in (3.10), (3.11), (3.12), (3.14) and (3.15), the ansatz (3.16) together with the Rellich type compactness theorem ($i : [H^1(\boldsymbol{\Omega}_R)]^3 \to \ [L^2(\boldsymbol{\Omega}_R)]^3$) yield to (II). $\square$

We are now ready to prove (II).

Proof. Take $\mathbf{f} \in [L^2(\boldsymbol{\Omega}_R)]^3$ such that

$$\mathbf{f} + B_\zeta(\sigma)\mathbf{f} = 0. \tag{3.18}$$

Then, the equation (3.17) yields to $\mathbf{h} = 0$. Therefore, the funcion $\mathbf{v}$ is solution of homogeneus system

$$\begin{cases} \mathbf{b}^2\, \boldsymbol{\Delta}\, \mathbf{v}(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\ \ \mathbf{v}(x)) + \sigma^2\mathbf{v}(x) = 0, & x \in \Omega, \\ \mathbf{v}\,(x) = 0, & x \in \partial\Omega \\ \text{Radiation condition.} & \end{cases}$$

This implies that $\mathbf{v} = 0$ on Ω (see uniqueness above). Hence, in particular we obtain

$$-\,\zeta\ \tilde{\mathbf{u}} = \mathbf{v}_0, \text{on } \Omega. \tag{3.19}$$

Now, from (3.19) we obtain $\mathbf{v}_0 = 0$ on $\Omega^R = \{x \in \mathbf{R}^3 : |x| > R\}$, since supp $\zeta \subset B(R)/B(R_0)$. Moreover, $\zeta = 0$on $\partial\overline{B}(R)$ implies $\mathbf{v}_0 = 0$ on $\partial\overline{B}(R)$. We now introduce on $\overline{B}(R)$ the following function

$$\vartheta(x) = \chi(x)\mathbf{v}_0(x) + (1 - \chi(x))\tilde{\mathbf{u}}(x), \tag{3.20}$$

where

$$\chi(x) = \begin{array}{l} 1, \text{ if } x \in \overline{D}, \\ 0, \text{ if } x \in \Omega_R\,.\cup\partial\overline{B}\,(R)\,. \end{array}$$

We note that $\vartheta \in [H^2(B(R))]^3$. Furthermore, $\vartheta(x) = \mathbf{v}_0(x)$, $x \in D$. Now, in a neighborhood of D we have $\mathbf{f}_0(x) = 0$, since $\psi(x) = 0$ when $x \notin \Omega_R$. This yields to $b^2\,\boldsymbol{\Delta}\vartheta\,(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\ \ \vartheta(x)) = -\sigma^2\mathbf{v}_0(x)$, if $x\in D$. Also, $\vartheta(x) = \tilde{\mathbf{u}}(x)(x) = \mathbf{w}(x)$, if $x \in \Omega_R$. Therefore, $b^2\,\boldsymbol{\Delta}\vartheta\,(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\ \ \vartheta(x)) = 0$, here. Note also that $\vartheta(x) = 0$ on $\partial\overline{B}(R)$, because $\mathbf{v}_0(x) = \tilde{\mathbf{u}}(x) = 0$ on $\partial\overline{B}(R)$. Now, using the Bettis-Green formula on $B(R)$, we obtain

$$\int_{B(R)} \overline{\vartheta}\cdot\widetilde{\boldsymbol{\Delta}}\vartheta dx + \int_{B(R)} e(\overline{\vartheta},\vartheta)dx = -\int_{\partial\ \overline{B}(R)} \overline{\vartheta}\cdot\mathbf{T}_n\vartheta\ ds = 0,$$

ie.,

$$\int_{B(R)} e(\overline{\vartheta}, \vartheta)dx = \sigma^2 \int_D \|\mathbf{v}_0\|^2 \, dx. \tag{3.21}$$

Thus,

$$\int_{B(R)} e(\overline{\vartheta}, \vartheta)dx = [\Re(\sigma)^2 - \Im(\sigma)^2] \int_D ||\mathbf{v}_0||^2 dx \tag{3.22}$$

and

$$0 = 2\Re(\sigma)\Im(\sigma) \int_D ||\mathbf{v}_0||^2 dx. \tag{3.23}$$

If $\Re(\sigma) \neq 0$, of (3.23) and $\Im(\sigma) > 0$ it is easy to see that $\mathbf{v}_0 = 0$, on D, since $\int_{B(R)} e(\overline{\vartheta}, \vartheta)dx \geq 0$. Therefore, for all $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$,the function $\vartheta \in [H^2(B(R))]^3$ in the ansatz (3.20) solves the system

$$\begin{array}{ll} b^2\, \mathbf{\Delta}\vartheta\,(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\;\; \vartheta(x) = 0, & x \in B(R), \\ \vartheta\,(x) = 0 & x \in \partial\overline{B}(R). \end{array}$$

