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STATIONARY SOLUTIONS OF MAGNETO-MICROPOLAR FLUID EQUATIONS IN EXTERIOR DOMAINS

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Abstract

We establish the existence and uniqueness of the solution for the magneto-micropolar fluid equations in the case of exterior domains in $I\!R^3$. First, we prove the existence of at least one weak solution of the stationary system. Then we discuss its uniqueness.

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1. Introduction

As in many non-bounded physical situations, the study of the dynamics of the magneto-micropolar fluid model considered on an unbounded domain plays an important and useful role. We often find solid-fluid structures in which a bounded obstacle stops or impedes the flow of the surrounding fluid and the spatial volume of the external environment of the body, namely exterior domain, is extensively much larger than the obstacle.

From the modeling point of view, this case may be regarded as a compact domain located in all of $I\!R^3$. Let K denote this compact subset, and let Ω denote its complement in $I\!R^3$, that is, $\Omega = K^c$. We deal with the existence of a weak solutions for equations that describe the motion of a viscous incompressible magneto-micropolar fluid in the exterior domain Ω. Such mathematical model (e.g., see [2]) reads: Find the three-dimensional fields $(\mathbf{u}, \mathbf{w}, \mathbf{h}) : \Omega \to \mathbf{R}^9$ and the scalar functions $(p, q) : \Omega \to \mathbf{R}^2$ which satisfy the system of equations:

$$-(\mu+\chi)\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla(p+\frac{r}{2}\mathbf{h}\cdot\mathbf{h}) = \chi \operatorname{rot}\mathbf{w}+r\,(\mathbf{h}\cdot\nabla)\mathbf{h}+\mathbf{f} \quad \text{in } \Omega, \tag{1.1}$$

$$-\gamma\Delta\mathbf{w} - (\alpha+\beta)\nabla \operatorname{div}\mathbf{w} + j\,(\mathbf{u}\cdot\nabla)\mathbf{w} + 2\chi\mathbf{w} = \chi \operatorname{rot}\mathbf{u} + \mathbf{g} \quad \text{in } \Omega, \tag{1.2}$$

$$-\nu\Delta\mathbf{h} + (\mathbf{u}\cdot\nabla)\mathbf{h} - (\mathbf{h}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{0} \quad \text{in } \Omega, \tag{1.3}$$

$$div\,\mathbf{u} = 0, \quad \operatorname{div}\mathbf{h} = 0 \qquad \text{in } \Omega, \tag{1.4}$$

$$\mathbf{u}(x) = \mathbf{w}(x) = \mathbf{h}(x) = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{1.5}$$

Here $\mathbf{u}(x), \mathbf{w}(x), \mathbf{h}(x) \in \mathbf{R}^3$ denote, respectively, the velocity, the microrotational velocity, and the magnetic field of the fluid at point $x \in \Omega$, and $p(x), q(x) \in \mathbf{R}$ denote the hydrostatic and magnetic pressures at the same place. The values $\mu, \chi, r, \alpha, \beta, \gamma, j$ and ν are constants associated to properties of the material. It follows that these constants satisfy $\min\{\mu, \chi, r, j, \gamma, \nu, (\alpha+\beta), \gamma\} > 0$, by physical reasons, and $\mathbf{f}(x)$, $\mathbf{g}(x) \in \mathbf{R}^3$ are given external fields.

To complete the system of equations, we establish the behaviour of the solution at infinity. We consider the classical homogeneous decay

$$\lim_{|x|\to\infty} \mathbf{u}(x) = \lim_{|x|\to\infty} \mathbf{w}(x) = \lim_{|x|\to\infty} \mathbf{h}(x) = \mathbf{0}\,. \tag{1.1}$$

It is important to remark that our case involves solutions in the space $J(\Omega)$ defined below, hence no weighted spaces are required. From a physical viewpoint, we note that the set K represents a non-perfect conductor body and q is an unknown function concerned with the motion of heavy ions (e.g. see [3]),

$$\nabla q \,=\, \frac{-1}{\sigma}\,\mathrm{rot}\,\mathbf{j}_0\,,$$

where $\mathbf{j}_0$ is the density of electric current and $\sigma\,>\,0$ is the constant electric conductivity.

Equation (1.1) has the familiar form of the Navier-Stokes equations though coupled with (1.2) and (1.3). Equation (1.2) describes the motion inside the macro-volumes as they undergo microrotational effects, which are represented by the microrotational velocity vector $\mathbf{w}$. For fluids with no micro structure, this velocity vanishes and we deal with a magneto-hydrodynamics system. For Newtonian fluids, where $\chi = 0$, equation (1.1) decouples from equation (1.2). Equation (1.3), which is the equation for $\mathbf{h}$, is none other than the Maxwell system in which the electrical field is determined in a posteriori way. It is also important to note that if $\mathbf{h}\,=\,\mathbf{0}$, we consider the well known stationary asymmetric fluid model.

It is worth citing some earlier work done on the boundary value problem (1.1)-(1.5) which is related to ours, and locating our contribution therein. When the magnetic field is absent (i. e. $\mathbf{h} \equiv \mathbf{0}$), the reduced problem on bounded domains was studied by Lukaszwicz [9] and Galdi & Rionero [10] ; for exterior domains, it was studied by Abid [1] and Padula & Russo [11]. In [9] under classical regularity assumptions, the existence of weak solutions for (1.1) – (1.5) was established, which was done by considering linearization and applying the Leray-Schauder Principle. In addition, using the regularity arguments of the Stokes equations for $\mathbf{u}$ and those of elliptic systems for $\mathbf{w}$, the regularity of solutions for the entire system is proved ; furthermore, conditions under which the uniqueness holds are determined. Again, when $\mathbf{h} \equiv \mathbf{0}$ and in the exteriro case, Abid [1] establishes similar results to those of Lukaszewicz's. In this case, they are deduced using results due to Girault and Sequeira [5] for the Navier-Stokes equations.

In this article, to prove existence of weak solutions we use "the extending domain method" as in Ladyzhenskaya [8] and Heywood [7]. We also discuss the uniqueness of solutions. Thus we study weak solutions the same way as in the classical study of the Navier-Stokes equations.

