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GREEN'S FUNCTION OF DIFFERENTIAL EQUATION WITH FOURTH ORDER AND NORMAL OPERATOR COEFFICIENT IN HALF AXIS

M. BAYRAMOGLU
and
K. OZDEN KOKLU
Yildiz Technical University, Turkey

Abstract

Let H *be an abstract seperable Hilbert space. Denoted by* $H_1 = L_2(0,\infty;H)$, *the all functions defined in* $[0,\infty)$ *and their values belongs to space* H, *which* $\int_0^\infty \|f(x)\|_H^2\, dx < \infty$. *We define inner product in* H_1 *by the formula*

$(f,g)_{H_1} = \int_0^\infty (f,g)_H dx \qquad f(x), g(x) \in H_1,$

H_1 *forms a seperable Hilbert space[3] where* $\|.\|_H$ *and* $(.,.)_H$ *are norm and scalar product, respectively in* H.

In this study, in space H_1, *it is investigated that Green's function (resolvent) of operator formed by the diferential expression*

$y^{IV} + Q(x)y, \qquad 0 \leq x < \infty,$

and boundary conditions

$y'(0) - h_1 y(0) = 0,$

$y'''(0) - h_2 y''(0) = 0,$

where $Q(x)$ *is a normal operator mapping in* H *and invers of it is a compact operator for every* $x \in [0,\infty)$. *Assume that domain of* $Q(x)$ *is independent from* x *and resolvent set of* $Q(x)$ *belongs to* $|\arg\lambda - \pi| < \varepsilon \;\; (0 < \varepsilon < \pi)$ *of complex plane* λ, h_1 *and* h_2 *are complex numbers. In addition assume that the operator function* $Q(x)$ *satisfies the Titchmarsh-Levitan conditions.*

1. INTRODUCTION

In this work, Green's function(resolvent) of operator L in $H_1 = L_2\left(0, \infty; H\right)$ space is investigated. Here, operator L is formed by the differential expression

$$y^{IV} + Q(x)y, \qquad 0 \leq x < \infty \tag{1}$$

and the boundary conditions

$$y'(0) - h_1 y(0) = 0, y'''(0) - h_2 y''(0) = 0 \tag{2}$$

where $Q(x)$ is a normal operator for every $x \in [0, \infty)$ in H and inverse of it is a compact operator, h_1, h_2 are arbitrary complex numbers. In the case of $Q^*(x) = Q(x)$, $h_1 = h_2$ and regular behavior of $Q(x)$ in infinity, Green function of operator L investigated in (Albayrak, Bayramoglu)[1].

Green's fuction of Sturm-Liouville equation given in $(-\infty, \infty)$ with unbounded self-adjoint operator coefficient was first investigated by B.M. Levitan [11].

In the space of $L_2\left(-\infty, \infty; H\right)$, Green's function and the asymptotic behaviour for the number of the eigenvalues of the operator formed by differential expression $\left(-1\right)^n y^{(2n)} + \sum_{j=2}^{2n} Q_j(x) y^{(2n-j)}$ was obtained by M. Bayramoğlu 1971 [4], where $Q_j(x)$ $(j = 2, \ldots, 2n)$ are the self-adjoint operators in H. Later on many studies (Boymatov, K.Ch.[6], Otelbayev, M.[12], Aslanov, G.I.[2], Kleyman, E.G.[8], Saito, Y.[15]) were published in this subject. Wide reference of these studies is given in (Kostyuchenko, A.G., Sargsyan, I.S. [10], Otelbayev, M. [13]). The main reference related to Green's function for the ordinary differential equation is the book Stakgold, I. [16].

2. DETERMINATION OF THE PROBLEM

In $H_1 = L_2\left(0, \infty; H\right)$, let consider the differential expression

$$y^{IV} + Q(x)y + \mu y \tag{3}$$

with boundary conditions

$$y'(0) - h_1 y(0) = 0 \tag{4}$$

$$y'''(0) - h_2 y''(0) = 0 \tag{5}$$

where $\mu \geq 0$ is a real number. It is assumed that $Q(x)$ is a normal operator mapping in H for every $x \in [0, \infty)$ and satisfies the following specification called Titchmarsh-Levitan conditions:

1) Let $Q(x)$ be a normal operator for each $x \in [0, \infty)$ in H with domain $D(Q(x)) \equiv D$ independent from x, $\overline{D} = H$ (Here $\overline{D}$ shows the closure of D).

2) Let $Q^{-1}(x)$ be compact operator for every x $(Q^{-1}(x) \in \sigma_\infty)$ and $1 \leq |\alpha_1(x)| \leq |\alpha_2(x)| \leq ... \leq |\alpha_n(x)| \leq ...$ where $\alpha_1(x), \alpha_2(x), \alpha_3(x), ...,$ are the eigenvalues of $Q(x)$.

