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ON THE INVARIANCE OF SUBSPACES IN SOME BARIC ALGEBRAS

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Abstract

In this article, we look for invariance in commutative baric algebras (A, ω) satisfying $(x^2)^2 = \omega(x)x^3$ and in algebras satisfying $(x^2)^2 = \omega(x^3)x$, using subspaces of kernel of ω that can be obtained by polynomial expressions of subspaces U_e e V_e of Peirce decomposition $A = Ke \oplus U_e \oplus V_e$ of A, where e is an idempotent element. Such subspaces are called p -subspaces. Basically, we prove that for these algebras, the p -subspaces have invariant dimension, besides that, we find out necessary and sufficient conditions for the invariance of the p-subspaces.

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1. Introduction

Let A be a commutative and not necessarily associative algebra with finite dimension over K, where K is a field with char (K)$\neq$ 2, 3. We consider in this paper two classes of baric algebras (A, w) satisfying respectively

$$(x^2)^2 = \omega(x)x^3 \tag{1.1}$$

or

$$(x^2)^2 = \omega(x^3)x \tag{1.2}$$

We present in the next two sections some well known results about these two classes of algebras. In particular we look for the idempotents in these classes.

1.1. Algebras satisfying $(x^2)^2 = \omega(x)x^3$

Let (A, ω) be a baric algebra satisfying $(x^2)^2 = \omega(x)x^3$ for all $x \in A$. From this identity, we have

$$(x^2)^2 = 0 \tag{1.3}$$

for all $x \in N = \ker \omega$. By linearization of (1.3), we deduce that

$$x_1^2(x_1x_2) = 0 \tag{1.4}$$

$$x_1^2(x_2x_3) + 2(x_1x_2)(x_1x_3) = 0 \tag{1.5}$$

for every x_1, x_2, x_3, $x_4 \in N$. It was proved in [1] that the set of idempotents of weight 1 in algebras satisfying (1.1) is given by $\mathrm{Ip}(A) = \{z^3 \, ; \, \omega(z) = 1\}$. Every $e \in \mathrm{Ip}(A)$ determines a decomposition $A = Ke \oplus U_e \oplus V_e$ called the Peirce decomposition of A, where $N = U_e \oplus V_e$ and

$$\begin{aligned} U_e &= \{u \in A \, ; \, eu = \tfrac{1}{2}u\} \\ V_e &= \{v \in A \, ; \, ev = 0\} \end{aligned}$$

and, moreover,

$$U_eV_e \subseteq U_e \quad U_e^2 \subseteq V_e \quad V_e^2 \subseteq V_e$$

The elements $u \in U_e$ and $v \in V_e$ satisfy the identities

$$\begin{aligned} u^3 &= 0 \\ v^3 &= 0 \\ uv^2 &= 2(uv)v \\ u^2v &= 2u(uv) \end{aligned} \tag{1.6}$$

and their linearizations

$$u_1^2u_2 + 2u_1(u_1u_2) = 0 \tag{1.7}$$
$$u_1(u_2u_3) + u_2(u_3u_1) + u_3(u_1u_2) = 0 \tag{1.8}$$
$$v_1(v_2v_3) + v_2(v_3v_1) + v_3(v_1v_2) = 0 \tag{1.9}$$
$$u(v_1v_2) = (uv_1)v_2 + (uv_2)v_1 \tag{1.10}$$
$$(u_1u_2)v = u_1(u_2v) + u_2(u_1v) \tag{1.11}$$

It was also proved in [1] that $\mathrm{Ip}(A) = \{e + u + u^2\ ;\ u \in U_e\}$ and if $f = e + u_0 + u_0^2$, for $u_0 \in U_e$ then

$$U_f = \{u + 2u_0u\ ;\ u \in U_e\} \tag{1.12}$$
$$V_f = \{v - 2u_0v\ ;\ v \in V_e\} \tag{1.13}$$

Using the previous identities, it is easy to prove that

$$x_1(x_1^2x_2) = 0 \tag{1.14}$$

for every $x_1,\ x_2 \in N$. Since A is commutative, the identities (1.4) and (1.14) imply that N is a Jordan algebra (in fact, of a very special kind: $x_1^2(x_1x_2) = x_1(x_1^2x_2) = 0$). For more information see [2]