We will formulate our main results, theorems 2.3 and 2.4, in Section 2. In it, we also state the basic assumptions and results which are used later on and rewrite system (1.1)-(1.6) in order to define the weak solution formally. In Section 3 we describe the truncation method used in the proof of Theorem 2.3, which is done in Section 4. The uniqueness of the weak solution will be discussed in Section 5.

2. Functional Spaces and Preliminaries

Throughout, the functions are defined on $\Omega$and are either $I\!R$ or $I\!R^3$-valued; we will distinguish these two situations in our notation.

We now give the precise definition of the exterior domain $\Omega$where our boundary value problem, that is original problem (1.1)-(1.6), has been formulated. Let Kbe a non-void compact and connected subset of $I\!R^3$whose boundary $\partial K$is of class C^2. The exterior domain Ωthat we consider is $\Omega = K^c$, and $\partial\Omega = \partial K$, of course.

The extending domain method was introduced by Ladyzhenkaya [8] to study the Navier-Stokes equations in unbounded domains. As was remarked by Heywood [7], the method is useful in the class of problems of exterior domain, which is where our problem is located. The principal idea of the method is as follows: the exterior domain Ωmay be approximated by interior domains $\Omega_k = B_k \cap \Omega$, for every $k \geq 1$, with $B_k$the ball of radius $k$and center at the origin. In each interior domain Ω_k, we prove the existence of a weak solution. For this we use the Galerkin method together with Brouwer's Fixed Point Theorem as in Heywood [7]. Next, using the estimates given in Ladyzhenskaya [8] together with the diagonal argument and Rellich's compactness theorem, we obtain the desired weak solution to the problem (1.1)-(1.6). Moreover, we are able to establish properties of regularity and uniqueness of this solution.

We establish here some spaces of vector-valued functions. For the sake of simplicity of the notation, we will call $D$the domain that represents Ωor Ω_k.

$$W^{r,p}(D) = \{ \mathbf{v} = (v_i)_{i=1,3} \ ; \ D^{\alpha} v_i \in L^p(D), \ |\alpha| \leq r,$$

$$\forall\, i = 1, .., n \ \}, \tag{2.1}$$

$$W_0^{r,p}(D) = \textit{Closure of } C_0^\infty(D) \textit{ in } W^{r,p}(D)\,, \tag{2.2}$$

$$W_0(D) = \textit{Closure of } C_0^\infty(D) \textit{ in norm } \|\nabla\varphi\|_{L^2(D)}\,, \tag{2.3}$$

$$C_{0,\sigma}^\infty(D) = \{\varphi \in C_0^\infty(D)\ ;\ \operatorname{div}\varphi = 0\}\,, \tag{2.4}$$

$$H(D) = \textit{Closure of } C_{0,\sigma}^\infty(D) \textit{ in norm } \|\varphi\|_{L^2(D)}\,, \tag{2.5}$$

$$J(D) = \textit{Closure of } C_{0,\sigma}^\infty(D) \textit{ in norm } \|\nabla\varphi\|_{L^2(D)}\,, \tag{2.6}$$

$$J_0(D) = \{\mathbf{v} \in W_0(D)\ ;\ \operatorname{div}\mathbf{v} = 0\}\,, \tag{2.7}$$

where $\|\cdot\|_{L^p(D)}$represents the $L^p(D)$-norm, $1 \le p \le \infty$. As usual $W^{r,2}(D) \equiv H^r(D)$. In particular, $W^{0,2}(D) \equiv L^2(D)$and its inner product is expressed solely by $(\cdot,\cdot)$,that is, without subscripts.

Since $D$is bounded or an exterior domain, we note that $J(D)$is equivalent to the space $J_0(D)$, which is proved by Heywood [6]. Also, it is clear that $W_0(\Omega_k) = H_0^1(\Omega_k)$. The following a priori estimations are crucial and may be found in Ladyzhenskaya [8].

Lemma 2.1 *Let $D \subseteq \mathbb{R}^3$be bounded or unbounded. Then*
(a) *For any $\mathbf{u} \in J(D)$(or $W_0(D)$,or $H_0^1(D)$), the following inequality holds:*

$$\|\mathbf{u}\|_{L^6(D)} \le C_L\, \|\nabla\mathbf{u}\|_{L^2(D)}\,, \tag{2.8}$$

where $0 < C_L \le (48)^{1/6}$.
(b) *(Hölder's inequality) If each integral makes sense, we have*

$$|\,(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})|\ \le\ 3^{\frac{1}{p}+\frac{1}{r}}\ \|\mathbf{u}\|_{L^p(D)}\ \|\nabla\mathbf{v}\|_{L^q(D)}\|\mathbf{w}\|_{L^r(D)}\,, \tag{2.9}$$

where $p, q, r > 0$ *and* $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 1.$ ■

We now state our problem rigorously establishing regularity assumptions on the boundary $\partial\Omega$ and on the external forces.

($\mathbf{S}_1$) *Let* O_0 *be a neighbourhood of the origin. Thus* $O_0 \subseteq \mathrm{int}K$ *and* $K \subseteq B(0,R)$, $R > 0$;
($\mathbf{S}_2$) $\partial\Omega = \partial K \in C^2$;
($\mathbf{S}_3$) *The given external fields are regulars:* $\mathbf{f} \in L^2(\Omega)$ *and* $\mathbf{g} \in L^2(\Omega)$.

In what follows, we introduce the classical forms
(2.10)
$$a(\mathbf{u},\mathbf{v}) = \sum_{i,j=1}^{3} \int_D \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i}\, dx \quad \text{and} \quad b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{3} \int_D u_j \frac{\partial v_i}{\partial x_j}\, w_i\, dx$$

defined for any vector-valued functions $\mathbf{u}, \mathbf{v}, \mathbf{w}$ for which the integrals make sense.