3) Assume that resolvent set of $Q(x)$ incluted domain $S_\varepsilon = \{\lambda : \pi - \varepsilon < \arg \lambda < \pi + \varepsilon,\ 0 < \varepsilon < \pi,\ \varepsilon = const\}$ of complex plane λ.

4) Suppose that function $F(x) = \sum\limits_{k=1}^{\infty} \frac{1}{|\alpha_k(x)|^{7/4}}$ belongs to $L(0, \infty)$:

   $\int_0^\infty F(x)dx < \infty$

5) $\left\| Q^{-1/4}(x).Q^{1/4}(s) \right\| \leq c$ and $\left\| Q^{1/4}(x).Q^{-1/4}(s) \right\| \leq c$ while $|x - s| \leq 1$, $c = constant$

6) Assume that $\|Q^{-a}(x) [Q(s) - Q(x)]\| \leq c\,|x - s|$ while $|x - s| \leq 1$, where $c = constant$ and $0 < a < \frac{5}{4}$.

Different constants are denoted with c.

Let $B(H)$ be a Banach space whose elements are bounded operators mapping in H [7]. $G(x, s; \mu)$ function which belongs to $B(H)$ for $0 \leq x, s < \infty$ and satisfies the following conditions is called Green's function of (3)-(5).

1) The operator function $G(x, s; \mu)$ itself and its two partial derivative according to s are continious functions for variables x and s $(0 \leq x, s \leq \infty)$.

2) When $s \neq x$ third derivative of $G(x, s; \mu)$ for s is continuous.

3) $\frac{\partial^3 G}{\partial s^3}(x, x+0, \mu) - \frac{\partial^3 G}{\partial s^3}(x, x-0, \mu) = I$ (I is identity operator in H)

4) When $s \neq x$, $\frac{\partial^4 G}{\partial s^4}(x, s; \mu) + G(x, s; \mu)Q(s) + \mu G(x, s; \mu) = 0$

5) $\frac{\partial G}{\partial s}(x, s; \mu)|_{s=0} - h_1 G(x, s; \mu)|_{s=0} = 0,$

$\frac{\partial^3 G}{\partial s^3}(x, s; \mu)|_{s=0} - h_2 \frac{\partial^2 G}{\partial s^2}(x, s; \mu)|_{s=0} = 0.$

According to parametrics method, the operator function $G(x, s; \mu)$ will be found as a solution of integral equation given by

$$G(x, s; \mu) = r(x-s)g(x, s; \mu) - \int_0^\infty \Big\{ r^{(IV)}(x-\xi)g(x, \xi; \mu)$$

$$+4r'''(x-\xi)g'(x, \xi; \mu) + 6r''(x-\xi)g''(x, \xi; \mu) + 4r'(x-\xi)g'''(x, \xi; \mu)$$

$$+r(x-\xi)g(x, \xi; \mu)\left[Q(\xi) - Q(x)\right]\Big\} G(\xi, s; \mu)d\xi \tag{6}$$

where

$$r(u) = \begin{cases} 1 & |u| \le \rho \\ 0 & |u| \ge 2\rho, \quad 0 < \rho < \frac{1}{2} \end{cases}$$

is any fixed sufficiently smooth function and

$$g(x, s; \mu) = \frac{\sqrt{2}}{8}\alpha^{-3}(1+i)e^{-\alpha\frac{\sqrt{2}}{2}[|x-s|+i|x-s|]}$$

$$-\frac{\sqrt{2}}{8}\alpha^{-3}(-1+i)e^{-\alpha\frac{\sqrt{2}}{2}[|x-s|-i|x-s|]}$$

$$+\frac{\left[\frac{1}{4}(1+i)h_2 + \frac{\sqrt{2}}{4}\alpha^{-1}(-1+i)h_1h_2 - \frac{\sqrt{2}}{4}\alpha(-1+i) + \frac{1}{4}(1+i)h_1\right]}{\alpha^2 i(-2h_1h_2 - \sqrt{2}\alpha(h_1+h_2) - 2\alpha^2)}e^{-\alpha\frac{\sqrt{2}}{2}(1+i)(x+s)}$$

$$+\frac{\left[-\frac{1}{2}h_2 + \frac{\sqrt{2}}{4}\alpha(-1+i) + \frac{1}{2}h_1\right]}{\alpha^2 i(-2h_1h_2 - \sqrt{2}\alpha(h_1+h_2) - 2\alpha^2)}e^{\alpha\frac{\sqrt{2}}{2}[-(x+s)+i(-x+s)]}$$