1.2. Algebras satisfying $(x^2)^2 = \omega(x^3)x$

Etherington showed in [4] that if (A, ω) satisfies the train equation in principal powers (of degree 3)

$$x^3 - (1+\gamma)\omega(x)x^2 + \gamma\omega(x^2)x = 0 \tag{1.15}$$

with $\gamma \in K$, then (A, ω) also satisfies the train equation in plenary powers (of degree 4)

$$(x^2)^2 - (1+2\gamma)\omega(x^2)x^2 + 2\gamma\omega(x^3)x = 0 \tag{1.16}$$

In [6], Walcher proved that (1.15) and (1.16) are equivalent, excepting for $\gamma = 0$ and $\gamma = -\frac{1}{2}$. If $\gamma = 0$ in (1.16), then $(x^2)^2 = \omega(x^2)x^2$ and, in this case, (A, ω) is a Bernstein algebra. Now, if $\gamma = -\frac{1}{2}$, then (A, ω) satisfies

$$(x^2)^2 = \omega(x^3)x$$

These algebras are also studied in [1], [5], and [6]. The following results are proved in [5]. Every algebra A satisfying (1.2) has an idempotent given by $e = (d^3)^3$ where $\omega(d) = 1$. Each idempotent e of A determines a Peirce decomposition $A = Ke \oplus U_e \oplus V_e$, where

$$\begin{aligned} U_e &= \{u \in \ker\omega \ ; \ eu = \tfrac{1}{2}u\} \\ V_e &= \{v \in \ker\omega \ ; \ ev = -\tfrac{1}{2}v\} \end{aligned}$$

These subspaces satisfy the inclusions

$$U_eV_e \subseteq U_e, \quad U_e^2 \subseteq V_e, \quad V_e^2 \subseteq V_e$$

As in the preceding section, we have $N = \ker\omega$ and $N = Ue \oplus Ve$. From (1.2), we have, for every $x \in N$,

$$(x^2)^2 = 0$$

Likewise, A also satisfies the identities (1.4) and (1.5). Another identity in algebras satisfying (1.2) is

$$x_1(x_1^2x_2) = 0 \tag{1.17}$$

From (1.4) and (1.17), N is a Jordan algebra of a very special kind, as remarked at the end of 1.1. For every $u \in U$ and $v \in V$ we have

$$\begin{aligned} u^3 &= 0 \\ v^3 &= 0 \\ u^2v &= 2u(uv) \\ uv^2 &= 2v(vu) \end{aligned} \tag{1.18}$$

By linearization of these identities we obtain

$$\begin{aligned} u_1(u_2u_3) + u_2(u_3u_1) + u_3(u_1u_2) &= 0 \\ v_1(v_2v_3) + v_2(v_3v_1) + v_3(v_1v_2) &= 0 \\ (u_1u_2)v &= u_1(u_2v) + u_2(u_1v) \\ u(v_1v_2) &= (uv_1)v_2 + (uv_2)v_1 \end{aligned} \tag{1.19}$$

It is proved in [1] that the set of idempotents of weight 1 of A is given by

$$\mathrm{Ip}(A) = \{e + u + \tfrac{1}{2}u^2 \ ; \ u \in U_e\}$$

and if $f = e + u_0 + \frac{1}{2}{u_0}^2, u_0 \in U_e$, is another idempotent, then

$$\begin{aligned} U_f &= \{u + u_0u \ ; \ u \in U_e\} \\ V_f &= \{v - u_0v \ ; \ v \in V_e\} \end{aligned}$$