We next define the meaning of a weak solution for (1.1)-(1.6) and set out the main results:

Definition 2.2 *We say a triplet of functions* $(\mathbf{u},\ \mathbf{w},\ \mathbf{h})$*, defined on* Ω*, is a weak solution of* (1.1)-(1.6) *if and only if*

i) $\mathbf{u},\ \mathbf{h} \in J(\Omega)$ *and* $\mathbf{w} \in W_0(\Omega)$;

ii) $\mathbf{u}$, $\mathbf{w}$, *and* $\mathbf{h}$ *satisfy the variational formulation*

$$(\mu+\chi)a(\mathbf{u},\varphi) - b(\mathbf{u},\varphi,\mathbf{u}) + rb(\mathbf{h},\varphi,\mathbf{h}) = (\mathbf{f},\varphi) + \chi(\mathbf{w},\ \mathrm{rot}\,\varphi)\,, \tag{2.11}$$

$$\gamma a(\mathbf{w},\xi) + (\alpha+\beta)(\mathrm{div}\,\mathbf{w},\ \mathrm{div}\,\xi) - jb(\mathbf{u},\xi,\mathbf{w}) + 2\chi(\mathbf{w},\xi) \tag{2.12}$$

$$= (\mathbf{g},\xi) + \chi(\mathbf{u},\mathrm{rot}\,\xi)\,, \tag{2.13}$$

$$\nu a(\mathbf{h},\psi) - b(\mathbf{u},\psi,\mathbf{h}) + b(\mathbf{h},\psi,\mathbf{u}) = 0\,,$$

for all $\varphi, \psi \in C^\infty_{0,\sigma}(\Omega)$ *and* $\xi \in C^\infty_0(\Omega)$. ■

Remark. If $\mathbf{u}$, $\mathbf{h} \in J(\Omega)$ and $\mathbf{w} \in W_0(\Omega)$, then $\mathbf{u}|_{\partial\Omega} = \mathbf{h}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega} = \mathbf{0}$. Moreover, for Lemma 2.1

$$\lim_{|x|\to\infty} \mathbf{u}(x) = \lim_{|x|\to\infty} \mathbf{w}(x) = \lim_{|x|\to\infty} \mathbf{h}(x) = \mathbf{0}\,.$$

We also see that the pressures are recovered by a standard application of De Rham's Theorem. ■

Theorem 2.3 (Existence) *Under the hypotheses* ($\mathbf{S}_1$), ($\mathbf{S}_2$) *and* ($\mathbf{S}_3$), *problem* (1.1)-(1.6) *has a stationary weak solution.* ■

Theorem 2.4 (Uniqueness) *Under hypotheses* ($\mathbf{S}_1$), ($\mathbf{S}_2$) *and* ($\mathbf{S}_3$), *if there exists a stationary weak solution satisfying the conditions*

$$\frac{\sqrt{3}\,C_L}{2\mu}\,\left(2\,\|\mathbf{u}\|_{L^3(\Omega)} + \|\mathbf{w}\|_{L^3(\Omega)} + 2r\,\|\mathbf{h}\|_{L^3(\Omega)}\right) < 1, \tag{2.14}$$

$$\frac{\sqrt{3}\,C_L}{\nu}(\|\mathbf{u}\|_{L^3(\Omega)} + \|\mathbf{h}\|_{L^3(\Omega)}) < 1, \quad \textit{and} \tag{2.15}$$

$$\frac{\sqrt{3}\,r\,C_L}{2\gamma}\|\mathbf{w}\|_{L^3(\Omega)} < 1\,, \tag{2.16}$$

where $0 < C_L \leq (48)^{1/6}$, *then the weak solution is unique.* ■

3. The Interior Problem

In this paragraph we are interested in considering the following interior problem, namely (P_k) in domains $\Omega_k = B_k \cap \Omega$, with $k \in \mathbf{N}$.

$$(P_k)\quad \begin{cases} -(\mu+\chi)\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(p + \frac{r}{2}\mathbf{h}\cdot\mathbf{h}) = \chi\,\mathrm{rot}\,\mathbf{w} + r\,(\mathbf{h}\cdot\nabla)\mathbf{h} + \mathbf{f} \\ \textit{in } \Omega_k\,, \\ -\gamma\Delta\mathbf{w} - (\alpha+\beta)\nabla\,\mathrm{div}\,\mathbf{w} + j\,(\mathbf{u}\cdot\nabla)\mathbf{w} + 2\chi\mathbf{w} \;=\; \chi\,\mathrm{rot}\,\mathbf{u} + \mathbf{g} \quad \text{in } \Omega_k\,, \\ -\nu\Delta\mathbf{h} + (\mathbf{u}\cdot\nabla)\mathbf{h} \;-\; (\mathbf{h}\cdot\nabla)\mathbf{u} \;+\; \nabla q \;=\; \mathbf{0} \quad \text{in } \Omega_k\,, \\ \mathrm{div}\,\mathbf{u} = 0, \quad \mathrm{div}\,\mathbf{h} = 0 \qquad \text{in } \Omega_k\,, \\ \mathbf{u}(x) = \mathbf{w}(x) = \mathbf{h}(x) = \mathbf{0} \qquad \text{on } \partial\Omega_k\,. \end{cases}$$

It is worth noting that $\partial\Omega_k = \partial\Omega \cup \partial B_k$. Also, it is straightforward to see that the mean of weak solution for (P_k) is completely similar to the one for (1.1)-(1.6).

Proposition 3.1 *Problem (P_k) admits at least one weak solution $(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k, \widetilde{\mathbf{h}}^k) \in J(\Omega_k) \times H_0^1(\Omega_k) \times J(\Omega_k)$.* ■

To prove existence of weak solutions for system (P_k), we use the Galerkin method together with Brouwer's Fixed Point Theorem as in Fujita [4] (see also Heywood [7]).

We first prove a priori estimates for any weak solution of (P_k).

Lemma 3.2 *Let $(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k, \widetilde{\mathbf{h}}^k)$ a weak solution of (P_k). Then it satisfies the estimate*

$$\mu \,\|\, \nabla\widetilde{\mathbf{u}}^k\|^2_{L^2(\Omega_k)} + \gamma \,\|\, \nabla\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)} + 2r\nu \,\|\, \nabla\widetilde{\mathbf{h}}^k\|^2_{L^2(\Omega_k)} \leq$$

$$\frac{1}{\mu}\,\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{1}{\gamma}\|\mathbf{g}\|^2_{W_0(\Omega)^*}. \tag{3.1}$$

