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RELATIVE INVARIANCE FOR MONOID ACTIONS

CARLOS BRAGA *

Universidade de Maringá, Brasil

Abstract

Let S be a topological monoid acting on the topological space M . Let J be a subset of M . Our purpose here is to study the subsets of M which correspond, under the action of S , to the relative (with respect to J) invariant control sets for control systems (see [4] section 3.3). The relation $x \sim y$ if $y \in \text{cl}(Sx)$ and $x \in \text{cl}(Sy)$ is an equivalence relation and the classes with respect to this relation with nonempty interior in M are the control sets for the action of S . It is given conditions for the existence and uniqueness of relative invariant classes. As it was done for the control sets, we define an order in the classes and relate it to the relative invariant classes. We also show under certain condition that the relative invariant classes are relatively closed in J .

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1. Introduction

One of the principal dynamical concepts in control theory is the study of the controllability of the control systems. Many questions about the control system, especially those related to its controllability depend, in fact, only on the action of the semigroup of the system, so that it can be abstracted to arbitrary semigroup actions and solved in a more general setting. The regions of the state space where the controllability occurs are called control sets. The control sets for control systems were studied by Colonius and Kliemann in [1],[2],[3] and [4]. In particular, Colonius and Kliemann introduced the concept of a control set which is relatively invariant with respect to a subset of the phase space of the control system (see Definition 3.1.9, pg. 50, in [4]). From a more general point of view, the theory of control sets for semigroup actions was developed by San Martin and Tonelli (see [6],[7] and [8]). Let S be a topological monoid and suppose that S acts on a topological space M . Since the control sets are the regions where S is approximate transitive it is natural to define an equivalence relation by saying that two points are equivalent if they are approximate attainable by the action of S . We consider equivalence classes in M with respect to this relation. We show that a class with nonempty interior in M is a control set for S . The purpose of this paper is to study the relative invariant classes in M . We define relatively invariant classes. In case S is the system semigroup of a control system these relatively invariant classes, with nonempty interior in M , are the relative invariant control sets defined by Colonius and Kliemann. We develop the theory of relative invariant classes. As it was done for the control sets, we define an order in the classes and relate it to the relative invariant classes. We give conditions for the existence and uniqueness of relative invariant classes. Under the hypothesis of accessibility, we show that a relative invariant class is relatively closed.

2. Relative invariance

First, we define the concept of a topological semigroup. We refer to [9] for the theory of topological semigroups. Throughout this paper we always assume that all topological spaces involved are Hausdorff.

Definition 1. Let S be a non-void topological space which is provided with an associative multiplication

$$\begin{aligned}\mu : S \times S &\rightarrow S \\ (x, y) &\mapsto \mu(x, y) = xy\end{aligned}$$

Then S is called a *topological semigroup* if μ is continuous as a map between the product space $S \times S$ and the space S . If S has an identity 1 then S is called a *topological monoid*.

Definition 2. Let M be a topological space and S a topological monoid. We say that S acts on M as a *transformation monoid* if there is a map

$$\begin{aligned}\pi : S \times M &\rightarrow M \\ (g, x) &\mapsto \pi(g, x) = g.x\end{aligned}$$

satisfying the following conditions:

1. π is continuous as a map between the product space $S \times M$ and the space M .
2. $g.(h.x) = (gh).x$ for all $g, h \in S$ and for all $x \in M$.
3. $1.x = x$ for all $x \in M$.

We suppose, in this paper, that S acts on a topological space M as a transformation monoid. We use the notation $Sx = \{gx : g \in S\}$ and call it the *orbit* of x under S . Assume that $x \in \text{cl}(Sy)$. Then

$$Sx \subset S\text{cl}(Sy) \subset \text{cl}(SSy) = \text{cl}(Sy)$$

because the action is continuous. Therefore $\text{cl}(Sx) \subset \text{cl}(Sy)$ if $x \in \text{cl}(Sy)$. We will use this fact frequently on the text.