2. Invariance of p -subspaces

We will use in this section the same terminology for p -subspaces found in [3]. Let $A = Ke \oplus Ue \oplus Ve$ be a Peirce decomposition of an algebra satisfying (1.1) or (1.2). Subspaces of A obtained by means of a monomial expression in Ue and Ve such as

$$U_e, \quad V_e \quad U_e^2, \quad V_e^2, \quad U_eV_e, \quad U_e(U_eV_e), \quad \text{etc}$$

are called p -monomials. If m denotes a p -monomial, then ∂m indicates the degree of m. The inclusions $UV \subseteq U$, $U^2 \subseteq V$ e $V^2 \subseteq V$, valid in both cases (1.1) and (1.2), imply that there are two possibilities for a p -monomial m : $m \subseteq U$ or $m \subseteq V$. A p -subspace of A is a sum of p -monomials. For instance,

$$U_e, \quad V_e, \quad U_e + V_e, \quad U_eV_e + V_e^2, \quad U_e^2 + V_e^3 + (U_eV_e)V_e$$

are examples of p -subspaces. A p -monomial is, of course, a particular case of p -subspace. In general, all p -subspaces can be obtained from an ordinary polynomial $p(x, y)$ in two commutative and non associative variables upon the substitution of x for U_e and y for V_e. We denote such p -subspaces by p_e or simply by p. Given a p -subspace p, there are two subspaces $g \subseteq U$ and $h \subseteq V$ of A such that $p = g \oplus h$. Choosing another idempotent $f \in \mathrm{Ip}(A)$ and proceeding as before for the same polynomial $p(x, y)$ we obtain a subspace p_f. If $p_e = p_f$ for every e, $f \in \mathrm{Ip}(A)$, we say that p is *invariant*. If $\dim p_e = \dim p_f$ for every e, $f \in \mathrm{Ip}(A)$, we say that p has *invariant dimension*. In the next section, we prove that all p -subspaces of algebras satisfying (1.1) or (1.2) have invariant dimension. We will also find a necessary and sufficient condition, of easy verification, for a p -subspace being invariant. Such a condition is also necessary and sufficient for a p -subspace being an ideal of A. These results allow us to introduce a large number of numerical invariants both in cases (1.1) and (1.2), namely the dimension of p -subspaces.

2.1. Invariance in algebras satisfying $(x^2)^2 = \omega(x)x^3$

We suppose in this section that (A, ω) satisfies the baric equation in the title. Given e, $f \in \mathrm{Ip}(A)$, let the functions $\sigma : U_e \to U_f$ and $\tau : V_e \to V_f$ be defined by

$$\begin{aligned} \sigma(u) &= u + 2u_0 u \\ \tau(v) &= v - 2u_0 v \end{aligned}$$

where $f = e + u_0 + u_0^2$ with $u_0 \in U_e$. From (1.12) and (1.13), σ and τ are surjective. If $u \in U_e$ and $v \in V_e$ are such that $\sigma(u) = \tau(v) = 0$, then $u = -2u_0u \in U_e \cap V_e$ and so $u = 0$. In the same way, $v = 0$ so that σ and τ are injective. Therefore σ and τ are isomorphisms of vector spaces. Consequently, U and V have invariant dimension. The isomorphism of vector spaces $\varphi : A \to A$ defined by $\varphi(\alpha e + u + v) = \alpha f + \sigma(u) + \tau(v)$ is called the *Peirce transformation of A associated to e and f. The linear operators $\xi : U_e \to U_e$ and $\zeta : V_e \to V_e$ defined by*

$$\begin{aligned} \xi(u) &= u - 2u_0^2 u \\ \zeta(v) &= v + 2u_0^2 v \end{aligned}$$

are also isomorphisms of vector spaces. In fact, if $\xi(u) = 0$, then $u = 2u_0^2 u$. Multiplying this equality by u_0 and using (1.14), we have that $u_0 u = 2u_0(u_0^2 u) = 0$. Likewise, from (1.7) we have $u = 2u_0^2 u = -4u_0(u_0 u) = 0$. Therefore, ξ is injective. Now, if $\zeta(v) = 0$, then $v = -2u_0^2 v$. In the same way, $u_0 v = 0$. Thus, from (1.6) we have $v = 0$.

Lemma 2.1. *The functions σ, τ, ξ and ζ satisfy the following identities, for $u, u_1, u_2 \in U$ and $v, v_1, v_2 \in V$:*

(a) $\sigma(u_1)\sigma(u_2) = \tau(\xi(u_1)\xi(u_2))$;

(b) $\sigma(u)\tau(v) = \sigma(\xi(u)\zeta(v))$;

(c) $\tau(v_1))\tau(v_2)) = \tau(\zeta(v_1)\zeta(v_2)))$.