Proof. Multiplying $(P_k)i$, $(P_k)ii$ and $(P_k)iii$ by $\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k$ and $r\widetilde{\mathbf{h}}^k$, respectively, and integrating by parts in Ω_k we obtain

$$(\mu+\chi)\,a(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{u}}^k) = \chi\,(\mathrm{rot}\,\widetilde{\mathbf{w}}^k, \widetilde{\mathbf{u}}^k) + rb\,(\widetilde{\mathbf{h}}^k, \widetilde{\mathbf{h}}^k, \widetilde{\mathbf{u}}^k) + (\mathbf{f}, \widetilde{\mathbf{u}}^k)\,,$$

$$\gamma\,a(\widetilde{\mathbf{w}}^k, \widetilde{\mathbf{w}}^k) + (\alpha+\beta)\,\|\mathrm{div}\,\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)} + 2\chi\,\|\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)} =$$

$$\chi\,(\mathrm{rot}\,\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k) + (\mathbf{g}, \widetilde{\mathbf{w}}^k)\,, r\nu\,a(\widetilde{\mathbf{h}}^k, \widetilde{\mathbf{h}}^k) \,=\, r\,b(\widetilde{\mathbf{h}}^k, \widetilde{\mathbf{u}}^k, \widetilde{\mathbf{h}}^k)\,.$$

Adding all the above equalities we have

$$(\mu+\chi)\,a(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{u}}^k) + \gamma\,a(\widetilde{\mathbf{w}}^k, \widetilde{\mathbf{w}}^k) + r\nu\,a(\widetilde{\mathbf{h}}^k, \widetilde{\mathbf{h}}^k) +$$

$$(\alpha+\beta)\,\|\mathrm{div}\,\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)}$$

$$+\,2\chi\,\|\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)} = 2\chi\,(\mathrm{rot}\,\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k) + (\mathbf{f}, \widetilde{\mathbf{u}}^k) + (\mathbf{g}, \widetilde{\mathbf{w}}^k)\,, \tag{3.2}$$

since $r\,b(\widetilde{\mathbf{h}}^k, \widetilde{\mathbf{h}}^k, \widetilde{\mathbf{u}}^k) + r\,b(\widetilde{\mathbf{h}}^k, \widetilde{\mathbf{u}}^k, \widetilde{\mathbf{h}}^k) = 0\,.$

We estimate the right side of equality (3.2) obtaining

$$2\chi\,(\mathrm{rot}\,\widetilde{\mathbf{u}}^k,\ \widetilde{\mathbf{w}}^k)$$

$$\leq 2\chi\,\|\mathrm{rot}\,\widetilde{\mathbf{u}}^k\|_{L^2(\Omega_k)}\,\|\widetilde{\mathbf{w}}^k\|_{L^2(\Omega_k)} = 2\chi\,\|\widetilde{\mathbf{w}}^k\|_{L^2(\Omega_k)}\,\|\nabla\widetilde{\mathbf{u}}^k\|_{L^2(\Omega_k)}$$

$$\leq \chi\,\|\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)} + \chi\,a(\widetilde{\mathbf{u}}^k,\widetilde{\mathbf{u}}^k)\,,$$

since $\|\mathrm{rot}\,\widetilde{\mathbf{u}}^k\|_{L^2(\Omega_k)} = \|\nabla\widetilde{\mathbf{u}}^k\|_{L^2(\Omega_k)}$. We also have

$$\begin{aligned}(\mathbf{f},\widetilde{\mathbf{u}}^k) &\leq \|\mathbf{f}\|_{J(\Omega_k)^*}\|\nabla\widetilde{\mathbf{u}}^k\|_{L^2(\Omega_k)} \leq \frac{1}{2\mu}\|\mathbf{f}\|^2_{J(_0\Omega)^*} + \frac{\mu}{2}\,a(\widetilde{\mathbf{u}}^k,\widetilde{\mathbf{u}}^k),\\ (\mathbf{g},\widetilde{\mathbf{w}}^k) &\leq \|\mathbf{g}\|_{W_0(\Omega_k)^*}\,\|\nabla\widetilde{\mathbf{w}}^k\|_{L^2(\Omega_k)} \leq \frac{1}{2\gamma}\|\mathbf{g}\|^2_{W_0(\Omega)^*} + \frac{\gamma}{2}a(\widetilde{\mathbf{w}}^k,\widetilde{\mathbf{w}}^k)\,.\end{aligned}$$

Consequently, using the above estimates in (3.2), we obtain

$$\mu\,a(\widetilde{\mathbf{u}}^k,\widetilde{\mathbf{u}}^k) + \gamma\,a(\widetilde{\mathbf{w}}^k,\ \widetilde{\mathbf{w}}^k) + 2r\,\nu\,a(\widetilde{\mathbf{h}}^k,\ \widetilde{\mathbf{h}}^k) + 2\,\chi\,\|\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)}$$

$$+\,2\,(\alpha+\beta)\,\|\mathrm{div}\,\widetilde{\mathbf{w}}^k\|^2_{L^2(\Omega_k)} \ \leq\ \frac{1}{\mu}\,\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{1}{\gamma}\,\|\mathbf{g}\|^2_{W_0(\Omega)^*}\,,$$

which immediately implies (3.1). ■

Remark. We note that estimate (3.1) is independent of k. ■

Now we prove the existence of a solution $(\widetilde{\mathbf{u}}^k,\widetilde{\mathbf{w}}^k,\widetilde{\mathbf{h}}^k)$ for (P_k). It will be a limit of a sequence obtained as follows: Let V_m be the finite dimensional subspace of $J(\Omega_k)$ spanned by $\{\varphi^1,...,\varphi^m\}$, and let M_m be the finite dimensional subspace of $H^1_0(\Omega_k)$ spanned by $\{\xi^1,...,\xi^m\}$. As mth-approximate solution of (P_k), we choose functions

$$\mathbf{u}^m(x) = \sum_{j=1}^m c_{mj}\,\varphi^j(x),\quad \mathbf{w}^m(x) = \sum_{j=1}^m d_{mj}\,\xi^j(x)\quad\text{and}$$

$$\mathbf{h}^m(x) = \sum_{j=1}^m e_{mj}\,\varphi^j(x)\,, \tag{3.3}$$

satisfying the following equalities, for every $j\in\{1,..,m\}$:

(3.4)

$$(\mu+\chi)\,a(\mathbf{u}^m,\varphi^j)+b(\mathbf{u}^m,\mathbf{u}^m,\varphi^j)-r\,b(\mathbf{h}^m,\mathbf{h}^m,\varphi^j) = \chi\,(\mathrm{rot}\,\mathbf{w}^m,\ \varphi^j)+(\mathbf{f},\ \varphi^j)\,,$$

$$\gamma\, a(\mathbf{w}^m, \xi^j) + (\alpha+\beta)\,(\operatorname{div}\mathbf{w}^m, \operatorname{div}\xi^j) + \; jb\,(\mathbf{u}^m, \mathbf{w}^m, \xi^j) + 2\chi\,(\mathbf{w}^m, \xi^j)$$

$$= \chi\,(\operatorname{rot}\mathbf{u}^m, \xi^j) + (\mathbf{g}, \xi^j)\,, \tag{3.5}$$

$$\nu\, a(\mathbf{h}^m, \varphi^j) + b(\mathbf{u}^m, \mathbf{h}^m, \varphi^j) - b(\mathbf{h}^m, \mathbf{u}^m, \varphi^j) = 0\,. \tag{3.6}$$

Note that the solutions $(\mathbf{u}^m, \mathbf{w}^m, \mathbf{h}^m)$ must satisfy estimate (3.1). In fact, this identity is obtained multiplying (3.4), (3.5), (3.6) by, respectively, the coefficients c_{mj}, d_{mj}, e_{mj}, next summing over j from 1 to m, and finally following the idea given in the proof of Lemma 3.2. Therefore the sequence $\{(\mathbf{u}^m, \mathbf{w}^m, \mathbf{h}^m)\}_{m\in\mathbf{N}}$ is uniformly bounded in $J(\Omega_k) \times H_0^1(\Omega_k) \times J(\Omega_k)$. Thus, assuming that the system (3.4,5,6) admits at least one weak solution, we have:

Proof of Proposition 3.1 Since $J(\Omega_k)$ (respectively $H_0^1(\Omega_k)$) is compactly embedded in $H(\Omega_k)$ (respectively $L^2(\Omega_k)$), we may choose subsequences which, again, we denote by $(\mathbf{u}^m, \mathbf{w}^m, \mathbf{h}^m)$, and elements $\widetilde{\mathbf{u}}^k \in J(\Omega_k)$, $\widetilde{\mathbf{w}}^k \in H_0^1(\Omega_k)$, and $\widetilde{\mathbf{h}}^k \in J(\Omega_k)$ such that

$$\left.\begin{array}{l} \mathbf{u}^m \to \widetilde{\mathbf{u}}^k \\ \mathbf{h}^m \to \widetilde{\mathbf{h}}^k \end{array}\right\} \text{ weakly in } J(\Omega_k) \text{ and strongly in } H(\Omega_k)\,,$$

$$\mathbf{w}^m \to \widetilde{\mathbf{w}}^k \quad \text{weakly in } H_0^1(\Omega_k) \text{ and strongly in } L^2(\Omega_k)\,.$$

It suffices to take the limit as m goes to infinity in equations (3.4), (3.5), (3.6) and the proof now follows straightforwardly. ■

Finally, to prove the solvability of system (3.4,5,6) for any k, $m \in \mathbf{N}$, we follow Heywood [7] in applying Brouwer's Fixed Point Theorem. For every $(\varphi, \xi, \psi) \in V_m \times M_m \times V_m$, we consider the unique solution of $L(\varphi, \xi, \psi) = (\mathbf{u}, \mathbf{w}, \mathbf{h}) \in V_m \times M_m \times V_m$ of the linearized equations

$$(\mu+\chi)\, a(\mathbf{u}, \varphi^j) + b(\varphi, \mathbf{u}, \varphi^j) - r\, b(\psi, \mathbf{h}, \varphi^j) - \chi\,(\operatorname{rot}\mathbf{w}, \varphi^j) = (\mathbf{f}, \varphi^j)\,, \tag{3.7}$$

$$\gamma\, a(\mathbf{w}, \xi^j) + (\alpha+\beta)\,(\operatorname{div}\mathbf{w}, \operatorname{div}\xi^j) + j\, b(\varphi, \mathbf{w}, \xi^j) + 2\chi\,(\mathbf{w}, \xi^j)$$

$$-\chi\,(\operatorname{rot}\mathbf{u},\xi^j) = (\mathbf{g},\xi^j)\,, \tag{3.8}$$

$$\nu\, a(\mathbf{h},\varphi^j) + b(\varphi,\mathbf{h},\varphi^j) - b(\psi,\mathbf{u},\varphi^j) = 0\,, \tag{3.9}$$

for all $1 \le j \le m$. This is a system of $3m$ linear equations for the coefficients in (3.3).

Equations (3.7), (3.8), and (3.9) have a unique solution, since the associated homogeneous system ($\mathbf{f}=\mathbf{0}$, $\mathbf{g}=\mathbf{0}$) has an unique one. In fact, if $(\mathbf{u},\mathbf{w},\mathbf{h})$ is a solution of the homogeneous system and proceeding as before, we obtain

$$\begin{aligned}
&(\mu+\chi)\,\|\nabla\mathbf{u}\|^2_{L^2(\Omega_k)} = \chi\,(\operatorname{rot}\mathbf{w},\mathbf{u}) + r\, b(\psi,\mathbf{h},\mathbf{u})\,,\\
&\gamma\,\|\nabla\mathbf{w}\|^2_{L^2(\Omega_k)} + (\alpha+\beta)\,\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega_k)} + 2\chi\,\|\mathbf{w}\|^2_{L^2(\Omega_k)} = \chi\,(\operatorname{rot}\mathbf{u},\mathbf{w}),\\
&r\nu\,\|\nabla\mathbf{h}\|^2_{L^2(\Omega_k)} = r\, b(\psi,\mathbf{u},\mathbf{h}).
\end{aligned}$$

Adding the above identities we obtain

$$\begin{aligned}
&(\mu+\chi)\,\|\nabla\mathbf{u}\|^2_{L^2(\Omega_k)} + \gamma\,\|\nabla\mathbf{w}\|^2_{L^2(\Omega_k)} + r\nu\,\|\nabla\mathbf{h}\|^2_{L^2(\Omega_k)}\\
&+2\chi\,\|\mathbf{w}\|^2_{L^2(\Omega_k)} + (\alpha+\beta)\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega_k)} = 2\chi(\operatorname{rot}\mathbf{u},\mathbf{w})\\
&\le 2\chi\|\nabla\mathbf{u}\|_{L^2(\Omega_k)}\,\|\mathbf{w}\|_{L^2(\Omega_k)}\\
&\le \chi\|\nabla\mathbf{u}\|^2_{L^2(\Omega_k)} + \chi\|\mathbf{w}\|^2_{L^2(\Omega_k)}\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\mu\,\|\nabla\mathbf{u}\|^2_{L^2(\Omega_k)} + \gamma\,\|\nabla\mathbf{w}\|^2_{L^2(\Omega_k)} + r\nu\,\|\nabla\mathbf{h}\|^2_{L^2(\Omega_k)}\\
&+\chi\,\|\mathbf{w}\|^2_{L^2(\Omega_k)} + (\alpha+\beta)\|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega_k)} \le 0\,,
\end{aligned}$$

hence $\mathbf{u}=\mathbf{0}$, $\mathbf{w}=\mathbf{0}$ and $\mathbf{h}=\mathbf{0}$.