The control sets for semigroup actions were defined by San Martin and Tonelli in [6]. Here, we recall the definition. Let S be a topological monoid acting on a topological space M .

Definition 3. A *control set* for S on M is a subset $D \subset M$ which satisfies

1. $\text{int}(D) \neq \emptyset$,
2. For every $x \in D$, $D \subset \text{cl}(Sx)$ and
3. D is maximal with these properties.

The action on M induces the pre-order relation defined by

$$x \preceq y \text{ if } y \in \text{cl}(Sx), \ x, y \in M.$$

Definition 4. We define $x \sim y$ if $x \preceq y$ and $y \preceq x$.

Therefore $[x] = [y]$ if and only if $y \in \text{cl}(Sx)$ and $x \in \text{cl}(Sy)$. Thus $\sim$ is the equivalence relation associated with $\preceq$. The pre-order in M induces a partial order in the quotient space $M / \sim$. This order in the quotient space is also denoted by $\preceq$. We denote by $[x] \in M / \sim$ the equivalence class of $x \in M$. From the control theoretic point of view a class $[x]$ with $\text{int}_M([x]) \neq \emptyset$ is a control set for the action of the monoid S .

Proposition 1. Let $D = [x]$ be a class with respect to the equivalence relation $\sim$. Suppose that $\text{int}_M(D) \neq \emptyset$. Then D is a control set for S on M .

Proof: We first show that $D = [x] \subset \text{cl}(Sy)$, for every $y \in [x]$. Take $y \in [x]$ then $x \sim y$. For $z \in [x]$, we have $z \sim x$. By the transitivity of the relation $\sim$ we have $z \sim y$ and $y \preceq z$. This implies that $z \in \text{cl}(Sy)$. Now, suppose $D \subseteq D'$ with D' satisfying the condition $D' \subset \text{cl}(Sz)$ for every $z \in D'$. Take $y \in D'$. Then $y \in \text{cl}(Sz)$, for every $z \in D'$, in particular for $x \in D$. Therefore $y \in \text{cl}(Sx)$ and $x \preceq y$. On the other hand, since $x \in D \subseteq D'$, we have $x \in \text{cl}(Sy)$ once $y \in D'$ and $y \preceq x$. Hence $x \sim y$ and $y \in [x] = D$, showing the maximality of D . $\square$

We define a maximal class.

Definition 5. A class $[x]$ is said maximal if every class $[y]$ with $[x] \preceq [y]$ satisfies $[x] = [y]$.

We will show later that a maximal class with $\text{int}_M([x]) \neq \emptyset$ is an invariant control set for the monoid S .

We observe that $[x] \subset \text{cl}(Sx)$ for every $x \in M$.

Definition 6. We say that a subset $A \subset M$ is S -invariant, or invariant for the monoid S , if for every $y \in A$ we have $\text{cl}(Sy) \subset A$.

A S -invariant class is always maximal. In fact, suppose $[x]$ is S -invariant. Then $\text{cl}(Sx) \subset [x]$. If we take $[x] \preceq [y]$ we have $x \preceq y$ and $y \in \text{cl}(Sx) \subset [x]$. Therefore $x \sim y$ and $[x] = [y]$, showing the maximality.

We define $\mathcal{O} = \{\text{cl}(Sx) \subset M : x \in M\}$. Now, we relate maximality and S -invariance.

Lemma 1. For $x \in M$ the following statements are equivalent:

1. $[x]$ is maximal
2. $\text{cl}(Sx)$ is minimal in $\mathcal{O}$ with respect to the inclusion of sets
3. $[x] = \text{cl}(Sx)$
4. $[x]$ is closed and S -invariant

