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A RADON NIKODYM THEOREM IN THE NON-ARCHIMEDEAN SETTING

MIRTA MORAGA *
Universidad de Magallanes, Chile
and
JOSÉ AGUAYO
Universidad de Concepción, Chile

Abstract

In this paper we define the absolutely continuous relation between nonarchimedean scalar measures and then we give and prove a version of the Radon-Nykodym Theorem in this setting. We also define the nonarchimedean vector measure and prove some results in order to prepare a version of this Theorem in a vector case.

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1. Introduction and notations

In the classical case, it is known that the Radon-Nikodym Theorem involves real or complex valued measures and it becomes a property if the measures involved are vector valued measures (see [12] and [7]).

In this paper we give a non-archimedean version of the Radon-Nikodym Theorem for **K**-valued measures, where **K** is a non-archimedean field with a non-trivial valuation for which it is complete. In the last part of the paper, we introduce some definitions and statements which involve scalar valued measures.

Throughout the paper, X will denote a nonempty set, Ω a cover ring of subset of X and τ the topology generated by the ring Ω. As we know, any element of Ω is a clopen (closed and open) for this topology, Ω is a base of it and (X, τ) becomes to be a zero-dimensional topological space.

All of the following results in this section can be found in [10] and [11] and we will give a sketch of the proof of few of them.

Definition 1. *A set function $\mu : \Omega \to \mathbf{K}$ is said to be a non-archime dean measure or simply measure if:*

(i) μ is finitely additive, that is, for $A, B \in \Omega$, with $A \cap B = \emptyset$,

$$\mu(A) + \mu(B) = \mu(A \cup B)$$

(ii) $(U_\alpha)_{\alpha \in I}$ is a net in Ω with $U_\alpha \downarrow \emptyset$, and if for any $\alpha \in I$, we choose $V_\alpha \in \Omega$ with $V_\alpha \subset U_\alpha$, then

$$\lim_{\lambda} \mu(V_\lambda) = 0$$

(iii) if for each $a \in X$, there exists $U \in \Omega$, such that $a \in U$ and

$$\{\mu(V) : V \in \Omega; V \subset U\} \text{ is bounded.}$$

Definition 2. *Let μ be a measure over a cover ring Ω of subsets of X and τ be the topology generated by Ω. For a $\tau-open$ $W \subset X$ and $a \in X$, we define:*

$$\|W\|_\mu = \sup\{|\mu(U)| : U \in \Omega; U \subset W\}$$

$$\mathcal{N}_\mu(a) = \inf\left\{\|W\|_\mu : W \text{ es } \tau\text{-open, } a \in W\right\}$$

These definitions satisfy the following properties (see [10] page 191):

Proposition 3. *(i) If $W_1, W_2 \in \tau$, with $W_1 \subseteq W_2$, then $\|W_1\|_\mu \leq \|W_2\|_\mu$*

(ii) If $W \in \tau$, then $\|W\|_\mu = \sup\left\{\|U\|_\mu : U \in \Omega; U \subset W\right\}$

(iii) If $a \in X$, then $\mathcal{N}_\mu(a) = \inf\left\{\|U\|_\mu : U \in \Omega; a \in U\right\}$

Lemma 4. *The function $\mathcal{N}_\mu$ is $\tau-$upper semicontinuous (u.s.c.).*
Proof. *Let us take $\epsilon > 0$ and consider $X_\epsilon = \{x \in X : \mathcal{N}_\mu(x) \geq \epsilon\}$. If $x \in X \setminus X_\epsilon$, then there exists $U \in \Omega$, $a \in U$, such that $\|U\|_\mu < \epsilon$. Now, for $z \in U$, $\mathcal{N}_\mu(z) \leq \|U\|_\mu < \epsilon$; hence $U \subset X \setminus X_\epsilon$. This proves that $\mathcal{N}_\mu$ is u.s.c.* ■

For a linear space $\mathcal{F}$ of functions $f : X \to \mathbf{K}$, the collection $\Omega(\mathcal{F})$ given by

$$\Omega(\mathcal{F}) = \{U \subseteq X : f\mathcal{X}_U \in \mathcal{F}, \forall\, f \in \mathcal{F}\}$$

is a cover ring of X. We will denote by $\tau(\Omega(\mathcal{F}))$ the corresponding topology.