The continuity of L follows from arguments similar to those used for taking the limit in (3.4,5,6). We also have the estimate

$$\mu \|\nabla \mathbf{u}\|^2_{L^2(\Omega_k)} + \gamma \|\nabla \mathbf{w}\|^2_{L^2(\Omega_k)} + 2r\nu \|\nabla \mathbf{h}\|^2_{L^2(\Omega_k)} \leq \frac{1}{\mu}\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{1}{\gamma}\|\mathbf{g}\|^2_{W_0(\Omega)^*},$$

which are shown exactly in the same way as was done for a solution $(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k, \widetilde{\mathbf{h}}^k)$ in Lemma 3.2. Then

$$\|\nabla \mathbf{u}\|^2_{L^2(\Omega_k)} \leq \frac{1}{\mu^2}\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{1}{\gamma\mu}\|\mathbf{g}\|^2_{W_0(\Omega)^*} \equiv \ell_1^2, \tag{3.10}$$

$$\|\nabla \mathbf{w}\|^2_{L^2(\Omega_k)} \leq \frac{1}{\mu\gamma}\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{1}{\gamma^2}\|\mathbf{g}\|^2_{W_0(\Omega)^*} \equiv \ell_2^2, \tag{3.11}$$

$$\|\nabla \mathbf{h}\|^2_{L^2(\Omega_k)} \leq \frac{1}{2r\nu\mu}\|\mathbf{f}\|^2_{J(\Omega)^*} + \frac{1}{2r\nu\gamma}\|\mathbf{g}\|^2_{W_0(\Omega)^*} \equiv \ell_3^2. \tag{3.12}$$

Thus (3.10), (3.11), and (3.12) define a continuous mapping L from the closed and convex set

$$S = \{(\varphi, \xi, \psi) \in V_m \times M_m \times V_m \;;$$

$$\|\nabla\varphi\|_{L^2(\Omega_k)} \leq \ell_1,\ \|\nabla\xi\|_{L^2(\Omega_k)} \leq \ell_2,\ \|\nabla\psi\|_{L^2(\Omega_k)} \leq \ell_3\}$$

into itself. We now state a result which finish the proof of existence of a weak solution $(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k, \widetilde{\mathbf{h}}^k)$ of (P_k).

Corollary 3.4 *The finite dimensional problem* (3.4)-(3.6) *admits at least one weak solution* $\forall\, k,\, m \in \mathbf{N}$.

Proof. By Brouwer's Fixed Point Theorem the map L has at least one fixed point, which is none other than a solution of (3.4), (3.5), (3.6). ∎

4. Proof of the Existence Theorem 2.3

We begin by extending the functions considered $(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k, \widetilde{\mathbf{h}}^k)_{k\in\mathbf{N}}$ to the whole domain Ω.

Lemma 4.1 *Let* $(\widetilde{\mathbf{u}}^k, \widetilde{\mathbf{w}}^k, \widetilde{\mathbf{h}}^k)$ *be a weak solution for* (P_k) *obtained in* Proposition 3.3. *Set*

$$\mathbf{u}^k(x) = \begin{cases} \widetilde{\mathbf{u}}^k(x) & \text{if } x \in \Omega_k, \\ \mathbf{0} & \text{if } x \in \Omega \setminus \Omega_k, \end{cases}$$

$$\mathbf{w}^k(x) = \begin{cases} \widetilde{\mathbf{w}}^k(x) & \text{if } x \in \Omega_k, \\ \mathbf{0} & \text{if } x \in \Omega \setminus \Omega_k, \end{cases}$$

$$\mathbf{h}^k(x) = \begin{cases} \widetilde{\mathbf{h}}^k(x) & \text{if } x \in \Omega_k, \\ \mathbf{0} & \text{if } x \in \Omega \setminus \Omega_k. \end{cases}$$

Then

$$(\mathbf{u}^k, \mathbf{w}^k, \mathbf{h}^k) \in J(\Omega) \times W_0(\Omega) \times J(\Omega)\,;$$

furthermore

$$\|\nabla \mathbf{u}^k\|_{L^2(\Omega_k)} \le \ell_1\,, \quad \|\nabla \mathbf{w}^k\|_{L^2(\Omega_k)} \le \ell_2\,, \quad \|\nabla \mathbf{h}^k\|_{L^2(\Omega_k)} \le \ell_3\,,$$

where the constants ℓ_1, ℓ_2 *and* ℓ_3 *were defined above and are independent of* k.

Proof. It is easy to show that $(\mathbf{u}^k, \mathbf{w}^k, \mathbf{h}^k) \in J(\Omega) \times W_0(\Omega) \times J(\Omega)$. The estimates are directly deduced from estimates (3.10) through (3.12) and the lower semi-continuity of the norm. ∎

Proof of Theorem 2.3 From the estimates given above in Lemma 4.1, by Rellich's compactness theorem, and from the diagonal argument, it follows that there exist subsequences, again denoted $(\mathbf{u}^k, \mathbf{w}^k, \mathbf{h}^k)$, and elements $\mathbf{u}, \mathbf{h} \in J(\Omega)$ and $\mathbf{w} \in W_0(\Omega)$ such that

$$\left.\begin{array}{l} \mathbf{u}^k \to \mathbf{u} \\ \mathbf{h}^k \to \mathbf{h} \end{array}\right\} \quad \text{weakly in } J(\Omega) \text{ and strongly in } L^2_{loc}(\Omega)\,, \tag{4.1}$$

$$\mathbf{w}^k \to \mathbf{w} \qquad \text{weakly in } W_0(\Omega) \text{ and strongly in } L^2_{loc}(\Omega)\,. \tag{4.2}$$