Proof: Let's assume that $\text{cl}(Sy) \subset \text{cl}(Sx)$ for some $y \in M$. Take $z \in \text{cl}(Sy)$. By the continuity of the action of S on M we have $\text{cl}(Sz) \subset \text{cl}(Sy)$. Since $z \in \text{cl}(Sx)$ the maximality of $[x]$ implies that $\text{cl}(Sx) \subset \text{cl}(Sz)$. Therefore $\text{cl}(Sx) \subset \text{cl}(Sy)$ showing that $\text{cl}(Sx)$ is minimal. Suppose that $\text{cl}(Sx)$ is minimal in $\mathcal{O}$. Then $\text{cl}(Sy) \subset \text{cl}(Sx)$ for all $y \in \text{cl}(Sx)$. By minimality $\text{cl}(Sy) = \text{cl}(Sx)$ for all $y \in \text{cl}(Sx)$. This implies that $\text{cl}(Sx)$ is entirely contained in an equivalence class so that $\text{cl}(Sx) = [x]$. If $[x] = \text{cl}(Sx)$ it is immediate that $[x]$ is closed and S -invariant. Finally, a S -invariant class is maximal. $\square$

The maximal classes with nonempty interior in M are the invariant control sets, more specifically, we have.

Corollary 1. Let $D = [x]$ be a maximal class with respect to the relation $\sim$. Assume $\text{int}_M(D) \neq \emptyset$. Then D is an invariant control set for S on M .

Proof: First we show that $\text{cl}(D) = \text{cl}(Sy)$ for every $y \in D$. Since $[x]$ is maximal we have by the Lemma 1 that $[x] = \text{cl}(Sx)$. Hence $D = \text{cl}(D)$. For $y, z \in [x]$ we have $z \in \text{cl}(Sy)$. Conversely, take $z \in \text{cl}(Sy)$, since $x \sim y$ we have $y \in \text{cl}(Sx)$. Therefore $z \in \text{cl}(Sx) = [x]$. Now we will show the maximality of D . Suppose $[x] = D \subseteq D'$ and D' satisfies the equality $\text{cl}(D') = \text{cl}(Sy)$ for every $y \in D'$. Take $z \in D'$. Then $\text{cl}(D') = \text{cl}(Sz)$. Since $x \in D \subseteq D' \subseteq \text{cl}(D')$ we have $x \in \text{cl}(Sz)$ and $z \preceq x$. On the other hand, since $x \in D \subseteq D'$ we have $\text{cl}(D') = \text{cl}(Sx)$. But, $z \in D' \subseteq \text{cl}(D') = \text{cl}(Sx)$ and therefore $x \preceq z$. It follows that $x \sim z$ and $z \in [x] = D$, showing the maximality of D . $\square$

On the existence of maximal classes we have the following proposition.

Proposition 2. *Let J be a S -invariant compact subset of M . Then for every $x \in J$ there exists a maximal equivalence class $[w] \subset \text{cl}(Sx)$.*

Proof: Fix $x \in J$ and consider the family of subsets

$$\mathcal{O}_x = \{\text{cl}(Sy) : \text{cl}(Sy) \subset \text{cl}(Sx)\}.$$

This family is not empty because it contains $\text{cl}(Sx)$. Let us order $\mathcal{O}_x$ by inclusion and show with the aid of Hausdorff's maximality principle that it contains minimal elements: Take a chain $\{\text{cl}(Sy)\}_{y \in \mathcal{I}}$ of subsets in $\mathcal{O}_x$, where $\mathcal{I}$ is an index set. Since J is S -invariant $\text{cl}(Sx) \subset J$. Therefore we have a chain of closed subsets of J . Hence they are compact which implies that the intersection $\bigcap_{y \in \mathcal{I}} \text{cl}(Sy)$ is not empty. Take $z \in \bigcap_{y \in \mathcal{I}} \text{cl}(Sy)$. Then $\text{cl}(Sz)$ belongs to $\mathcal{O}_x$ and is contained in $\text{cl}(Sy)$ for all $y \in \mathcal{I}$. These means that $\text{cl}(Sz)$ is a lower bound of the chain. Applying the maximality principle we conclude that $\mathcal{O}_x$ contains a minimal element, say $\text{cl}(Sw)$. Any element of $\mathcal{O} = \{\text{cl}(Sx) \subset M : x \in M\}$ contained in $\text{cl}(Sw)$ is an element of $\mathcal{O}_x$ because $\text{cl}(Sw) \subset \text{cl}(Sx)$. Hence $\text{cl}(Sw)$ is also minimal in $\mathcal{O}$ so the proof follows from Lemma 1. $\square$