$$+\frac{\left[-\frac{1}{2}h_1 - \frac{\sqrt{2}}{4}\alpha i + \frac{1}{2}h_2\right]}{\alpha^2 i(-2h_1h_2 - \sqrt{2}\alpha(h_1+h_2) - 2\alpha^2)}e^{\alpha\frac{\sqrt{2}}{2}[-(x+s)+i(x-s)]}$$

$$+\frac{\left[\frac{1}{4}(-1+i)h_1+\frac{\sqrt{2}}{4}\alpha^{-1}(1+i)h_1h_2-\frac{\sqrt{2}}{4}\alpha(1+i)+\frac{1}{4}(-1+i)h_2\right]}{\alpha^2 i(-2h_1h_2-\sqrt{2}\alpha(h_1+h_2)-2\alpha^2)}e^{\alpha\frac{\sqrt{2}}{2}(-1+i)(x+s)}$$

$$0 \le x, s < \infty. \tag{7}$$

$\alpha = \sqrt[4]{Q(x)+\mu I}$ and defined by formula of spectral expansion

$$\sqrt[4]{Q(x)+\mu I} = \sum_{n=1}^{\infty} \sqrt[4]{\alpha_n(x)+\mu}(., e_n)e_n$$

where $e_1(x), e_2(x), ..., e_n(x), ...$ are the ortonormalized eigenvectors corresponding to eigenvalues $\alpha_1(x), \alpha_2(x), ..., \alpha_n(x), ...$ of $Q(x)$. Branch of the $\sqrt[4]{\alpha_j(x)+\mu}$ is defined with $|\arg(\alpha_n(x)+\mu)| < \pi$. Note that the operator function $g(x, s; \mu)$ is founded by writing $\alpha = \sqrt[4]{Q(x)+\mu I}$ instead of α in Green function defined with diferential expression

$$y^{IV} + \alpha^4 y \qquad (\alpha > 0)$$

and boundary conditions

$$y^{(j)}(0) - h_{(j+1)/2}y^{(j-1)}(0) = 0, \qquad j = 1, 3$$

in space $L_2[0, \infty)$. Let's say that

$$g = \sum_{i=1}^{6} g_i.$$

It will be shown that integral equation (6) has only one solution and this solution is Green's fuction of the problem (3)-(5).

Equation (6) will be investigated in these spaces; X_2, $X_3^{(1)}$, $X_4^{(-1/4)}$, X_5. These are shown that they are Banach spaces and given by Levitan, B.M. [11].

Consider that integral Eq.(6) is in the space X_2. Let us show that this equation has one solution in X_2 for $\mu >> 0$ (μ is a big enough positive value) and the solution can be found by successive aproximation method. For this it is enough to show that $g(x, s; \mu) \in X_2$ and the operator of

$$NA = \int_0^\infty \Big\{ r^{(IV)}(x-\xi)g(x,\xi;\mu) + 4r'''(x-\xi)g'(x,\xi;\mu)$$
$$+6r''(x-\xi)g''(x,\xi;\mu) + 4r'(x-\xi)g'''(x,\xi;\mu)$$
$$+r(x-\xi)g(x,\xi;\mu)\left[Q(\xi)-Q(x)\right]\Big\} A(\xi,s;\mu)d\xi$$

is constriction operator in X_2 for $\mu >> 0$.

LEMMA 1: If operator function $Q(x)$ satisfies the conditions 4-) and 6-) for $\mu >> 0$, operator N is constriction operator in the space X_2.

PROOF: If it is shown that the norm of operator N for $\mu >> 0$ are small enough, it is demonstrated that N is constriction operator at large values of $\mu > 0$.

$NA = \sum_{i=1}^{5} N_j A$
$N_1 A = \int_0^\infty \left[r(x-\xi) g(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] \right] A(\xi,\eta) d\xi$
$N_2 A = \int_0^\infty 4r'(x-\xi) g'''(x,\xi;\mu) A(\xi,\eta) d\xi$
$N_3 A = \int_0^\infty 6r''(x-\xi) g''(x,\xi;\mu) A(\xi,\eta) d\xi$
$N_4 A = \int_0^\infty 4r'''(x-\xi) g'(x,\xi;\mu) A(\xi,\eta) d\xi$
$N_5 A = \int_0^\infty r^{(IV)}(x-\xi) g(x,\xi;\mu) A(\xi,\eta) d\xi$

$\|N\| \leq \sum_{i=1}^{5} \|N_i\|$ can be written from nature of norm. Let's do the operations for operator $N_1 A$.