Proof. From (1.5) and (1.8) we have
$\sigma(u_1)\sigma(u_2) = (u_1)(u_2) - 2u_0(u_1)(u_2) - 2(u_0{}^2)((u_1)(u_2))$. Using (1.3), (1.5) and (1.11) we have $\xi(u_1)\xi(u_2) = (u_1)(u_2) - 2u_0^2((u_1)(u_2))$. Hence,
$\sigma(u_1)\sigma(u_2) = \tau(\xi(u_1)\xi(u_2))$ by (1.14). Now, from (1.5) and (1.11),
$\sigma(u)\tau(v) = uv + 2u_0(uv) + 2u_0{}^2(uv)$. It follows from (1.3), (1.5) and (1.10) that $\xi(u)\zeta(v) = uv + 2u_0 2(uv)$. From (1.14) we obtain
$\sigma(u)\tau(v) = \sigma(\xi(u)\zeta(v))$. Finally (1.5) and (1.10) imply that $\tau(v_1)\tau(v_2) = v_1v_2 - 2u_0(v_1v_2) - 2u_0{}^2(v_1v_2)$. Using (1.3), (1.5) and (1.9) we prove that $\zeta(v_1)\zeta(v_2) = v_1v_2 - 2u_0{}^2(v_1v_2)$. Therefore, $\tau(v_1)\tau(v_2) = \tau(\zeta(v_1)\zeta(v_2))$ by (1.14).

The next corollary follows immediately from the previous lemma.

Corollary 2.2. *Let X, X_1, $X_2 \subseteq U$ and W, W_1, $W_2 \subseteq V$ be subspaces of $A = Ke \oplus U \oplus V$. Then* (*a*) $\sigma(X_1)\sigma(X_2) = \tau(\xi(X_1)\xi(X_2))$;

(*b*) $\sigma(X)\tau(W) = \sigma(\xi(X)\zeta(W))$;

(*c*) $\tau(W_1)\tau(W_2) = \tau(\zeta(W_1)\zeta(W_2))$.

In the following proposition we show that p -subspaces of A absorb products by V.

Proposition 2.3. *Every p -subspace p of A satisfies $Vp \subseteq p$.*

Proof. It is enough to prove the statement for monomials. We have, when the degree of m is 1,

$$\begin{aligned} VU &\subseteq U \\ VV &\subseteq V \end{aligned}$$

If $\partial m \geq 2$ then one of the 3 next possibilities might occur:

$$\begin{aligned} m &= \mu\nu \\ m &= \mu_1\mu_2 \\ m &= \nu_1\nu_2 \end{aligned}$$

where μ, μ_1, $\mu_2 \subseteq U$ and ν, ν_1, $\nu_2 \subseteq V$ are p -monomials with lower degree than ∂m. A generator of $V(\mu\nu)$ has the form $v(uw)$, where $v \in V$, $u \in \mu$ and $w \in \nu$. From (1.10),

$$v(uw) = u(vw) - w(uv) \in \mu(V\nu) + \nu(V\mu)$$

We have that $V(\mu_1\mu_2) = \langle v((u_1)(u_2))\,;\ v \in V,\, u_1 \in \mu_1,\, u_2 \in \mu_2\rangle$. Using (1.11) we obtain

$$v(u_1u_2) = u_1(u_2v) + u_2(u_1v) \in \mu_1(\mu_2V) + \mu_2(\mu_1V)$$

Finally, $V(\nu_1\nu_2)$ is spanned by elements having the form $v(w_1w_2)$, where $v \in V$, $w_1 \in \nu_1$ and $w_2 \in \nu_2$.