Now, thanks to Lemma 4, we obtain $\vartheta(x) = 0$, for any $x \in B(R)$, i.e., $\tilde{\mathbf{u}}(x) = 0$, on Ω_R. From this together with $-\zeta(x)\;\tilde{\mathbf{u}}(x) = \mathbf{v}_0(x)$, for all $x \in \Omega_R \subset \Omega$. Thus,

$$0 = b^2 \mathbf{\Delta} v_0(x) + (a^2 - b^2)\,\mathrm{grad}\;(\mathrm{div}\;\mathbf{v}_0(x)) + \sigma^2\,\mathbf{v}_0(x) = \mathbf{f}(x), x \in \Omega_R.$$

Now, from the Fredholm theory, the equation $\mathbf{f} + B_\zeta(\sigma)\mathbf{f} = \mathbf{h}$ is uniquely solvable and proof is complet. □

4. Meromorphic Extension

In the previous sections the existence and uniqueness of solutions for the system

$$\left\{ \begin{array}{ll} b^2\,\mathbf{\Delta}\,\mathbf{v}(x) + (a^2 - b^2)\,\mathrm{grad}(\mathrm{div}\;\;\mathbf{v}(x)) + \sigma^2\,\mathbf{v}(x) = \mathbf{h}(x), & x \in \Omega, \\ \mathbf{v}(x) = 0, & x \in \partial\Omega, \\ \text{Radiation condition,} & \end{array} \right. \tag{4.1}$$

with $\sigma \in \mathbf{C}$ such that $\Im(\sigma) > 0$ is proved. Now, in this section we present the extension of the solution for all $\sigma \in \mathbf{C}$ such that $\Im(\sigma) \leq 0$ except, for some countable number of complex singularities, called "resonant frequencies". Our approach follows the main ideas of the previous sections and the subjet iniciated in [5] and [6], but it is related to some other works mainly [3], [4], [17] among other. The basic tools for the proof is the Steinberg theorem [20] about families of compact operators depending on a complex parameter (see, also [19]). With the same notations of the section §2 and §3, we stablish the following

Lemma 5. *Let $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$. Fix $\zeta \in C_0^\infty(\mathbf{R}^3)$, with properties $(\zeta 1 - \zeta 3)$ (see section §2). Then for any $\mathbf{h} \in [L^2(\mathbf{\Omega})]^3$ such that supp $\mathbf{h} \subset \Omega_R$ the function $\mathbf{v}(x) = \mathbf{v}_0(x) + \zeta(x)\tilde{\mathbf{u}}(x)$, $x \in \mathbf{R}^3$, solves the system (4.1) if only if, $\mathbf{f} \in [L^2(\mathbf{\Omega}_R)]^3$ solves*

$$\mathbf{h} = \mathbf{f} + B_\zeta(\sigma)\mathbf{f}. \tag{4.2}$$

Here, $B_\zeta(\sigma)$ is given by

$$B_\zeta(\sigma)\mathbf{f} = -\mathbf{G}_\zeta(\sigma)P(\sigma)\Lambda_n F_R(\sigma)\widetilde{A}(\sigma)\mathbf{f}, \tag{4.3}$$

where the operators $\mathbf{G}_\zeta(\sigma)$, $P(\sigma)$, Λ_n, $F_R(\sigma)$, $\widetilde{A}(\sigma)$ are given in (3.10), (3.11), (3.12), (3.14) and (3.15) respectively.

Proof. The proof is implicit in Theorem 1. $\square$

Lemma 6. *The set operators $\{B_\zeta(\sigma)\}$, $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$ given in (3.16) is an analytic family of compact operators of $[L^2(\mathbf{\Omega}_R)]^3$ onto itself.*

Proof. *Since the solution $\mathbf{v}_0$ from system*

$$b^2\mathbf{\Delta}\mathbf{v}_0(x) + (a^2 - b^2)\,\textit{grad}\ (\textit{div}\ \mathbf{v}_0(x)) + \sigma^2\,\mathbf{v}_0(x) = \mathbf{f}_0(x),\ x \in \mathbf{R}^3.$$

depend analitically of $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$, the operators $\mathbf{G}_\zeta(\sigma)$, $P(\sigma)$, Λ_n, $F_R(\sigma)$, $\widetilde{A}(\sigma)$ given in (3.10), (3.11), (3.12), (3.14) and (3.15) have this property. From this and (4.1), the operators $\{B_\zeta(\sigma)\}$ depend analitically of $\sigma \in \mathbf{C}$. The compactness follow from (I) above. $\square$

Theorem 2. *The inverse operators $[\mathbf{I}+B_\zeta(\sigma).]^{-1}$ have an analytic extension from $\Im(\sigma) > 0$ to all the complex plane except foe a countable set of poles, called resonant frequencies. Furthermore, σ is a resonant* *frequency of the operator* $[\mathbf{I} + B_\zeta(\sigma).]^{-1}$ *if and only if the system* (4.1) *with* $\mathbf{h} = 0$ *has non zero solutions.*