Once we obtain these convergences and limits, we can show that $(\mathbf{u}, \mathbf{w}, \mathbf{h})$ is the desired stationary weak solution for (1.1)-(1.6). Indeed, let (φ, ξ, ψ) be any arbitrary test function. Then we find a bounded domain Ω' and k_0 such that supp φ, supp ξ, supp $\psi \subseteq \Omega' \subseteq \Omega_{k_0} \subseteq \Omega_k$, for all $k \ge k_0$. Moreover, by Lemmas 2.1 and 3.2

$$|((\mathbf{u}^k\cdot\nabla)\varphi, \mathbf{w}^k) - ((\mathbf{u}\cdot\nabla)\varphi, \mathbf{w})| \le |(((\mathbf{u}^k - \mathbf{u})\cdot\nabla)\varphi, \mathbf{w})| + |((\mathbf{u}^k\cdot\nabla)\varphi, \mathbf{w} - \mathbf{w}^k)|$$

$$\leq \sqrt[3]{9}\, \|\mathbf{u}^k - \mathbf{u}\|_{L^2(\Omega')}\, \|\nabla\varphi\|_{L^3(\Omega')} \|\mathbf{w}\|_{L^6(\Omega')}$$

$$+\ \sqrt[3]{9}\, \|\mathbf{u}^k\|_{L^6(\Omega')} \|\nabla\varphi\|_{L^3(\Omega')}\, \|\mathbf{w} - \mathbf{w}^k\|_{L^2(\Omega')}\,,$$

$$\leq \sqrt[3]{9}\, C_L \left(\ell_2 \|\mathbf{u}^k - \mathbf{u}\|_{L^2(\Omega')}\, \|\nabla\varphi\|_{L^3(\Omega')} + \ell_1 \|\mathbf{w} - \mathbf{w}^k\|_{L^2(\Omega')}\, \|\nabla\varphi\|_{L^3(\Omega')} \right).$$

By convergences (4.1) and (4.2)

$$|((\mathbf{u}^k \cdot \nabla)\varphi, \mathbf{w}^k)_\Omega - ((\mathbf{u} \cdot \nabla)\varphi, \mathbf{w})_\Omega| \to 0\,,$$

as $k \to \infty$. The other convergences are analogously established. Thus $(\mathbf{u}, \mathbf{w}, \mathbf{h})$ is a stationary weak solution for (1.1)-(1.6). ■

5. Proof of the Uniqueness Theorem 2.4

Let $(\mathbf{u}_1, \mathbf{w}_1, \mathbf{h}_1)$ and $(\mathbf{u}_2, \mathbf{w}_2, \mathbf{h}_2)$ be two different weak solutions of (1.1)-(1.6). Setting $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2$ and $\mathbf{h} = \mathbf{h}_1 - \mathbf{h}_2$ we have

$$\mu + \chi)\, (\nabla\mathbf{u}, \nabla\varphi) + (\mathbf{u} \cdot \nabla\mathbf{u}_1, \varphi) + (\mathbf{u}_2 \cdot \nabla\mathbf{u}, \varphi) = \chi\, (\operatorname{rot}\mathbf{w}, \varphi)$$

$$+\, r\, (\mathbf{h} \cdot \nabla\mathbf{h}_1, \varphi) + r\, (\mathbf{h}_2 \cdot \nabla\mathbf{h}, \varphi)\,,$$

$$\gamma\, (\nabla\mathbf{w}, \nabla\xi) + (\alpha + \beta)\, (\operatorname{div}\mathbf{w}, \operatorname{div}\xi) + 2\chi\, (\mathbf{w}, \xi)$$

$$+\, j\, (\mathbf{u} \cdot \nabla\mathbf{w}_1, \xi) + j\, (\mathbf{u}_2 \cdot \nabla\mathbf{w}, \xi) = \chi\, (\operatorname{rot}\mathbf{u}, \xi)\,,$$

$$\nu\, (\nabla\mathbf{h}, \nabla\psi) + (\mathbf{u} \cdot \nabla\mathbf{h}_1, \psi) + (\mathbf{h}_2 \cdot \nabla\mathbf{h}, \psi) - (\mathbf{h} \cdot \nabla\mathbf{u}_1, \psi) - (\mathbf{h}_2 \cdot \nabla\mathbf{u}, \psi) = 0\,.$$

We take $\varphi = \mathbf{u}$, $\xi = \mathbf{w}$ and $\psi = r\,\mathbf{h}$ in the above equalities and obtain

(5.1)

$$(\mu+\chi)\, \|\nabla\mathbf{u}\|^2_{L^2(\Omega)} = \chi\, (\operatorname{rot}\mathbf{w}, \mathbf{u}) - (\mathbf{u}\cdot\nabla\mathbf{u}_1, \mathbf{u}) + r\, (\mathbf{h}\cdot\nabla\mathbf{h}_1, \mathbf{u}) + r\, (\mathbf{h}_2\cdot\nabla\mathbf{h}, \mathbf{u})\,,$$

(5.2)

$$\gamma\, \|\nabla\mathbf{w}\|^2_{L^2(\Omega)} + (\alpha+\beta)\, \|\operatorname{div}\mathbf{w}\|^2_{L^2(\Omega)} + 2\chi\, \|\mathbf{w}\|^2_{L^2(\Omega)} = \chi\, (\operatorname{rot}\mathbf{u}, \mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w}_1, \mathbf{w})\,,$$

$$r\nu \left\|\nabla \mathbf{h}\right\|^2_{L^2(\Omega)} = r\left(\mathbf{h}\cdot\nabla\mathbf{u}_1,\mathbf{h}\right) + r\left(\mathbf{h}_2\cdot\nabla\mathbf{u},\mathbf{h}\right) - r\left(\mathbf{u}\cdot\nabla\mathbf{h}_1,\mathbf{h}\right). \tag{5.3}$$

