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DIFFERENTIABILITY OF SOLUTIONS OF THE ABSTRACT CAUCHY PROBLEM

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Abstract

In this note we establish a result of differentiability for the mild solution of the inhomogeneous abstract Cauchy problem when the underlying space is reflexive.

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1. Introduction.

In this work we are concerned with regularity properties of solutions of the first order abstract Cauchy problem (in short, ACP). We refer the reader to [3, 10] for the theory of strongly continuous semigroup operators and the associated ACP.

Let X be a Banach space endowed with a norm $\|\cdot\|$. Henceforth $T(t)$ is a strongly continuous semigroup of operators on X with infinitesimal generator A.

The existence of solutions of the first order abstract Cauchy problem

$$x'(t) = Ax(t) + h(t), \quad 0 \le t \le a, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

it has been treated in several works. We only mention here the texts [3, 10] and the references cited therein. Similarly, the existence of solutions of the semilinear abstract Cauchy problem it has been discussed in [1, 9].

Assuming that $h : [0, a] \to X$ is integrable the function given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)h(s)\,ds, \quad 0 \le t \le a, \tag{1.3}$$

is said mild solution of (1.1)-(1.2). In the case in which h is continuous , the function $x(\cdot)$ is called a classic solution on $[0, a]$ of (1.1)-(1.2) if x is a function of class C^1, $x(t) \in D(A)$ and (1.1) is verified.

The existence of classical solutions of (1.1)-(1.2) as well as some weaker forms of differentiability of solutions have been studied in a number of works. We refer the reader to [2, 8, 10, 12, 13, 14] and the references therein indicated.

The purpose of this note is to establish a new condition in order to the mild solution $x(\cdot)$ turn to be a classical solution .

Next the notation $C([0, a]; X)$ stands for the space of continuous functions from $[0, a]$ into X, whilst $BV([0, a]; X)$ represents the space of functions with bounded variation from $[0, a]$ into X. For a function $h \in BV([0, a]; X)$ we denote by $V(h)$ the variation of h on $[0, a]$ and by $v(t, h)$ the variation of h on $[0, t]$, for $0 \le t \le a$. Additional terminology and notations are those generally used in functional analysis. In particular, X^* denotes the dual space of X.

2. Results.

In this section h denotes a continuous function of bounded variation on a fixed interval $[0,a]$, $a>0$. We define the translation of h by

$$\mathcal{T}_t h(s) = \begin{cases} h(s+t), & s \le a-t, \\ h(a), & s \ge a-t, \end{cases}$$

for $t \ge 0$. Let $\mu(t,h) = V(\mathcal{T}_t h - h)$.

We introduce the following condition for a function $h \in C([0,a];X) \cap BV([0,a];X)$.

(H_0) $\mu(t,h) \to 0$, as $t \to 0^+$.

Initially we discuss some examples.

Example 1. If $h \in W^{1,1}([0,a];X)$, then $\mu(t,h) \to 0$, $t \to 0^+$.

Example 2. Let $h : [0,1] \to \mathbb{R}$ be the function defined in [4], Exercise 4.19. Let E be a perfect nowhere dense set with measure 0 included in $[0,1]$. Let (a_k,b_k), $k \in \mathbb{N}$, be disjoint intervals such that $(0,1) \setminus E = \bigcup_{k=1}^{\infty}(a_k,b_k)$ and let $\sum_{k=1}^{\infty} c_k$ be a convergent series of positive number with sum equal to 1. For each $x \in [0,1]$ let

$$I(x) = \{k : [a_k,b_k] \cap [0,x] \ne \Phi\}$$

and define

$$h(x) = \sum_{k \in I(x)} c_k. \tag{2.1}$$

It is clear that $h(0)=0$, $h(1)=1$. Moreover, h is continuous and nondecreasing with $h'=0$, a.e. Thus h is a singular function . Now we establish that h does not satisfy (H_0). In fact, from (2.1) it follows easily that for each $t>0$ and $0 \le s \le 1-t$,

$$h(s+t) - h(s) = \sum_{k \in I} c_k, \tag{2.2}$$

where $I = \{k \in \mathbb{N} : a_k \in (s, s+t]\}$. In addition, for $n \in \mathbb{N}$ we can choose $t > 0$ small enough such that $\bigcup_{i=1}^{n}[a_i, b_i] \subseteq [0, 1-t]$, $a_i + 3t < b_i$, and, for each $k = 1, \cdots, n$, $a_k - t \notin \bigcup_{i=1,\, i\neq k}^{n} [a_i, b_i]$.

Defining $\alpha_i = a_i - t/2$ and $\beta_i = a_i + 2t$ it follows from (2.2) that $h(\alpha_i + t) - h(\alpha_i) = c_i$ and $h(\beta_i + t) - h(\beta_i) = 0$. From this we obtain that

$$\begin{aligned} V(\mathcal{T}_t h - h) &\geq \sum_{i=1}^{n} |(\mathcal{T}_t h - h)(\beta_i) - (\mathcal{T}_t h - h)(\alpha_i)| \\ &= \sum_{i=1}^{n} |h(\beta_i + t) - h(\beta_i) - (h(\alpha_i + t) - h(\alpha_i))| \\ &= \sum_{i=1}^{n} c_i \end{aligned}$$

which implies that $\mu(t, h)$ does not converge to 0 as $t \to 0^+$.