Definition 5. *We will say that $\mathcal{F}$ is a Wolfheze space if each $f \in \mathcal{F}$ is τ-continuous and for each $a \in X$, there exists $f \in \mathcal{F}$, with $f(a) \neq 0$.*

Definition 6. *A linear functional $I : \mathcal{F} \to \mathbf{K}$ will be called an integral if :*

(I) for each net $(f_\alpha)_{\alpha\in\Gamma}$, $f_\alpha \in \mathcal{F}$ with $f_\alpha \downarrow 0$ and for any $\alpha \in \Gamma$,

$$\lim_{\alpha\in\Gamma} I(g_\alpha) = 0$$

for any net $(g_\alpha)_{\alpha\in\Gamma}$ in $\mathcal{F}$ with $|g_\alpha| \leq |f_\alpha|$.

Lemma 7. *Fix $f \in \mathcal{F}$ and define*

$$\begin{array}{rccl} \mu_f : & \Omega & \to & \mathbf{K} \\ & U & \mapsto & \mu_f(U) = I\left(f\mathcal{X}_U\right) \end{array}$$

Then, μ_f is a non-archimedean measure.
Proof. *It follows easily from the fact that if $(U_\alpha)_{\alpha\in I}$ is a net in Ω with $U_\alpha \downarrow \emptyset$, then $f * f_\alpha = f * \mathcal{X}_{U_\alpha} \downarrow 0$.* ■

The next lemma gives a relation between those measures given in the previous lemma (see [10], page 193)

Lemma 8. *There exists a unique function $\mathcal{N}_I : X \to [0, \infty)$, such that*
$|f| \mathcal{N}_I = \mathcal{N}_{\mu_f}$ *for each $f \in \mathcal{F}$. Even more, $\mathcal{N}_I$ is $\tau - u.s.c.$*

Let I be an integral over a Wolfheze space $\mathcal{F}$. We denote by $\Omega = \Omega(\mathcal{F})$, $\tau = \tau(\mathcal{F})$. For each $g : X \to \mathbf{K}$, we write

$$\|g\|_I = \sup \{|g(x)| \mathcal{N}_I(x) : x \in X\}$$

Then, we have:

Definition 9. *A function g is said to be $I-$integrable if for each $\delta > 0$, there exists $f \in \mathcal{F}$ such that,*

$$\|f - g\|_I < \delta$$

We denote by $L(I)$ the class of the integrable functions, that is,

$$L(I) = \{g : X \to \mathbf{K} : g \text{ is integrable}\}$$

This class is a linear space over the non-archimedean field $\mathbf{K}$.

The following theorem gives us a characterization of integrable functions.

Theorem 10. *A function $f : X \to \mathbf{K}$ is integrable if and only if f satisfies the following conditions:*

(i) f is $\tau(\mathcal{F})-$ continuous on each $X_t = \{x \in X : \mathcal{N}_I(x) \geq t\}$

(ii) For each $\delta > 0$, there exists a $\tau(\mathcal{F})-$compact P contained in some X_t such that $|f| \mathcal{N}_I < \delta$ off P.

Proof. *See* [10], *pages 195-96.* ■

Theorem 11. *The space $L(I)$ is a Wolfheze space and $\mathcal{F}$ is $\|\|_I -$dense in $L(I)$.*

Definition 12. *Let $\Theta : \Omega \to \mathbf{K}$ be set function and μ be an integral over $\mathcal{F}$. For each $a \in X, \alpha \in \mathbf{K}$, $c \in]0, 1[$ and $r \in \mathbf{R}$, we write*

1. $\underset{U\to\alpha}{LIM}\ \Theta(U)=\alpha$, if

   $(\forall\epsilon>0)\,(\exists V\in\Omega; a\in V)\,(U\subset V; U\in\Omega\Rightarrow|\Theta(U)-\alpha|<\epsilon)$