Remark: We comment that the Proposition 2 has already been proved in the context of topological dynamics. Since an invariant

class $[x]$ is a minimal set the Proposition 2 is the classical result from topological dynamics which says that every compact invariant set has a minimal set. (see [5] Theorem 2.22).

We define a maximal class relatively to a subset J contained in M .

Definition 7. Given a subset $J \subset M$ a class $[x] \subset J$ is said to be *J-maximal*, if every class $[y] \subset J$ with $[x] \preceq [y]$ satisfies $[x] = [y]$.

For subsets J of M which are compact and S -invariant we have.

Corollary 2. Suppose that $J \subset M$ is compact and S -invariant. Then there are *J-maximal* classes and they are the maximal classes contained in J .

Proof: Take $x \in J$. Since J is compact and invariant under the action of S we have by the Lemma 2 that there exist a maximal class, say, $[w] \subset \text{cl}(Sx) \subset J$. We also have that $[w]$ is *J-maximal*. Conversely, a maximal class contained in J is *J-maximal*. $\square$

Now, we define a class which is invariant with respect to a subset J in M .

Definition 8. For a subset $J \subset M$ a class $[x] \subset J$ is called *SJ-invariant*, if $z \in \text{cl}(Sx)$ with $z \notin [x]$ implies $z \notin J$.

Therefore, if a class $[x] \subset J$ is *SJ-invariant* it cannot leave by the closure of an orbit without leaving J .

Remark: The relative invariant control sets for control systems were studied by Colonius and Kliemann in [4], pg.70, section 3.3. The definition of a *SJ-invariant* class was motivated by the definition of a relative invariant control set (see [4], pg. 50, Definition 3.1.9). Roughly speaking, a control set D for a control system contained in a subset J of the phase space is said to be a *relative invariant control set*, if $x \in D$ and a trajectory of the system starting at x leaves D , for some time and for some control, then the trajectory also leaves J .

It follows immediately that a *SJ-invariant* class is *J-maximal*. As a consequence a *SM-invariant* class is *M-maximal* and therefore it is a maximal class. Conversely, by the Lemma 1, a maximal class is

SM -invariant. Therefore, if $\text{int}_M([x]) \neq \emptyset$ a class $[x]$ is an invariant control set if and only if $[x]$ is SM -invariant.

On the existence of SJ -invariant classes we have.

Corollary 3. *Suppose that $J \subset M$ is compact and S -invariant. Then $[x]$ is a SJ -invariant class if and only if $[x]$ is J -maximal. In this case there exist SJ -invariant classes.*

Proof: Suppose $[x]$ is J -maximal. By the Corollary 2 $[x]$ is an S -invariant class contained in J . Therefore $[x]$ is a SJ -invariant class. $\square$

The no-return condition defined bellow was introduced, in the context of control systems, by Colonius and Kliemann (see [4], pg. 50, Theorem 3.1.10) in the study of relatively invariant control sets.

Definition 9. *We say that a subset $J \subset M$ satisfy the no-return condition if $z \in \text{cl}(Sx)$ for some $x \in J$ and $\text{cl}(Sz) \cap J \neq \emptyset$, then $z \in J$.*

This condition says that if we leave J we cannot go back to J again thorough the closure of an orbit of S .

Now, we translate the no-return condition in terms of an union of equivalence classes.