By Lemma 2.1 we have

$$\begin{aligned}
|(\mathbf{u}\cdot\triangledown\mathbf{u}_1,\mathbf{u})| = |(\mathbf{u}\cdot\triangledown\mathbf{u},\mathbf{u}_1)| &\le \sqrt{3}\left\|\mathbf{u}\right\|_{L^6(\Omega)}\left\|\triangledown\mathbf{u}\right\|_{L^2(\Omega)}\left\|\mathbf{u}_1\right\|_{L^3(\Omega)}\\
&\le \sqrt{3}C_L\left\|\triangledown\mathbf{u}\right\|^2_{L^2(\Omega)}\left\|\mathbf{u}_1\right\|_{L^3(\Omega)},\\
|r\left(\mathbf{h}\cdot\triangledown\mathbf{h}_1,\mathbf{u}\right)| = |r\left(\mathbf{h}\cdot\triangledown\mathbf{u},\mathbf{h}_1\right)| &\le \sqrt{3}r\left\|\mathbf{h}\right\|_{L^6(\Omega)}\left\|\triangledown\mathbf{u}\right\|_{L^2(\Omega)}\left\|\mathbf{h}_1\right\|_{L^3(\Omega)}\\
&\le \sqrt{3}rC_L\left\|\triangledown\mathbf{h}\right\|_{L^2(\Omega)}\left\|\triangledown\mathbf{u}\right\|_{L^3(\Omega)}\left\|\mathbf{h}_1\right\|_{L^3(\Omega)},\\
|\mathcal{X}\left(rot\mathbf{w},\mathbf{u}\right)| = |\mathcal{X}\left(\mathbf{w},rot\mathbf{u}\right)| &\le \mathcal{X}\left\|\mathbf{w}\right\|_{L^2(\Omega)}\left\|\triangledown\mathbf{u}\right\|_{L^2(\Omega)}\\
&\le \tfrac{\mathcal{X}}{2}\left(\left\|\mathbf{w}\right\|^2_{L^2(\Omega)} + \left\|\triangledown\mathbf{u}\right\|^2_{L^2(\Omega)}\right),\\
|(\mathbf{u}\cdot\triangledown\mathbf{w}_1,\mathbf{w})| = |(\mathbf{u}\cdot\triangledown\mathbf{w},\mathbf{w}_1)| &\le \sqrt{3}\left\|\mathbf{u}\right\|_{L^6(\Omega)}\left\|\triangledown\mathbf{w}\right\|_{L^2(\Omega)}\left\|\mathbf{w}_1\right\|_{L^3(\Omega)}\\
&\le \sqrt{3}C_L\left\|\triangledown\mathbf{u}\right\|_{L^2(\Omega)}\left\|\triangledown\mathbf{w}\right\|_{L^2(\Omega)}\left\|\mathbf{w}_1\right\|_{L^3(\Omega)},\\
|r\left(\mathbf{h}\cdot\triangledown\mathbf{u}_1,\mathbf{h}\right)| = |r\left(\mathbf{h}\cdot\triangledown\mathbf{h},\mathbf{u}_1\right)| &\le \sqrt{3}r\left\|\mathbf{h}\right\|_{L^6(\Omega)}\left\|\triangledown\mathbf{h}\right\|_{L^2(\Omega)}\left\|\mathbf{u}_1\right\|_{L^3(\Omega)}\\
&\le \sqrt{3}rC_L\left\|\triangledown\mathbf{h}\right\|^2_{L^2(\Omega)}\left\|\mathbf{u}_1\right\|_{L^3(\Omega)},\\
|(\mathbf{u}\cdot\triangledown\mathbf{h}_1,\mathbf{h})| = |r\left(\mathbf{u}\cdot\triangledown\mathbf{h},\mathbf{h}_1\right)| &\le \sqrt{3}r\left\|\mathbf{u}\right\|_{L^6(\Omega)}\left\|\triangledown\mathbf{h}\right\|_{L^2(\Omega)}\left\|\mathbf{h}_1\right\|_{L^3(\Omega)}\\
&\le \sqrt{3}rC_L\left\|\triangledown\mathbf{u}\right\|_{L^2(\Omega)}\left\|\triangledown\mathbf{h}\right\|_{L^2(\Omega)}\left\|\mathbf{h}_1\right\|_{L^3(\Omega)}
\end{aligned}$$

Consequently, adding equalities (5.1) through (5.3), using Young's inequality, and the above estimates we obtain

$$\begin{aligned}
&\mu\left\|\nabla\mathbf{u}\right\|^2_{L^2(\Omega)} + \gamma\left\|\nabla\mathbf{w}\right\|^2_{L^2(\Omega)} + (\alpha+\beta)\left\|\mathrm{div}\,\mathbf{w}\right\|^2_{L^2(\Omega)} + r\,\nu\left\|\nabla\mathbf{h}\right\|^2_{L^2(\Omega)}\\
&+\chi\left\|\mathbf{w}\right\|^2_{L^2(\Omega)} \;\le\; \tfrac{\sqrt{3}}{2}\,C_L\left(2\|\mathbf{u}_1\|_{L^3(\Omega)}+\|\mathbf{w}_1\|_{L^3(\Omega)}+2r\left\|\mathbf{h}_1\right\|_{L^3(\Omega)}\right)\left\|\nabla\mathbf{u}\right\|^2_{L^2(\Omega)}\\
&+\sqrt{3}\,r\,C_L\left(\left\|\mathbf{u}_1\right\|_{L^3(\Omega)} + \left\|\mathbf{h}_1\right\|_{L^3(\Omega)}\right)\left\|\nabla\mathbf{h}\right\|^2_{L^2(\Omega)}\\
&+\tfrac{\sqrt{3}}{2}\,r\,C_L\left\|\mathbf{w}_1\right\|_{L^3(\Omega)}\left\|\nabla\mathbf{w}\right\|^2_{L^2(\Omega)},
\end{aligned}$$

and together with hypothesis (2.14) - (2.16) we have

$$\left\|\nabla\mathbf{u}\right\|_{L^2(\Omega)} = \mathbf{0},\;\left\|\nabla\mathbf{w}\right\|_{L^2(\Omega)} = \mathbf{0},\;\left\|\nabla\mathbf{h}\right\|_{L^2(\Omega)} = \mathbf{0}.$$

Hence $\mathbf{u} = \mathbf{0}$, $\mathbf{w} = \mathbf{0}$, and $\mathbf{h} = \mathbf{0}$, and the proof of the uniqueness is now complete. ∎

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