2. $\underset{U\to a}{LIM_{\mu,c}}\ \Theta(U)=\alpha$, if

   for each $\epsilon>0$, there exists a neighborhood V of a, such that for all $U\in\Omega$, with $U\subset V$, and $|\mu(U)|\geq c\mathcal{N}_\mu(a)$ we have

$$|\Theta(U)-\alpha|<\epsilon$$

3. $\underset{U\to a}{LIM_{\mu}}\ \Theta(U)=\alpha$, if

$$\overline{\underset{U\to a}{LIM}}_{\mu,c}\ \Theta(U)=\alpha,\quad \forall\ c\in\,]0,1[$$

4. $\overline{\underset{U\to a}{LIM}}\ |\Theta(U)|=r$, if

   $(\forall\epsilon>0)$
   $(\exists U\in\Omega; a\in U)\,(r-\epsilon\leq\sup\{|\Theta(V)|:V\in\Omega; V\subset U\}\leq r+\epsilon)$

The proof of the next lemmas follows directly from the definitions (see [11] , page 80).

Lemma 13. *For a measure* $\mu:\Omega\to\mathbf{K}$ *and* $a\in X$ *we have*

1. $\mathcal{N}_\mu(a)=\overline{\underset{U\to a}{LIM}}\ |\mu(U)|$

2. $\mathcal{N}_\mu(a)=0\Longleftrightarrow\underset{U\to a}{LIM}\ \mu(U)=0$

Lemma 14. *If* $\Theta:\Omega\to\mathbf{K}$ *is additive and* $0<c<1$, *then*

$$\underset{U\to a}{LIM}_{\mu,c}\ \Theta(U)=0\Longleftrightarrow\underset{U\to a}{LIM}\ \Theta(U)=0$$

2. The Radon-Nikodym Theorem

Throughout this section X will be a set and $\mathcal{F}$ a Wolfheze space of functions from X into $\mathbf{K}$ with the assumption that $\mathcal{F}$ contains the constant functions. It follows from this assumption that $\mathcal{X}_U \in \mathcal{F}$ for all $U \in \Omega(\mathcal{F})$. Since $1 \in \mathcal{F}$, we have that if $I : \mathcal{F} \to \mathbf{K}$ is an integral, then $\mu : \Omega(\mathcal{F}) \to \mathbf{K}$ defined by $\mu(U) = I(\mathcal{X}_U)$ is a non-archimedean measure. In the sequel, we will use Greek letter to denote integral. We start with the following lemma [11]:

Lemma 1. *If $\mu : \mathcal{F} \to \mathbf{K}$ is an integral, $f \in L(\mu)$ and $a \in X$, then*

$$\underset{U \to a}{LIM} \left[\mu_f(U) - f(a)\mu(U)\right] = 0$$

Definition 2. *Let μ, ν be two measures over a cover ring Ω. We will say that ν is absolutely continuous with respect to μ, $\nu \ll \mu$, if for each $a \in X$ there exists $\alpha \in \mathbf{K}$, such that*

$$\mathcal{N}_{\nu - \alpha\mu}(a) = 0$$

Example 3. *Let $\mu : \mathcal{F} \to \mathbf{K}$ be an integral functional. Let $j : X \to \mathbf{K}$ be a $\mu-$integrable function. We define*

$$\begin{array}{rccl} j\mu : & \mathcal{F} & \to & \mathbf{K} \\ & f & \mapsto & j\mu(f) = \mu(fj) = \int fj d\mu \end{array}$$

It is well defined since $fj \in L(\mu)$ and is an integral. Note that its corresponding measure $j\mu : \Omega(\mathcal{F}) \to \mathbf{K}$ is the following:

$$j\mu(U) = \mu(\mathcal{X}_U j) = \mu_j(U)$$

(see Lemma 7). We claim that $j\mu$ is absolutely continuous with respect to μ. In fact, by Lemma 1

$$\begin{array}{rcl} \underset{U \to a}{LIM} \left[j\mu(U) - j(a)\mu(U)\right] & = & \underset{U \to a}{LIM} \left[\mu_j(U) - j(a)\mu(U)\right] \\ & = & 0 \end{array}$$