Proposition 3. *Suppose that $J \subset M$ satisfy the no-return condition. Then J is exhaustive for the equivalence relation $\sim$, i.e., any class $[x]$ is entirely contained in J or in J^c .*

Proof: Let $[x]$ be a class such that $[x] \cap J \neq \emptyset$ and $[x] \cap J^c \neq \emptyset$ and we will obtain a contradiction. Take $z \in [x] \cap J$ and $y \in [x] \cap J^c$. Then $z \sim y$, i.e., $z \in \text{cl}(Sy)$ and $y \in \text{cl}(Sz)$. By the no-return condition we have that $y \in J$ which is a contradiction. $\square$

Corollary 4. *Suppose J satisfies the no-return condition. Then*

$$J = \bigcup_{x \in J} [x]$$

Proof: Take $x \in J$. Then by the Proposition 3 we have $x \in [x] \subset J$. Thus $\bigcup_{x \in J} [x] \subset J$. Conversely, since $x \in [x]$ we have $J \subset \bigcup_{x \in J} [x]$. $\square$

The converse of the last corollary is not always true. In fact, suppose that $J = \bigcup_{[x] \in J} [x]$. Take $x \in J$ and $z \in \text{cl}(Sx)$ with $\text{cl}(Sz) \cap J \neq \emptyset$. Pick $w \in \text{cl}(Sz) \cap J$. Then $x \preceq z \preceq w$. It follows that $[x] \preceq [z] \preceq [w]$. Now, It is not always possible to conclude that $z \in [z] \subset J$.

The next theorem gives conditions for the existence of SJ -invariant classes. It also generalizes Proposition 3.3.3, pg. 72, in [4].

Theorem 1. *Let J be a subset of M satisfying the no-return condition. Suppose there is an $x \in J$ and a compact set $K \subset J$ such that for all $y \in \text{cl}(Sx) \cap J$*

$$\text{cl}(Sy) \cap K \neq \emptyset$$

Then there exists a SJ -invariant class $[w] \subset \text{cl}(Sx)$.

Proof: For $y \in \text{cl}(Sx) \cap J$ we define the compact $K_y = \text{cl}(Sy) \cap K$. Since $1 \in S$ we have that K_x is defined. Now, consider the family

$$\mathcal{T} = \{K_y : y \in K_x\}$$

define the following order on $\mathcal{T}$

$$K_{y_1} \preceq K_{y_2} \text{ if } y_1 \preceq y_2$$

thus if we assume $K_{y_1} \preceq K_{y_2}$ then $y_2 \in \text{cl}(Sy_1)$ and $K_{y_2} = \text{cl}(Sy_2) \cap K \subset \text{cl}(Sy_1) \cap K = K_{y_1}$. Now, observe that $K_{y_1} \subset K_{y_2}$ implies that $K_{y_2} \preceq K_{y_1}$. In fact, $y_1 \in K_{y_1} \subset K_{y_2} = \text{cl}(Sy_2) \cap K$. Thus $y_1 \in \text{cl}(Sy_2)$ and $K_{y_2} \preceq K_{y_1}$. Let $\{K_{y_i} : i \in I\}$ be a linearly ordered set. The intersection $\bigcap_{i \in I} K_{y_i}$ is a compact set since it is the intersection of decreasing compact subsets of the compact set K . Take $p \in \bigcap_{i \in I} K_{y_i}$.

Since $p \in \text{cl}(Sy_i)$ for every $i \in I$ and $y_i \in \text{cl}(Sx) \cap J$ we have that $p \in \text{cl}(Sy_i) \subset \text{cl}(Sx)$ and $K_p = \bigcap_{i \in I} K_{y_i}$, showing that $K_p \in \mathcal{T}$. The Zorn's lemma implies that the family $\mathcal{T}$ has a maximal element K_r . Since $r \in Sr \cap K$ we have $r \in K_r \subset J$. Let's define