Example 3. Let $h : [0, 1] \to \mathbb{R}$ be the singular function defined in [6], Example 18.8. As above, $h(0) = 0$, $h(1) = 1$, h is continuous and strictly increasing and $h' = 0$, a.e. We will show that this function satisfies the assumption (H_0). Initially, for completeness we include here the construction carried out in [6].

Let $(t_n)_n$ be a sequence in $(0, 1)$. Set $F_1(0) = 0$, $F_1(1) = 1$, $F_1(\frac{1}{2}) = \dfrac{1 + t_1}{2}$ and define F_1 to be linear on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Suppose that F_1, $F_2, \cdots, F_n$ have been defined. Then define

$$F_{n+1}\left(\frac{k}{2^n}\right) = F_n\left(\frac{k}{2^n}\right), \quad \text{for } k = 0, 1, \cdots, 2^n,$$

$$\mathrm{F}_{n+1}\left(\tfrac{2k+1}{2^{n+1}}\right) = \tfrac{1-t_{n+1}}{2} F_n\left(\tfrac{k}{2^n}\right) + \tfrac{1+t_{n+1}}{2} F_n\left(\tfrac{k+1}{2^n}\right),$$

for $k = 0, 1, \cdots, 2^n - 1$,
and complete the definition of F_{n+1} as a continuous linear function in the intervals $[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}]$, for $k = 0, 1, \cdots, 2^{n+1} - 1$. It

is shown in [6] that $(F_n)_n$ is a nondecreasing sequence. Thus this sequence converges to a function h which satisfies the properties already mentioned. Applying now the Dini's theorem ([11]) we obtain that the convergence of $(F_n)_n$ is uniform. Since $(\mathcal{T}_t F_n - F_n)_n$ converges uniformly to $\mathcal{T}_t h - h$, as $n \to \infty$, and this convergence is also uniform on $t \geq 0$, it is follows that $\mu(t, F_n) \to \mu(t,h)$, $n \to \infty$, and this convergence is uniform on t. In view of F_n is absolutely continuous , from Example 1 we infer that $\mu(t, F_n) \to 0$, as $t \to 0^+$, which implies that $\mu(t,h) \to 0$, as $t \to 0^+$.

To study the regularity of solutions of the abstract Cauchy problem (1.1)-(1.2) we begin by establishing some preliminary lemmas.

In the sequel we denote by M a positive constant such that $\|T(t)\| \leq M$, $0 \leq t \leq a$. Moreover, for a fixed h, we use the notation

$$u(t) = \int_0^t T(t-s)h(s)\,ds, \tag{2.3}$$

Lemma 2.1. *Assume that X is a reflexive space . Let $T(\cdot)$ be a strongly continuous semigroup of operators on X and let $h : [0,a] \to X$ be a continuous function of bounded variation which satisfies the assumption (H_0). Then the Riemann-Stieltjes integral*

$$w(t) = \int_0^t T(t-s)\,d_s h = \int_0^t T(s)\,d_s h(t-s)$$

exists in the weak topology and define a continuous function $w : [0,a] \to X$.

Proof. Let $\Lambda : X^* \to \mathbb{C}$ be defined by

$$\Lambda(x^*) = \int_0^t < T(t-s)^* x^*,\, d_s h > .$$

The Riemann-Stieltjes integral in the above expression exists because $T(\cdot)^* x^*$ is a continuous function ([10]) and h has bounded variation ([7]). Moreover, Λ is linear and

$$|\Lambda(x^*)| \leq M \|x^*\| V(h).$$

Consequently, $\Lambda \in X^{**}$ and in view of that X is reflexive we infer the existence of $w(t) \in X$ such that $\Lambda(x^*) =< x^*, w(t) >$, for all $x^* \in X^*$.

On the other hand, for $t < 1$ and τ small enough, from the relations

$$\begin{aligned} w(t+\tau) - w(t) &= \int_0^{t+\tau} T(s)\, d_s h(t+\tau-s) - \int_0^t T(s)\, d_s h(t-s) \\ &= \int_0^t T(t-s)\, d_s[h(\tau+s) - h(s)] \\ &\quad + \int_0^\tau T(t+s)\, d_s h(\tau - s) \end{aligned}$$

we deduce that

$$\|w(t+\tau) - w(t)\| \leq M\mu(\tau,h) + Mv(\tau,h).$$

Since $\mu(\tau,h) \to 0, \tau \to 0,$ because the condition (H_0) holds and $v(\tau,h) \to 0,\ \tau \to 0,$ by the Proposition I.2.9 in [7]) the previous estimation shows that $w(\cdot)$ is right continuous at t. Similarly, one can prove that w is left continuous at $t > 0$.

Next we denote by χ_E the characteristic function of a set E.