Proof. From Lemma 6 we have that the set $\{B_\zeta(\sigma)\}$ with $\sigma \in \mathbf{C}$ and $\Im(\sigma) > 0$ is a analityc family of compact operators of $[L^2(\mathbf{\Omega}_R)]^3$onto itself. By the Steinberg theorem **??**, either (a) the operators $[\mathbf{I} + B_\zeta(\sigma)]^{-1}$ are never invertible for $\sigma \in \mathbf{C}$, or (b) there is $\sigma_0 \in \mathbf{C}$ such that the operator $[\mathbf{I} + B_\zeta(\sigma_0)]^{-1}$ is invertible. From Theorem 1 we have the existence and uniqueness of the solution for the system (4.1) for all $\sigma \in \mathbf{C}$ with $\Im(\sigma) > 0$, by the equivalence established in Lemma 5 we are in the (b) case. In this case, Steinberg Theorem also states that $[\mathbf{I} + B_\zeta(\sigma)]^{-1}$ is defined analytically on $\mathbf{C}$ except for a countable set of poles. Now, Lemma 5 yields to the equivalence stament. □

Acknowledgement 1. *The author is very grateful to his thesis advisor Prof. Dr. Gustavo Alverto Perla Menzala for suggesting the subjet and his useful discussions.*

References

[1] J. F Ahner, The exterior time harmonic elasticity problem with prescribed displacement vector on the boundary, Arch. Math. **27**, pp. 106–111, (1976).

[2] J.F Ahner and G.C. Hsiao, On the two dimensional exterior boundary value problems of elasticity, J. Apll. Math. **31**, pp. 677-685, (1976).

[3] M.A. Astaburruaga and R. Coimbra Charao, C. Fernández and G. Perla Menzala, Scattering Frequencies for a perturbed system of elastic wave equations, J. Math. Anal Appl. **219**, pp. 52–75, (1998).

[4] R. Coimbra Charao and G. Perla Menzala, Scattering Frequencies and a class of perturbed systems of elastic waves, Math. Meth. In the Appl. Sci. **19**, pp. 699–716, (1996).

[5] L. A. Cortés–Vega, The exerior time-harmonic elasticity problem with prescribed stress- traction operator on the boundary, J. Math. Anal. Appl. (submitted).

[6] L. A. Cortés–Vega, " Frequências de espalhamento e a propagacão de ondas elásticas no exterior de um corpo tridimensional ". Ph. D. Thesis, (In portuguesse). University of São Paulo, Outubro–2000.

[7] G. Duvaut and J. L. Lions, "Les inequations in Mecanique et en Physique ", Dunond, Paris, (1972).

[8] G. Fichera, " Existence theorems in elasticity ", Handbuch der Physik, Springer-Verlag, Berlin, Heidelberg, New York, (1973).

[9] P. Hähner and G.C Hsiao, Uniqueness theorems in inverse obstacle scattering of elastic waves, Inverse Prob. **9**, pp. 525–534, (1993).

[10] P. Hähner, A uniqueness theorem in inverse scattering of elastic waves, IMA J. Of Appl. Math. **51**, pp. 201–215, (1993).

[11] V.D. Kupradze, " Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity", Amsterdam, North–Holand, (1973).

[12] J. T. Marti, " Introduction to Sobolev spaces and finite element solution of elliptic boundary value problems ", Academic Press, New York, (1986).

[13] M. Costabel and E.P. Stephan, Integral equations for transmission problems in linear elasticity, J. Int. Eq. and Appl. **2**, pp. 211–223, (1996).

[14] H. J–P. Morand and R. Ohayon, " Fluid–Structure interactions ". John Wiley–Sons, New York, (1995).

[15] J. Necăs, " Les méthodos directes em théoria des équations elliptiques ", Masson, Paris, (1967).

[16] Y–H. Pao, F. Santosa, W.W. Symes and C. Holland, eds., " Inverse problems of acoustic and elastic waves ". SIAM, (1984).

[17] O. Poisson, Calcul des pôles de résonance associés à la diffraction d'ondes acoustiques par un obstacle en dimension deux, C. R. Acad. Sci. Paris, **I. 315**, pp. 747–752, (1992).

[18] A. G. Ramm, " Scattering by obstacles ", Riedel, Dordrecht, (1986).

[19] R. T. Seeley, Integral equations depending analytycally on parameter, Indag. Math. **24**, pp. 434–442, (1962).

[20] S. Steinberg, Meromorphic families of compact operators, Arch. Rational. Mech. Anal. **31**, pp. 372–379, (1968).

Received : July, 2001.

Luis Cortés Vega
Departamento de Matemáticas
Facultad de Ciencias
Universidad del Bío–Bío
Casilla 5–C
Concepción
Chile
email : lcortes@triton.ciencias.ubiobio.cl