$g(x,s;\mu) = \sum_{i=1}^{6} g_i$
$N_1 A = \int_0^\infty \left[r(x-\xi) g(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] \right] A(\xi,\eta) d\xi$
$= \int_0^\infty \left[r(x-\xi) \sum_{i=1}^{6} g_i(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] \right] A(\xi,\eta) d\xi$
$N_1 A = \sum_{i=1}^{6} N_{1i} A$
$\|N_1\| \leq \sum_{i=1}^{6} \|N_{1i}\|$
$N_{11} A = \int_0^\infty r(x-\xi) g_1(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] A(\xi,\eta) d\xi$
$= \int_{|x-\xi| \leq 1} r(x-\xi) g_1(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] A(\xi,\eta) d\xi$
$+ \int_{|x-\xi| > 1} r(x-\xi) g_1(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] A(\xi,\eta) d\xi$
$= b_1 + b_2$
where $b_2 = 0$ according to $r(u) = 0$, $|u| > 1$.

$$\|N_{11} A(x,\xi)\|_2 = \|b_1\|_2$$

($\|.\|_2$ is a norm in space X_2)
$\|b_1\|_2^2 = \left\| \int_{|x-\xi| \leq 1} r(x-\xi) g_1(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] A(\xi,\eta) d\xi \right\|_2^2$
$\leq \left[\int_{|x-\xi| \leq 1} \| r(x-\xi) g_1(x,\xi;\mu) \left[Q(x) - Q(\xi) \right] A(\xi,\eta) \|_2 \, d\xi \right]^2$ (8)
$\|b_1\|_2^2 \leq c^2 \mu^{2q} \left[\int_{|x-\xi| \leq 1} |x-\xi|^{-\varepsilon} \|A(\xi,\eta)\|_2 \, d\xi \right]^2$

is found. Hence

$$\|b_1\|_2^2 \leq \int_0^\infty \int_0^\infty c\mu^{2q} \left[\int_{|x-\xi|\leq 1} |x-\xi|^{-\varepsilon} \|A(\xi,\eta)\|_2 \, d\xi \right]^2 dx d\eta$$

is obtained, or

$$\|b_1\|_2 \leq c\mu^{2q} \|A(x,\eta)\|_2 < \infty.$$

Therefore,

$$\|N_{11}A\|_2^2 \leq c\mu^{2q} \|A(x,\eta)\|_2$$

is derived. Here, $q < 0$ is constant. Thus, operator $N_{11}A(x,\xi)$ is bounded with small enough norm of large $\mu > 0$. In a similar way, operators $N_{1i}A$ $(i = 1, ..., 6)$ are bounded with small enough norm of large $\mu > 0$. Then it is obtained that operator NA is constriction operator in space X_2 for $\mu >> 0$. Thus Lemma 1 is proved.

If it can be shown that $r(x-s)g(x,s;\mu)$ belongs to the space X_2, then, it is obtained that Eq.(6) has only one solution belonging to X_2 for $\mu >> 0$.

$\|g\|_2 \leq \sum_{i=1}^{6} \|g_i\|_2$

can be written.

Now let us show that rg belongs to space X_2 assuming that the condition 4-) of function $Q(x)$ is fulfilled. Let us perform the operation for any term included by rg, for example the term $r(x-s)g_1(x,s;\mu)$. That is, let us show that $rg_1 \in X_2$. In a same manner, it is indicated that other terms also belong to X_2. Since $r(x-s)g_1(x,s;\mu)$ is a function of normal operator valued function $Q(x)$, using the spectral expansion formula for normal operators [17]:

$\|rg_1\|_2^2 =$

$$\sum_{j=1}^{\infty} \left(\frac{\sqrt{2}}{8}\right)^2 \left| r(x-s)(1+i)\left(\alpha_j(x)+\mu\right)^{-3/4} e^{-(\alpha_j(x)+\mu)^{1/4}\frac{\sqrt{2}}{2}(1+i)|x-s|} \right|^2$$

$\|rg_1\|_2^2 \leq (1/16) \sum_{j=1}^{\infty} |\alpha_j(x)+\mu|^{-3/2} e^{-\sqrt{2}\delta|x-s|\,\mathrm{Re}(\alpha_j(x)+\mu)^{1/4}}$\$

$(\delta = const > 0)$

is implied. From the fourth property of $Q(x)$

$$(c = const > 0) \qquad \int_0^\infty \int_0^\infty \|rg_1\|_2^2 \, dsdx = c \int_0^\infty \sum_{j=1}^\infty \frac{dx}{|\alpha_j(x) + \mu|^{7/4}} < \infty,$$

is obtained. Thus it is denoted that $rg_1 \in X_2$. Therefore the following theorem has been proved.