From (1.9),

$$v(w_1w_2) = -w_1(w_2v) - w_2(vw_1) \in \nu_1(\nu_2V) + \nu_2(V\nu_1)$$

Now, by induction on ∂m, we obtain $Vm \subseteq m$. □

Corollary 2.4. *For all p -subspaces $g \subseteq U$ and $h \subseteq V$ we have*

$$\xi(g) = g$$
$$\zeta(h) = h$$

Proof. Let $u \in g$. From the preceding proposition, $\xi(u) = u - 2u_0^2 u \in g + Vg = g$. Next, $\xi(g) \subseteq g$ and, as ξ is an isomorphism of vector spaces, $\xi(g) = g$. In the same way, it is shown that $\zeta(h) = h$. □

Subsequently, we have the main result referring to algebras satisfying (1.1).

Theorem 2.5. *Let A be a baric algebra satisfying* (1.1)*. Then:*

(1) *Every p -subspace p of A satisfies $\varphi(p_e) = p_f$, where φ is the Peirce transformation of A associated to the idempotents e and f. In particular, every p -subspace has invariant dimension.*

(2) *The following statements are equivalent relative to the p -subspace p of A :*

(2.1) *p is invariant;*

(2.2) *$Up \subseteq p$;*

(2.3) *p is an ideal of A.*

Proof. It suffices to show the statement (1) for a p -monomial m and, for this, we use induction on ∂m. Let e be an idempotent of weight 1 in A and, for all $u \in U_e$, we consider $f = e + u + u^2$. Since σ and τ are isomorphisms, we have

$$U_f = \sigma(U_e) = \varphi(U_e)$$
$$V_f = \tau(V_e) = \varphi(V_e)$$

Let us suppose that the result is true for all p-monomials with degree $\leq k$ and let m be a p -monomial with $\partial m = k+1$. There are 3 possibilities: $m = \mu\nu$, $m = \mu_1\mu_2$, $m = \nu_1\nu_2$, where μ, μ_1, $\mu_2 \subseteq U$ and ν, ν_1, $\nu_2 \subseteq V$ are p -monomials with degree $\leq k$. If $m = \mu\nu$ then,

$$m_f = \mu_f \nu_f = \sigma(\mu_e)\tau(\nu_e)$$

From corollaries 2.2 and 2.4 we have

$$m_f = \sigma(\xi(\mu_e)\zeta(\nu_e)) = \sigma(\mu_e\nu_e) = \sigma(m_e) = \varphi(m_e)$$

The other cases are similar. To prove part (2), we let $p = g \oplus h$, where g and h are p -subspaces with $g \subseteq U$ and $h \subseteq V$. From part (1), $p_f = g_f \oplus h_f = \{\sigma(x) + \tau(w)\ ;\ x \in g_e,\ w \in h_e\}$. Therefore,

$$p_f = \{(x - 2uw) + (w + 2ux)\ ;\ x \in g_e,\ w \in h_e\} \tag{2.20}$$

Let us suppose that p is invariant. Then, $p_f = p_e$ for every $e, f \in \mathrm{Ip}(A)$. It follows from (2.20) that for $x \in g_e$ and $w \in h_e$, there are $x' \in g_e$ and $w' \in h_e$ such that $x - 2uw = x'$ and $w + 2ux = w'$. Then,

$$uw = \frac{1}{2}(x - x')$$
$$ux = \frac{1}{2}(w' - w)$$

It means that $U_e g_e \subseteq h_e$ and $U_e h_e \subseteq g_e$ and so $U_e p_e \subseteq p_e$. Reciprocally, let us suppose that $Up \subseteq p$. We can state (2.20) as

$$p_f = \{x + w + 2u(x - w)\ ;\ x \in g_e,\ w \in h_e\}$$

and so we obtain $p_f \subseteq p_e + U e p_e$. Now, using the hypothesis, we conclude that $p_f \subseteq p_e$. Since this inclusion is valid for every pair of idempotents, then $p_e = p_f$ and p is invariant. Finally, as we know that $A = Ke \oplus U \oplus V$, $ep = g \subseteq p$ and $Vp \subseteq p$, and so p is an ideal if and only if $Up \subseteq p$. □

2.2. Invariance in algebras satisfying $(x^2)^2 = \omega(x^3)x$

We suppose now that the baric algebra (A, ω) satisfies the baric equation in the title. The proof of next proposition is the same as that of Proposition 2.3 and will be ommitted.