and then, by Lemma 13-(2),

$$\mathcal{N}_{\mu_j - j(a)\mu}(a) = 0.$$

Theorem 4. *(The Radon-Nikodym Theorem)*

Let ν, μ be two measures defined on $\Omega(\mathcal{F})$. If $\nu \ll \mu$, then there exists a $\mu-$integrable function j such that

$$\nu = j\mu$$

Let $a \in X$. Since $\nu \ll \mu$, there exists $\alpha_a \in \mathbf{K}$ such that

$$\mathcal{N}_{\nu-\alpha_a\mu}(a) = 0$$

Using Lemma 13, we can prove the uniqueness of α_a. Thus, we define the function j by:

$$\begin{aligned} j \quad &: \quad X \to \mathbf{K} \\ & \quad a \longmapsto j(a) = \begin{cases} \alpha_a & , \text{if } \mathcal{N}_\mu(a) > 0 \\ 0 & , \text{if } \mathcal{N}_\mu(a) = 0 \end{cases} \end{aligned}$$

The first step is to prove that j is μ-integrable. To do that we will use Theorem 10, that is, we have to prove that (i) $j_{|X_t}$ is continuous and (ii) for each $\delta > 0$, there exists a compact set P of X contained in some $X_t = \{x \in X : \mathcal{N}_\mu(x) \geq t\}$ such that $|j|\mathcal{N}_\mu < \delta$ off P. Take $t > 0$ and $a \in X_t$. By the assumption, $\mathcal{N}_{\upsilon-j(a)\mu}(a) = 0$ or, equivalently, $\underset{U\to a}{LIM} \; (\nu\upsilon - j(a)\mu)(U) = 0$. Thus, if $\epsilon > 0$ is given, there exists $U_0 \in \Omega$, with $a \in U_0$, such that

$$|(\nu - j(a)\mu)(V)| < \epsilon, \quad \forall V \in \Omega, V \subset U_0$$

Take $b \in U_0 \cap X_t$; hence $\mathcal{N}_\mu(b) \geq t$. Since $\underset{U\to b}{LIM} \; (\nu - j(b)\mu)(U) = 0$, there exists $U_1 \in \Omega$, with $b \in U_1$, such that

$$|(\nu - j(b)\mu)(V)| < \epsilon, \quad \forall V \in \Omega, V \subset U_1$$

Thus, if $V \subset U_0 \cap U_1 = U_2$ and $V \in \Omega$, then

$$\begin{aligned}
|j(a) - j(b)|\, t &\leq |j(a) - j(b)|\, \mathcal{N}_\mu(b) \\
&\leq |j(a) - j(b)|\, \|U_2\|_\mu \\
&= \sup\{|j(a) - j(b)|\, |\mu(V)| : V \in \Omega, V \subset U_2\} \\
&\leq \sup\{|v - j(b)\,\mu(V) + j(a)\,\mu(V) - \nu| : \\
&\quad V \in \Omega, V \subset U_2\} \\
&\leq \max\left\{\sup_{V \subset U_2} |(\nu - j(b)\,\mu)(V)|\,,\right. \\
&\quad \left.\sup_{V \subset U_2} |(\nu - j(a)\,\mu)(V)|\right\} \\
&\leq \epsilon
\end{aligned}$$

therefore, $j_{|X_t}$ is continuous. To prove (ii), take $\delta > 0$ and $a \in X$, with $\mathcal{N}_\mu(a) > 0$. The set $P = \{x \in X : \mathcal{N}_\nu(x) \geq \delta\}$ is $\tau-$compact in X. Since $\mathcal{N}_{\nu - j(x)\mu}(x) = 0$ for all $x \in X$ with $\mathcal{N}_\mu(x) > 0$ and

$$\left|\mathcal{N}_\nu(x) - \mathcal{N}_{j(x)\mu}(x)\right| \leq \mathcal{N}_{\nu - j(x)\mu}(x) = 0$$

we have

$$\mathcal{N}_\nu(x) = \mathcal{N}_{j(x)\mu}(x) = |j(x)|\, \mathcal{N}_\mu(x)$$

Then, for $x \notin P$, we have $|j(x)|\, \mathcal{N}_\mu(x) < \delta$. Now, we finish this part proving that there is $t > 0$ such that $P \subset X_t$. If $a \in P$, then there exists $U_a \in \Omega$, with $a \in U_a$, such that

$$|(\nu - j(a)\,\mu)(V)| < \frac{\delta}{2}, \quad \forall V \in \Omega, V \subset U_a$$