THEOREM 1: If the conditions 4-) and 6-) of operator $Q(x)$ are satisfied, then, for $\mu >> 0$, there exists a solution in the space X_2 for Eq.(6) and it is unique. This solution can be found by successive aproximation method.

The following lemma can be proved.

LEMMA 2: If operator fuction $Q(x)$ satisfies the conditions in Lemma 1 then for $\mu >> 0$, operator N is a constriction operator in every spaces X_2, $X_3^{(1)}$, $X_4^{(-1/4)}$ and X_5. At the same time in addition to the conditions 1-) and 6-), if operator fuction $Q(x)$ satisfies the condition $\left\| Q^{1/4}(x) Q^{-1/4}(s) \right\| \leq c$, $c = constant$, then $g \in X_4^{(-1/4)}$.

3. DERIVATIONS OF GREEN'S FUNCTION

Let us try to show that operator function $G(x, s; \mu)$ has the derivatives $\frac{\partial^J G(x,s;\mu)}{\partial s^J}$ $(j = 1, 2, 3)$. If the derivatives of both sides of Eq.(6) is calculated according to s

$$\frac{\partial^J G(x, s; \mu)}{\partial s^J} = \frac{\partial^J [r(x - s) g(x, s; \mu)]}{\partial s^J} - \int_0^\infty \Big\{ r^{(IV)}(x - \xi) g(x, \xi; \mu)$$

$$+4\mathrm{r}'''(x - \xi) g'(x, \xi; \mu) + 6r''(x - \xi) g''(x, \xi; \mu) + 4r'(x - \xi) g'''(x, \xi; \mu)$$
$$+r(x - \xi) g(x, \xi; \mu) \left[Q(\xi) - Q(x) \right] \Big\} \frac{\partial^J G(\xi, s; \mu)}{\partial s^J} d\xi (9)$$

$$(10) \quad K_j(x, s; \mu) = \frac{\partial^J [r(x - s) g(x, s; \mu)]}{\partial s^J} - N K_j(\xi, s; \mu) \qquad (j = 1, 2, 3)$$

can be written. Let us investigate integral Eq.(10) in Banach space $X_3^{(1)}$. In Lemma 2, N was denoted is a constriction operator in the space $X_3^{(1)}$ for $\mu >> 0$. If it is implied that operator function $\frac{\partial^J [r(x-s) g(x,s;\mu)]}{\partial s^J}$ $(j = 1, 2, 3)$ belongs to $X_3^{(1)}$, it is shown that there exists a solution for Eq.(10) in $X_3^{(1)}$

for $\mu >> 0$. It is seen that $\frac{\partial^J rg}{\partial s^J} \in X_3^{(1)}$, that is,
$\underset{0\leq x<\infty}{Sup} \int_0^\infty \left\| \frac{\partial^J (rg)}{\partial s^J} \right\|_H ds < \infty$ from clear expression of operator function $g(x,s;\mu)$. It is demonstrated that $\frac{\partial^J G(x,s;\mu)}{\partial s^J} - \frac{\partial^J [r(x-s)g(x,s;\mu)]}{\partial s^J}$

(j=1,2,3) is a continuous function for s, $(s \neq x)$, performing the similar operations as in [10] and [4]. On the other hand, since $\frac{\partial^J [r(x-s)g(x,s;\mu)]}{\partial s^J}$ $(j = 1,2)$ is continuous, function $\frac{\partial^J G(x,s;\mu)}{\partial s^J}$ $(j = 1,2)$ is also continuous for according to s. From $\frac{\partial^3 [r(x-s)g(x,s;\mu)]}{\partial s^3}$, it is denoted that this function satisfies the condition $\frac{\partial^3 [rg(x,x+0;\mu)]}{\partial s^3} - \frac{\partial^3 [rg(x,x-0;\mu)]}{\partial s^3} = I$ at the point $s = x$. This results in that operator function $\frac{\partial^3 G}{\partial s^3}$ fulfilles the condition 4-) from the continuity of $\frac{\partial^3 G}{\partial s^3} - \frac{\partial^3 rg}{\partial s^3}$.

The following Lemma can be proved.