Proposition 2.6. *If p is a p -subspace of any algebra satisfying (1.2) then $Vp \subseteq p$.*

□

For any $u_0 \in U$ and $\alpha,\ \beta \in K$ we consider the linear operator $T_{(\alpha,\beta)}\ : N \to N$ defined by

$$T_{(\alpha,\beta)}(x) = x + \alpha u_0 x + \beta u_0^2 x$$

Such operators satisfy the following properties:

Lemma 2.7. *For every $u_0 \in U$ and $\alpha,\ \beta \in K$, we have:*
$T_{(\alpha,\beta)}$ is an automorphism of vector spaces;
$T_{(0,\beta)}(p) = p$ for every p -subspace p of A.

Proof. Let $x = u + v \in N$ ($u \in U$ and $v \in V$) be such that $T_{(\alpha,\beta)}(x) = 0$. Then $u + v + \alpha u_0(u + v) + \beta u_0^2(u + v) = 0$, and so

$$\begin{aligned} u + \alpha u_0 v + \beta u_0^2 u &= 0 \\ v + \alpha u_0 u + \beta u_0^2 v &= 0 \end{aligned}$$

Multiplying these identities by u_0 and using (1.17), (1.18) and (1.19), we obtain

$$\begin{aligned} u_0 u + \tfrac{1}{2}\alpha u_0^2 v &= 0 \\ u_0 v - \tfrac{1}{2}\alpha u_0^2 u &= 0 \end{aligned}$$

Again, multiplying by u_0 the latest 2 equalities and using (1.17), (1.18) and (1.19), we have ${u_0}^2 u = {u_0}^2 v = 0$. Then $u_0 u = u_0 v = 0$. Therefore $u = v = 0$, $T_{(\alpha,\beta)}$ is injective and so it is an isomorphism. Let $x \in p$; we know that ${u_0}^2 \in V$, and so from Proposition 2.6, we have

$$T_{(0,\beta)}(x) = x + \beta u_0^2 x \in p + Vp = p$$

Hence $T_{(0,\beta)}(p) \subseteq p$ and, since $T_{(\alpha,\beta)}$ is injective, we have $T_{(0,\beta)}(p) = p$. $\square$

Given $e,\ f \in \mathrm{Ip}(A)$, where $f = e + u_0 + \frac{1}{2}u_0 2$, $u_0 \in Ue$, we use the following notations:
$\sigma = T_{(1,0)}|U_e$, $\tau = T_{(-1,0)}|V_e$, $\xi = T_{(0,-\frac{1}{2})}|U_e$, $\zeta = T(0, \frac{1}{2})|V_e$. We have that $\sigma\ : U_e \to U_f$, $\tau\ : V_e \to V_f$, $\xi\ : U_e \to U_e$ and $\zeta\ : V_e \to V_e$. The vector space isomorphism $\varphi\ : A \to A$ defined by $\varphi(\alpha e + u + v) = \alpha f + \sigma(u) + \tau(v)$ is the *Peirce transformation* of A associated to e and f. The next result is proved similarly to Lemma 2.1:

Lemma 2.8. *The functions σ, τ, ξ and ζ satisfy the identities:*

(*a*) $\sigma(u_1)\sigma(u_2) = \tau(\xi(u_1)\xi(u_2))$;

(*b*) $\sigma(u)\tau(v) = \sigma(\xi(u)\zeta(v))$;

(*c*) $\tau(v_1))\tau(v_2)) = \tau(\zeta(v_1)\zeta(v_2))$;

for every u, u_1, $u_2 \in U_e$ *and* v, v_1, $v_2 \in V_e$.

Finally, as we have done before, we can prove the next theorem.

Theorem 2.9. *Let* A *be a baric algebra satisfying* (1.2). *Then*

(1) *Every* p*-subspace* p *of* A *satisfies* $\varphi(p_e) = p_f$. *In particular, every* p*-subspace has invariant dimension.*

(2) *The following statements about a* p *-subspace* p *of* A *are equivalent:*

  (2.1) p *is invariant;*

  (2.2) $Up \subseteq p$;

  (2.3) p *is an ideal.*

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