$$D = \text{cl}(Sr) \cap J.$$

We will show that $D \subset \text{cl}(Sx)$ is a SJ -invariant class. We know that $r \in Sr \cap J \subset D$. By its only definition, every $z \in D$ is approximately reachable from r and $r \preceq z$. Conversely, take $z \in D$ and we will show that $z \preceq r$. Since $z \in D$ we have $z \in \text{cl}(Sr)$ and by the definition of K_r we have $r \in \text{cl}(Sx)$. Thus $z \in \text{cl}(Sx) \cap J$. It follows from the hypothesis that $K_z = \text{cl}(Sz) \cap K \neq \emptyset$. Pick an element $z_0 \in K_z$. It is obvious that $z \preceq z_0$. Then $K_{z_0} \in \mathcal{T}$ and by the maximality of K_r one obtains $z_0 \preceq r$. The transitivity of $\preceq$ implies that $z \preceq r$. Therefore we have approximate transitivity in D . We also have that D is a class. Otherwise, there exists a class $[w] \supset D$ containing a point $w \notin D = \text{cl}(Sr) \cap J$. The no-return condition and Proposition 3 implies that $D \subset [w] \subset J$. It follows that $w \sim r$ and $w \in \text{cl}(Sr) \cap J$ contradicting the choice of w . It remains to show the SJ -invariance of D . Take $z \in D = \text{cl}(Sr) \cap J$ and suppose that there exists $k \in \text{cl}(Sz) \cap J$. Then $k \in \text{cl}(Sz) \subset \text{cl}(Sr)$ and $k \in \text{cl}(Sr) \cap J$, showing the SJ -invariance of D . $\square$

The theorem above allows us to show that the J -maximal and SJ -invariant classes coincide.

Corollary 5. *Let $J \subset M$ be a subset satisfying the no-return condition. Take $x \in J$ and assume that there exists a compact set $K \subset J$ such that for all $y \in \text{cl}(Sx) \cap J$*

$$\text{cl}(Sy) \cap K \neq \emptyset$$

Then for a class $[w] \subset \text{cl}(Sx) \cap J$ we have that $[w]$ is J -maximal if and only if $[w]$ is SJ -invariant.

Proof: Suppose $[w]$ is J -maximal. Assume that there exists $y \in [w] \subset \text{cl}(Sx)$ and $z \in \text{cl}(Sy) \cap J$ with $z \notin [w]$. By the last proposition there exist a SJ -invariant class $[w_1] \subset \text{cl}(Sy) \subset \text{cl}(Sx)$. Thus $[w] \preceq [w_1]$ and $[w] \neq [w_1]$. This contradicts the J -maximality of $[w]$. Hence $[w]$ is SJ -invariant. $\square$

We will see next that a closed subset is a maximal class if and only if it is the closure of the orbit of its elements.

Proposition 4. *Let C be a closed subset of M . Then C is maximal class if and only if $C = \text{cl}(Sx)$ for every $x \in C$.*

Proof: Suppose $C = \text{cl}(Sx)$ for every $x \in C$. Then for $x, y \in C$ we have $y \in \text{cl}(Sx)$ and $x \in \text{cl}(Sy)$, i.e., $x \sim y$ and therefore C is contained in a equivalence class, say, $[x] \supset C$. Furthermore, for $y \in [x]$ we have $y \in \text{cl}(Sx)$, and therefore $[x] \subset \text{cl}(Sx) = C$, showing that $[x] = C$. Now, suppose that $C = [x] \preceq [y]$. Then $x \preceq y$ and $y \in \text{cl}(Sx) = [x]$. Therefore $C = [x] = [y]$ is a maximal class. Conversely, assume that $C = [x]$ is a maximal class. Thus by the Lemma 1 we have $C = [x] = \text{cl}(Sx)$ for every $x \in C$. $\square$

Now, we give conditions for the existence of a unique SJ -invariant class.

Proposition 5. *Let $J \subset M$ be a subset satisfying the no-return condition. Take $x \in J$ and assume that there exists a compact set $K_x \subset J$ which is S -invariant and such that for all $y \in \text{cl}(Sx) \cap J$*

$$\text{cl}(Sy) \cap K_x \neq \emptyset$$

Suppose that

$$C = \bigcap_{x \in J} \bigcap_{y \in \text{cl}(Sx) \cap J} (\text{cl}(Sy) \cap K_x) \neq \emptyset$$

Then C is a unique SJ -invariant class contained in J .