LEMMA 3: Assume that operator function $Q(x)$ satisfies the conditions 1-) and 3-) and

$$\left\| Q^{1/4}(x) Q^{-1/4}(s) \right\| \leq c \tag{11}$$

while $|x - s| \leq 1$. In this case;

$$\frac{\partial^4 [r(x-s)g(x,s;\mu)]}{\partial s^4} \in X_4^{(-1/4)}$$

that is

$$\underset{0\leq x<\infty}{Sup} \int_0^\infty \left\| \frac{\partial^4 [r(x-s)g(x,s;\mu)]}{\partial s^4} Q^{-1/4}(s) \right\| ds < \infty.$$

4. THE FOURTH DERIVATIVE OF GREEN'S FUNCTION

In previous part it has been shown that the derivative $\frac{\partial^3 G}{\partial s^3}$ of Green's function $G(x,s;\mu)$ belongs to the space X_3 and it satisfies the continuity $(x \neq s)$ for the variable s and the following expression

$$\frac{\partial^3 G(x,s;\mu)}{\partial s^3} = \frac{\partial^3 [r(x-s)g(x,s;\mu)]}{\partial s^3} - \int_0^\infty P(x,\xi;\mu) \frac{\partial^3 G(\xi,s;\mu)}{\partial s^3} d\xi \tag{12}$$

where

$P(x,\xi;\mu) = r^{(IV)}(x-\xi)g(x,\xi;\mu) + 4r'''(x-\xi)g'(x,\xi;\mu)$

$+6r''(x-\xi)g''(x,\xi;\mu)+4r'(x-\xi)g'''(x,\xi;\mu)$
$+r(x-\xi)g(x,\xi;\mu)\,[Q(\xi)-Q(x)]$.

Let us write Eq.(9) as follows

$$L(x,s;\mu)=l(x,s;\mu)-\int_0^\infty P(x,\xi;\mu)L(\xi,s;\mu)d\xi. \tag{13}$$

Here
$L(x,s;\mu)=\frac{\partial^3 G}{\partial s^3}-\frac{\partial^3 (rg)}{\partial s^3}$
and
$l(x,s;\mu)=-\int_0^\infty P(x,\xi;\mu)\frac{\partial^3 (rg)(\xi,s;\mu)}{\partial s^3}d\xi$.

Let us derive the Eq.(13) according to s as formal. From this
$\frac{\partial L(x,s;\mu)}{\partial s}=\frac{\partial l(x,s;\mu)}{\partial s}-\int_0^\infty P(x,\xi;\mu)\frac{\partial L(\xi,s;\mu)}{\partial s}d\xi$
is obtained. If the expression

$$\frac{\partial^3[r(x-(x+0))g(x,x+0;\mu)]}{\partial s^3}-\frac{\partial^3[r(x-(x-0))g(x,x-0;\mu)]}{\partial s^3}=I$$

is used and if we write as
$l(x,s;\mu)=-\left(\int_0^{s-0} P(x,\xi;\mu)\frac{\partial^3 (rg)(\xi,s;\mu)}{\partial s^3}d\xi+\int_{s+0}^\infty P(x,\xi;\mu)\frac{\partial^3 (rg)(\xi,s;\mu)}{\partial s^3}d\xi\right)$,
$\frac{\partial l(x,s;\mu)}{\partial s}=-P(x,s;\mu)-\int_0^\infty P(x,\xi;\mu)\frac{\partial^4 (rg)(\xi,s;\mu)}{\partial s^4}d\xi$
is found. Let us say that

$$\frac{\partial l(x,s;\mu)}{\partial s}=l_1(x,s;\mu).$$

If it can be shown that element l_1 belongs to $X_4^{(-1/4)}$, according to Lemma 2. It is obtained that there exists a derivative of the function $\frac{\partial^3 G}{\partial s^3}-\frac{\partial^3 (rg)}{\partial s^3}$ according to s and $\frac{\partial}{\partial s}\left(\frac{\partial^3 G}{\partial s^3}-\frac{\partial^3 (rg)}{\partial s^3}\right)\in X_4^{(-1/4)}$. From this point according to Lemma 3, $\frac{\partial^4 G}{\partial s^4}\in X_4^{(-1/4)}$ is obtained. It is found that the element l_1 belongs to $X_4^{(-1/4)}$ by the studies [11], [4].

5. SATISFYING DIFFERENTIAL EQUATION OF GREEN'S FUNCTION

Let us show that Green's function $G(x,s,\mu)$ for $x\neq s$ satisfies the equation
$\frac{\partial^4 G}{\partial s^4}+G(x,s,\mu)\,[Q(s)+\mu I]=0$.