Thus,

$$P \subset \bigcup_{a \in P} U_a$$

and since P is compact, there exist $a_1, a_2, ..., a_n \in P$ such that

$$P \subset \bigcup_{i=1}^{n} U_{a_i}$$

Now, denote by $M = \max\{|j(a_1)|, |j(a_2)|, ..., |j(a_n)|\}$ and define $t = \delta M^{-1}$. We claim that $P \subset X_t$; in fact, since

$$\begin{aligned}
|\nu(V)| &\leq |\nu(V) - j(a_i)\,\mu(V) + j(a_i)\,\mu(V)| \\
&\leq \max\{|\nu(V) - j(a_i)\,\mu(V)|, |j(a_i)\,\mu(V)|\} \\
&\leq \max\left\{\frac{\delta}{2}, \delta t^{-1}\, |\mu(V)|\right\}
\end{aligned}$$

Then, by Lemma 13, we have

$$\begin{aligned}
\delta \leq \mathcal{N}_\nu(a) &= \overline{\underset{V\to a}{LIM}}\, |\nu(V)| \\
&\leq \max\left\{\frac{\delta}{2}, \delta t^{-1}\, \overline{\underset{V\to a}{LIM}}\, |\mu(V)|\right\} \\
&\leq \max\left\{\frac{\delta}{2}, \delta t^{-1}\mathcal{N}_\mu(a)\right\}
\end{aligned}$$

and, therefore

$$\mathcal{N}_\mu(a) \geq t$$

To prove that $\nu = j\mu$, we consider the measure

$$\begin{array}{rccl}
\nu - j\mu : & \Omega & \to & \mathbf{K} \\
& U & \to & (\nu - j\mu)(U) = \nu(U) - \int \mathcal{X}_U j d\mu
\end{array}$$

Fix $a \in X$, then

$$\underset{U\to a}{LIM}\, |(\nu - j\mu)(U)| = \underset{U\to a}{LIM} \left|\nu(U) - \int \mathcal{X}_U j d\mu\right|$$

Now,

$$\begin{aligned}
\left|\nu(U) - \int \mathcal{X}_U j d\mu\right| &= \left|\nu(U) - j(a)\mu(U) + j(a)\mu(U) - \int \mathcal{X}_U j d\mu\right| \\
&\leq \begin{array}{l} \max\{|\nu(U) - j(a)\mu(U)|, \\ |j(a)\mu(U) - \int \mathcal{X}_U j d\mu|\} \end{array}
\end{aligned}$$

Therefore,

$$\underset{U\to a}{LIM} \left|\nu(U) - \int \mathcal{X}_U j d\mu\right| \leq$$

$$\max\left\{\underset{U\to a}{LIM}\, |\nu(U) - j(a)\mu(U)|, \underset{U\to a}{LIM} \left|j(a)\mu(U) - \int \mathcal{X}_U j d\mu\right|\right\} = 0$$

Thus,

$$\underset{U\to a}{LIM} \left|\nu(U) - \int \mathcal{X}_U j d\mu\right| = 0$$

and then,

$$\mathcal{N}_{\nu - j\mu}(a) = 0$$

Since $a \in X$ is arbitrary,

$$\mathcal{N}_{\nu - j\mu}(a) = 0, \forall\ a \in X$$

Therefore,

$$\nu - j\mu = 0$$

or,

$$\nu = j\mu$$

■

3. Non-archimedean vector measures

In this section we will briefly analyze $E-$valued measures, where E is a normed space over the field $\mathbf{K}$. As in the previous sections, X will be a nonempty set, Ω a cover ring of X and E a Banach space, with E'$\neq \{\theta\}$ over a non-archimedean field $\mathbf{K}$ with a nontrivial valuation and complete. As we know, the function $\| \|_0 : E \to \mathbf{R}$ defined by

$$\|x\|_0 = \sup \{|x'(x)| : x' \in E'; \|x'\| \leq 1\}$$

is equivalent to the original norm of E.