Lemma 2.2. *Let $h : [0,a] \to X$ be the step function $h = \sum_{i=1}^n x_i \chi_{I_i}$, where I_i are intervals and $\{I_1, \cdots, I_n\}$ is a partition of $[0,a]$. Then the function u given by (2.3) is piecewise smooth, $u(t) \in D(A)$, $Au(\cdot)$ is continuous on $[0,a]$ and $u'(t) = Au(t) + h(t)$, $t \notin P$, where P denotes the set formed by the extreme points of intervals I_i, $i = i, \cdots, n$.*

Proof. Applying the linearity of u in terms of h, it is sufficient to prove the assertion for a function $h = x\chi_I$ where I is an interval of type $[t_1, t_2]$. In fact, in this case, $u(t)$ is given by

$$u(t) = \begin{cases} 0, & 0 \leq t \leq t_1, \\ \displaystyle\int_0^{t-t_1} T(s)x\, ds, & t_1 \leq t \leq t_2, \\ \displaystyle\int_{t-t_2}^{t-t_1} T(s)x\, ds, & t_2 \leq t. \end{cases}$$

From the properties of semigroups we infer that $u(t) \in D(A)$ and that

$$Au(t) = \begin{cases} 0, & 0 \leq t \leq t_1, \\ T(t-t_1)x - x, & t_1 \leq t \leq t_2, \\ T(t-t_1)x - T(t-t_2)x, & t_2 \leq t. \end{cases}$$

This shows that $Au(\cdot)$ is continuous . Moreover, it is immediate to verify that $u'(t) = Au(t) + h(t),\ t \neq t_1,\ t_2$.

Now we can prove the main result of this note.

Theorem 2.1. *Assume that X is a reflexive space and let h be a continuous function of bounded variation on $[0, a]$ which satisfies assumption (H_0). Let $x_0 \in D(A)$. Then the mild solution of (1.1)-(1.2) is a classical solution .*

Proof. We consider a sequence $(h_n)_n$ of step functions , where each h_n is given by

$$h_n = \sum_{i=1}^{n} h(t_i)\chi_{I_i}.$$

In this expression we have denoted $I_i = [t_{i-1}, t_i),\ i = 1, \cdots, n-1,$ and $I_n = [t_{n-1}, t_n],$ where the points t_i have been chosen as $t_i = \frac{a}{n} i,\ \ i = 0,\ 1, \cdots, n.$

It is clear that the sequence $(h_n)_n$ converge uniformly to h. Let u_n be the function given by (2.3), with h_n instead of h. Then, $u_n \to u,\ n \to \infty,$ uniformly on $[0, a]$. Moreover, by Lemma 2.2 we have that $u_n(t) \in D(A)$ and, if we fix $0 \leq t \leq a$ and $n \in \mathbb{N}$, then $t \in I_k$, for some $k = 1, \cdots, n$. From our definitions we can write

$$\begin{aligned} Au_n(t) &= A\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} T(t-s)h(t_i)\,ds \ + \ A\int_{t_{k-1}}^{t} T(t-s)h(t_k)\,ds \\ &= \sum_{i=1}^{k-1}[T(t-t_{i-1}) - T(t-t_i)]h(t_i) \ + \ [T(t-t_{k-1}) - I]h(t_k) \\ &= \sum_{i=1}^{k-1} T(t-t_{i-1})[h(t_i) - h(t_{i-1})] \ + \ T(t-t_{k-1}) \end{aligned}$$

$$
\begin{aligned}
& [h(t)-h(t_{k-1})]+T(t-t_{k-1})[h(t_k)-h(t)]\ + \\
(2.4) \qquad & T(t)h(0)-h(t_k)
\end{aligned}
$$

so that

$$\|Au_n(t)\| \leq MV(h)\ +\ (M+1)\|h\|_\infty.$$

This shows that $(Au_n(t))_n$ is a bounded sequence. Consequently, there is a subsequence which converges to $z(t) \in X$ in the weak topology. Moreover, from (2.4) it follows that

$$z(t) = w(t) - h(t) + T(t)h(0).$$

An standard argument shows that the full sequence $(Au_n(t))_n$ converges to $w(t)$. As A is a closed operator this implies that $u(t) \in D(A)$ and $z(t) = Au(t)$.

An application of Lemma 2.1 yields that $Au(\cdot)$ is a continuous function . On the other hand, from Lemma 2.2 we have

$$u_n'(t) = Au_n(t)\ +\ h_n(t),\ \ n \in I\!N,\ \ t \neq i/n,\ i = 1, \cdots, n-1,$$

so that for each $x^* \in X^*$ we obtain

$$< x^*, u_n(t) >= \int_0^t < x^*, Au_n(s) + h_n(s) >\ ds$$

and taking limit as $n \to \infty$, it follows that

$$< x^*, u(t) >= \int_0^t < x^*, Au(s) + h(s) >\ ds$$

which implies that

$$u(t) = \int_0^t Au(s)\, ds\ +\ \int_0^t h(s)\, ds.$$

This shows that $u(\cdot)$ is a function of class C^1 that satisfies (1.1)-(1.2).

A similar result holds for the second order abstract Cauchy problem ([5]).

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