Let $f\in D$. Then,

$\frac{\partial^4 G}{\partial s^4}(f) + rg\left[Q(x) + \mu I\right](f) =$
$-rg\left[Q(s) - Q(x)\right](f) - \int_0^\infty P(x,\xi;\mu)\frac{\partial^4 G(\xi,s;\mu)}{\partial s^4}(f)d\xi$
or

$$(14) \qquad \frac{\partial^4 G}{\partial s^4}(f) = -rg\left[Q(s) + \mu I\right](f) - \int_0^\infty P(x,\xi;\mu)\frac{\partial^4 G(\xi,s;\mu)}{\partial s^4}(f)d\xi$$

is obtained Let $\left[Q(s) + \mu I\right] f = \varphi$. From this, Eq.(14) becomes as follows,

$$\frac{\partial^4 G}{\partial s^4}\left[Q(s) + \mu I\right]^{-1}\varphi = -rg\varphi - \int_0^\infty P(x,\xi;\mu)\frac{\partial^4 G(\xi,s;\mu)}{\partial s^4}\left[Q(s) + \mu I\right]^{-1}\varphi d\xi.$$

Compairing this equation with Eq.(6),

$$\frac{\partial^4 G}{\partial s^4}\left\{-\left[Q(s) + \mu I\right]^{-1}\varphi\right\} = G(x,s;\mu)\varphi$$

is found. From the last expression, fourth property is obtained as elements' set of φ for every constant $s \geq 0$ is dence everywhere in H.

6. SATISFACTION OF BOUNDARY CONDITIONS

Let us show that $G(x,s;\mu)$ satisfies the conditions
$\frac{\partial G(x,s;\mu)}{\partial s}\mid_{s=0} - h_1 G(x,s;\mu)\mid_{s=0} = 0$
$\frac{\partial^3 G(x,s;\mu)}{\partial s^3}\mid_{s=0} - h_2\frac{\partial^2 G(x,s;\mu)}{\partial s^2}\mid_{s=0} = 0$
that is Green's function fulfilles the condition 5-).

$$(15) \qquad G(x,s;\mu) = r(x-s)g(x,s;\mu) - \int_0^\infty P(x,\xi;\mu)G(\xi,s;\mu)d\xi$$

$$(16) \qquad \frac{\partial G(x,s;\mu)}{\partial s} = \frac{\partial[r(x-s)g(x,s;\mu)]}{\partial s} - \int_0^\infty P(x,\xi;\mu)\frac{\partial G(\xi,s;\mu)}{\partial s}d\xi$$

From the Eq.15 and 16;

$$\frac{\partial[r(x-s)g(x,s;\mu)]}{\partial s}\mid_{s=0} - \int_0^\infty P(x,\xi;\mu)\frac{\partial G(\xi,s;\mu)}{\partial s}d\xi\mid_{s=0} -$$

-h$_1$ $\left[r(x-s)g(x,s;\mu) - \int_0^\infty P(x,\xi;\mu)G(\xi,s;\mu)d\xi\right]\mid_{s=0} = 0(17)$

is obtained. Considering that
$\frac{\partial(rg)}{\partial s}\mid_{s=0} - h_1 rg\mid_{s=0} = 0$
from Eq.(17);

$$\int_0^\infty P(x,\xi;\mu)\left[\frac{\partial G(\xi,s;\mu)}{\partial s} - h_1 G(\xi,s;\mu)\right]_{s=0} d\xi = 0 \tag{18}$$

can be written. Homogen Eq.(18) can be written as below,
$(N+I)\left[\frac{\partial G}{\partial s} - h_1 G\right]_{s=0} = 0.$

Since operator N is constriction operator for $\mu >> 0$, then
$\frac{\partial G(\xi,s;\mu)}{\partial s}\mid_{s=0} - h_1 G(\xi,s;\mu)\mid_{s=0} = 0$
is obtained. Thus the first boundary condition of 5-) is satisfied.

Now let us calculate the second and third derivation of $G(x,s;\mu)$ according to s.
$\frac{\partial^j G}{\partial s^j}\mid_{s=0} = \frac{\partial^j (rg)}{\partial s^j}\mid_{s=0} - \int_0^\infty r(x-\xi)g(x,\xi;\mu)\left[Q(\xi)-Q(x)\right]\frac{\partial^j G(\xi,s;\mu)}{\partial s^j}d\xi\mid_{s=0}$
$(j=2,3)$

$$\frac{\partial^3 G}{\partial s^3}\mid_{s=0} - \int_0^\infty P(x,\xi;\mu)\frac{\partial^3 G(\xi,s;\mu)}{\partial s^3}d\xi\mid_{s=0} -$$

$$-h_2\left[\frac{\partial^2 (rg)}{\partial s^2}\mid_{s=0} - \int_0^\infty P(x,\xi;\mu)\frac{\partial^2 G(\xi,s;\mu)}{\partial s^2}d\xi\mid_{s=0}\right]$$

$$= \frac{\partial^3 G}{\partial s^3}\mid_{s=0} - h_2\frac{\partial^2 G}{\partial s^2}\mid_{s=0} \tag{19}$$