Definition 1. *Let $m : \Omega \to E$ a finitely additive and bounded set function. We will say that m is a non-archimedean vector measure or, simply, a vector measure if for each net $\{U_\lambda\}_{\lambda\in\Lambda}$ in Ω such that $U_\lambda \downarrow \emptyset$ and if, for each $\lambda \in \Lambda$, we choose $V_\lambda \subset U_\lambda$ and $V_\lambda \in \Omega$, then*

$$\lim_{\lambda} m(V_\lambda) = 0$$

that is,

$$(\forall \varepsilon > 0)\,(\exists \lambda_0 \in \Lambda)\,(\lambda > \lambda_0 \Rightarrow \|m(V_\lambda)\|_0 < \varepsilon)$$

Definition 2. *Let $m : \Omega \to E$ a finitely additive set function. For $U \in \Omega$, we define*

$$\|U\|_m = \sup \{\|m(B)\|_0 : B \subset U; B \in \Omega\}$$

$$\mathcal{N}_m\ (x) = \inf \{\|U\|_m : U \in \Omega;\ x \in U\}$$

It is not difficult to prove that this function is $\tau\,(\Omega)\,-$u.s.c.

Definition 3. *Let* $\Theta : \Omega \to E$ *a set function,* $\mu : \Omega \to \mathbf{K}$ *a measure,* $c \in \,]0, 1[\,$, $a \in X$ *and* $r \in \mathbf{R}$.

1. $\underset{U \to a}{LIM}\, \Theta(U) = e$ *means*

   $$(\forall \epsilon > 0)\,(\exists U \in \Omega; a \in U)\,(V \subset U; V \in \Omega \Rightarrow \|\Theta(V) - e\|_0 < \epsilon)$$

2. $\underset{U \to a}{LIM_{\mu,c}}\, \Theta(U) = e$ *means*
   $(\forall \epsilon > 0)\,(\exists U \in \Omega; a \in U)$
   $(V \subset U; V \in \Omega; |\mu(V)| \geq c\mathcal{N}_\mu(a) \Rightarrow \|\Theta(V) - e\|_0 < \epsilon)$

3. $\underset{U \to a}{LIM_{\mu}}\, \Theta(U) = e$ *means*

   $$\underset{U \to a}{LIM_{\mu,c}}\, \Theta(U) = e$$

   for all $c \in \,]0, 1[$

4. $\underset{U \to a}{\overline{LIM}}\, \|\Theta(U)\| = r$ *means*
   $(\forall \epsilon > 0)\,(\exists U \in \Omega; a \in U)$
   $(r - \epsilon \leq \sup\{\|\Theta(V)\|_0 : V \in \Omega; V \subset U\} \leq r + \epsilon)$

Lemma 4. *Let* $m : \Omega \to E$ *a vector measure and* $a \in X$. *Then,*

1. $\mathcal{N}_m(a) = \underset{U \to a}{\overline{LIM}}\, \|m(U)\|$
2. $\mathcal{N}_m(a) = 0 \Leftrightarrow \underset{U \to a}{LIM}\, m(U) = \theta$

Proof. *In both cases, the proof follows easily from the definition and we omit them.* ■

We will denote by $\mathcal{S}(X)$ the vector space spanned by $\langle\{\mathcal{X}_U : U \in \Omega\}\rangle$, that is, $f \in \mathcal{S}(X)$, if

$$f = \sum_{i=1}^{n} \alpha_i \mathcal{X}_{U_i}$$

where $\alpha_i \in \mathbf{K}$, $U_i \in \Omega$, and $\bigcup_{i=1}^{n} U_i = X$. We can assume that $\{U_i\}_1^n$ is pairwise disjoint.