Proof: First, we show that C is a SJ -invariant class. It is easy to see that C is closed and it is contained in J . By the Proposition 4 and Corollary 5 it is enough to show that $C = \text{cl}(Sx)$ for every $x \in C$. Take $x \in J$ and $y \in \text{cl}(Sx) \cap J$. Then $\text{cl}(Sy) \subset \text{cl}(Sx)$ and $\text{cl}(Sx) \cap K_x \subset \text{cl}(Sx)$. Therefore $C \subset \bigcap_{x \in J} (\text{cl}(Sx)) \subset \text{cl}(Sx)$ for every $x \in C$. Now take $w \in \text{cl}(Sz)$ with $z \in C$. Then for every $x \in J$ and every $y \in \text{cl}(Sx) \cap J$ we have $z \in \text{cl}(Sy) \cap K_x$. Since K_x is S -invariant we have $w \in \text{cl}(Sz) \subset \text{cl}(Sy) \cap K_x$. Thus $w \in C$. It remains to show the uniqueness of C . Suppose that $C_1 \subset J$ is a maximal class. By the

Proposition 4, $\text{cl}(Sx) = C_1$ for every $x \in C_1$. Thus $C_1 = \bigcap_{x \in C_1} \text{cl}(Sx)$.

Therefore

$$C = \bigcap_{x \in J} \bigcap_{y \in \text{cl}(Sx) \cap J} (\text{cl}(Sy) \cap K_x) \subset \bigcap_{x \in J} (\text{cl}(Sx)) \subset \bigcap_{x \in C_1} \text{cl}(Sx) = C_1.$$

By the maximality of C it follows that $C = C_1$. $\square$

As the invariant control sets are closed in M under the hypothesis of accessibility of S , we show that the SJ -invariant classes are relatively closed in J .

Definition 10. We say that a subset $J \subset M$ satisfies the J -accessibility condition for a monoid S if for all $y \in J$, $\text{int}_M(Sy \cap J) \neq \emptyset$.

Remark: The hypothesis of J -accessibility was also considered in the context of control systems (see [4], pg. 50, Theorem 3.1.10).

The next theorem generalizes Proposition 3.3.4, pg. 72, in [4].

Theorem 2. Suppose that $J \subset M$ satisfy the J -accessibility condition for S . Then any SJ -invariant class $[w] \subset J$ is relatively closed in J , i.e., $(\partial([w]) - [w]) \cap J = \emptyset$.

Proof: Assume that $[w]$ is a SJ -invariant class and $(\partial([w]) - [w]) \cap J \neq \emptyset$. Pick $y \in (\partial([w]) - [w]) \cap J$. We have that $Sy \cap [w] = \emptyset$. Otherwise, there exist $g \in S$ such that $gy \in [w]$. Thus $[w] \subset \text{cl}(Sgy) \subset \text{cl}(Sy)$ and we would have $y \in [w]$, cause if $y \in \partial([w])$ is not in $[w]$ then $\text{cl}(Sy)$ do not contains $[w]$. Since $y \in J$ the J -accessibility condition guarantees the existence of $z \in (\text{int}(Sy \cap J)) - [w]$. There exists $g \in S$ such that $gy = z$. Let V be a neighborhood of z contained in $Sy \cap J$ and outside $[w]$. Now, $g^{-1}V$ is an open neighborhood of $y \in \partial([w])$. Therefore there exists $x \in [w]$ such that $gx \in V$, which is a contradiction with the SJ -invariance of $[w]$. $\square$

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Carlos Braga

Departamento de Matemática

Universidade Estadual de Maringá

Av. Colombo 5790

87020-900 Maringá-PR

Brazil

e-mail : cjbbarros@uem.br