From the expression of $g(x,s;\mu)$, considering that
$\frac{\partial^3(rg)}{\partial s^3}\mid_{s=0} - h_2\frac{\partial^2(rg)}{\partial s^2}\mid_{s=0} = 0$
from the Eq.19
$-\int_0^\infty P(x,\xi;\mu)\left[\frac{\partial^3 G(\xi,s;\mu)}{\partial s^3} - h_2\frac{\partial^2 G(\xi,s;\mu)}{\partial s^2}\right]\mid_{s=0} d\xi$
$= \frac{\partial^3 G(\xi,s;\mu)}{\partial s^3} - h_2\frac{\partial^2 G(\xi,s;\mu)}{\partial s^2}$
is found. This homogen equation can be expressed by

$$(N+I)\left[\frac{\partial^3 G}{\partial s^3} - h_2\frac{\partial^2 G}{\partial s^2}\right]\mid_{s=0} = 0.$$

Since N is constriction operator for $\mu >> 0$ in $X_3^{(1)}$

$$\frac{\partial^3 G(x,s;\mu)}{\partial s^3} - h_2\frac{\partial^2 G(x,s;\mu)}{\partial s^2}\mid_{s=0} = 0$$

is obtained. Thus the second condition of 5-) is also fulfilled.

Consequently, it is shown that operator function $G(x, s; \mu)$ satisfies all properties of Green's function.

If integral operator

$Af = \int_0^\infty G(x, s; \mu) f(s) ds, \qquad \mu > 0$

is formed in H_1 by using Green's function obtained, it is seen that A is a Hilbert-Schmidt (H-S) type operator from the property proved

$\int_0^\infty \int_0^\infty \|G(x, s; \mu)\|_2^2 \, dxds < \infty.$

If $Q(x) = Q^*(x)$, $h_1 = h_2$, are real numbers then $G^*(x, s; \mu) = G(s, x; \mu)$ can be proved.

Appendix:

Example. Let $\Omega \subset 211d^m$ $(m \geq 1)$ be any finite region with uniformly smooth boundary and $R^+ = [0, \infty)$.

Let us consider the boundary value problem of

$\frac{\partial^4 u}{\partial x^4} + q(x)(-\triangle_y)^s u = \lambda u \qquad$ (a)

$\frac{\partial u(x,y)}{\partial x} |_{x=0} + h_1 u(0, y) = 0 \qquad$ (b)

$\frac{\partial^3 u(x,y)}{\partial x^3} |_{x=0} + h_2 u(0, y) = 0 \qquad$ (c)

$u |_{\partial\Omega} = \frac{\partial u}{\partial \gamma} |_{\partial\Omega} = ... = \frac{\partial^{s-1}}{\partial \gamma^{s-1}} |_{\partial\Omega} = 0 \qquad$ (d)

in space $L_2(R^+ \times \Omega)$. Here $y = (y_1, y_2, ..., y_m)$,

$-\triangle_y = -\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} - ... - \frac{\partial^2}{\partial y_m^2},$

s is any integer $> \frac{2m}{7}$, $\partial\Omega$ is the boundary of Ω region, γ is the normal of $\partial\Omega$ and $q(x)$ is a complex valued function with values in $C \backslash S_\varepsilon$ satisfying the conditions

$c_1(1 + x^\alpha) \leq |q(x)| \leq c_2(1 + x^\alpha)$

where $\alpha > \frac{4}{7}$, c_1, c_2 are positive constants and h_1, h_2 are arbitrary complex constants.

Let us define self-adjoint A operator (like in [14]) in space $H = L_2(\Omega)$ by $(-\triangle_y)^s$ with boundary conditions (d).

Therefore the problem (a)-(d) in the space $H_1 = L_2(R^+ \times \Omega) = L_2(R^+, H)$ can be writed as a boundary value problem with operator coefficient as follows:

$\frac{\partial^4 u}{\partial x^4} + Q(x)u - \lambda u = 0$

$u^{'}(0) - h_1 u(0) = 0$

$u^{'''}(0) - h_2 u'(0) = 0$

where $u(x) = u(x, .)$, $Q(x) = q(x)A$.

Resolvent set of operator function $Q(x)$ defined like this consists of region S_ε and it can be shown that conditions 1-) - 6-) are satisfied (See also [9], [5]). Applying the founded results in the theoretical part, Green function of the problem (a)-(d) can be examined.

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Kevser Ozden Koklu

Mathematical Enginering Department

Yıldız Technical University

P. O. Box

Davutpasa Campus, Istanbul, Turkey

email:ozkoklu@yildiz.edu.tr

and

M. Bayramoglu
Mathematical Enginering Department
Yıldız Technical University
P. O. Box
Davutpasa Campus, Istanbul, Turkey
email: mbayram@yildiz.edu.tr