Definition 5. *Let m be a non-archimedean vector measure. We will define the E-valued linear operator $\int$ on $\mathcal{S}(X)$ by*

$$\int \mathcal{X}_U dm = m(U)$$

Lemma 6. *This operator satisfies*

$$\left\| \int f dm \right\| \leq \|f\|_{\mathcal{N}}$$

Proof. *It follows from the fact that*

$$\begin{aligned} \left\| \int \mathcal{X}_U dm \right\| &= \|m(U)\| \\ &\leq \|U\|_m \\ &\leq \sup_{x \in U} \mathcal{N}(x) \\ &\leq \sup_{x \in X} |\mathcal{X}_U| \mathcal{N}(x) \\ &\leq \|\mathcal{X}_U\|_{\mathcal{N}} \end{aligned}$$

■

Definition 7. *A function $f : X \to \mathbf{K}$ is said to be m-integrable if there exists a sequence $\{f_n\}_{n \in IN}$ of $\mathcal{S}(X)$ such that*

$$\lim_{n \to \infty} \|f - f_n\|_{\mathcal{N}} = 0$$

We will denote by $L(m)$ the space of all m-integrable functions. Since $\mathcal{S}(X)$ is $\|\|_{\mathcal{N}}$ −dense in $L(m)$, we have that the linear operator $\int$ has a unique extension to $L(m)$ and this extension satisfies the condition:

$$\left\| \int f dm \right\| \leq \|f\|_{\mathcal{N}}, \quad \forall f \in L(m)$$

For $U \in \Omega$ and $e \in E$, we define the function $\mathcal{X}_U \otimes e$ by

$$\begin{aligned} \mathcal{X}_U \otimes e : \quad X &\to E \\ x &\mapsto \mathcal{X}_U \otimes e(x) = \begin{cases} e & \text{if} \quad x \in U \\ \theta & if \quad x \notin U \end{cases} \end{aligned}$$

We denote by

$$\mathcal{S}(X,E) = \langle \{\mathcal{X}_U \otimes e : U \in \Omega; e \in E\}\rangle$$

Thus, $f \in \mathcal{S}(X,E)$ means

$$f = \sum_{i=1}^{n} \mathcal{X}_{U_i} \otimes e_i$$

Now, if $\mu : \Omega \to \mathbf{K}$ a measure and $f \in \mathcal{S}(X,E)$, then we define the set function

$$\begin{array}{rccl} \mu \otimes f : & \Omega & \to & E \\ & U & \mapsto & \mu \otimes f(U) = \sum\limits_{i=1}^{n} \mu\left(U_i \cap U\right) e_i \end{array}$$

Lemma 8. *$\mu \otimes f$ is a vector measure and satisfies the following property*

$$\mathcal{N}_{\mu\otimes f}(a) = \|f(a)\|\, \mathcal{N}_{\mu\otimes f(a)}(a)$$

Proof. Trivially, $\mu \otimes f$ is finitely additive. To see $\mu \otimes f$ is bounded, take $U \in \Omega$. Then,

$$\|U\|_{\mu\otimes f:} = \sup\left\{\|\mu \otimes f(V)\|_0 : V \in \Omega; V \subset U\right\}$$

Now, since

$$\|\mu \otimes f(V)\|_0 = \sup\left\{|x'\left(\mu \otimes f(V)\right)| : \|x'\| \leq 1\right\}$$

where,

$$\begin{aligned} x'\left(\mu \otimes f(V)\right) &= x'\left(\sum_{i=1}^{n} \mu\left(U_i \cap V\right) e_i\right) \\ &= \sum_{i=1}^{n} \mu\left(U_i \cap V\right) x'(e_i) \end{aligned}$$

we have,

$$\begin{aligned} |x'\left(\mu \otimes f(V)\right)| &\leq \max_{1\leq i\leq n} |\mu\left(U_i \cap V\right)|\, |x'(e_i)| \\ &\leq \max_{1\leq i\leq n} |\mu\left(U_i \cap V\right)|\, \|e_i\|_0 \\ &\leq M \max_{1\leq i\leq n} \|U_i \cap V\|_\mu \\ &\leq M \left\|\cup_{i=1}^{n} U_i \cap V\right\|_\mu \\ &= M \|U\|_\mu \end{aligned}$$

where $M = \sup\{\|e_i\|_0 : i = 1, 2, ..., n\}$.

From this,

$$\|U\|_{\mu\otimes f:} \leq M\,\|U\|_{\mu}$$

and since μ is a measure, we have

$$\left\{\|U\|_{\mu\otimes f:} : U \in \Omega\right\} \text{ is bounded.}$$

Now, consider a net $\{U_\lambda\}_{\lambda\in\Lambda}$ in Ω such that $U_\lambda \downarrow \emptyset$ and choose $V_\lambda \subset U_\lambda$ with $V_\lambda \in \Omega$ for any $\lambda \in \Lambda$. Then,

$$\begin{aligned}
\lim_{\lambda} \mu \otimes f(V_\lambda) &= \lim_{\lambda} \sum_{1\leq i\leq n} \mu\left(U_i \cap V_\lambda\right) e_i \\
&= \sum_{1\leq i\leq n} \lim_{\lambda} \mu\left(U_i \cap V_\lambda\right) e_i \\
&= \theta
\end{aligned}$$

since μ is measure and $\{U_i \cap V_\lambda\}_{\lambda\in\Lambda}$ is a net in Ω with $U_i \cap V_\lambda \subset U_\lambda$ for each λ. This proves that $\mu \otimes f$ is a vector measure

For the second part, let $a \in X$; hence there exists a unique $i \in \{1, 2, 3, ..., n\}$ such that $a \in U_i$, where $f = \sum_{i=1}^{n} \mathcal{X}_{U_i} \otimes e_i$. By Lemma 4, we have

$$\mathcal{N}_{\mu\otimes f}(a) = \overline{\underset{U\to a}{LIM}}\ \|\mu \otimes f(U)\|$$

Thus, if $U \in \Omega$, with $a \in U$, is small enough, that is, $U \subset U_i$ for some i, then

$$\mu \otimes f(U) = \mu(U)e_i$$

and therefore,

$$\begin{aligned}
\mathcal{N}_{\mu\otimes f}(a) &= \overline{\underset{U\to a}{LIM}}\ \|\mu(U)e_i\| \\
&= \|e_i\|\ \overline{\underset{U\to a}{LIM}}\ |\mu(U)| \\
&= \|f(a)\|\,\mathcal{N}_{\mu}(a)
\end{aligned}$$

■

Definition 9. *Let $m : \Omega \to E$ and $\mu : \Omega \to \mathbf{K}$ two non-archimedean measures. We will say that m is absolutely continuous with respect to μ if:*

$$(\forall a \in X)\,(\exists e \in E)\,(\mathcal{N}_{m-\mu\otimes e}(a) = 0)$$

Lemma 10. *For any $f \in \mathcal{S}(X, E)$, $\mu \otimes f$ is absolutely continuous with respect to μ.*

Proof. Let $a \in X$; hence, $a \in U_i$, where $f = \sum_{i=1}^{n} \mathcal{X}_{U_i} \otimes e_i$. Now, if $U \in \Omega$, with $a \in U$, is small enough, then

$$\mu \otimes f(U) = \mu(U)e_i$$

where $f(a) = e_i$. Thus,

$$\mu \otimes f(U) - \mu \otimes f(a)(U) = 0$$

and then,

$$LIM_{U \to a} [\mu \otimes f(U) - \mu \otimes f(a)(U)] = 0$$

Therefore, by Lemma 4, we get

$$\mathcal{N}_{\mu \otimes f - \mu \otimes f(a)}(a) = 0$$

■

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Mirta Moraga
Departamento de Matemática y Física
Facultad de Ciencia
Universidad de Magallanes
Casilla 113-D
Punta Arenas
Chile
e-mail : mmoraga@um.cl

and

José Aguayo
Departamento de Matemáticas
Facultad de Ciencias Físicas y Matemáticas
Universidad de Concepción
Casilla 160-C
Concepción
Chile
e-mail : jaguayo@udec.cl