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NONRESONANCE BELOW THE SECOND EIGENVALUE FOR A NONLINEAR ELLIPTIC PROBLEM

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Abstract

We study the solvability of the problem

$$-\Delta_p u = g(x,u) + h \quad in\ \Omega;\ \ u = 0 \quad on\ \partial\Omega,$$

when the nonlinearity g is assumed to lie asymptotically between 0 and the second eigenvalue λ_2 of $-\Delta_p$. We show that this problem is nonresonant.

Key words *Eigenvalue, nonresonance, p-laplacian, variational approach.*

1. Introduction

In this paper we consider nonresonant problems of the form

$$(1.1) \qquad \begin{cases} -\Delta_p u &= g(x,u)+h \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $\Delta_p =$div $(|\nabla u|^{p-2}\nabla u)$ denotes the p-laplacian, $h \in W^{-1,p'}(\Omega)$ and $g: \ \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that

$$(g_0) \qquad m_R(x) = \sup_{|s|\le R} |g(x,s)| \in L^{p'}(\Omega) \quad \text{for each} \quad R>0.$$

We are interested in the conditions to be imposed on g and on the primitive G ($G(x,s) = \int_0^s g(x,t)\,dt$) in order to have the nonresonance i.e. the solvability of (1.1) for every h in $W^{-1,p'}(\Omega)$.
First we introduce some notations.
$\lambda_1(m), \ \lambda_2(m)$ denote the first and the second eigenvalue of the weighted nonlinear eigenvalue problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in} \quad \Omega; \ u=0 \quad \text{on} \quad \partial\Omega,$$

where $m(.) \in L^\infty(\Omega)$ is a weight function which is positive on subset of positive measure. λ_1(resp λ_2) denotes $\lambda_1(1)$ (resp $\lambda_2(1)$).
It is known that $\lambda_1(m) > 0$ is a simple eigenvalue, φ_1 the normalized λ_1-eigenfuction does not change sign in Ω and $\sigma(-\Delta_p, m(.)) \cap]\lambda_1(m), \lambda_2(m)[= \emptyset$, where $\sigma(-\Delta_p)$ is the spectrum of $-\Delta_p$(cf [2], [4]).
The inequality $\alpha(x) \lneq \beta(x)$ means that $\alpha(x) \le \beta(x)$ for a.e. $x \in \Omega$ with a strict inequality $\alpha(x) < \beta(x)$ holding on subset of positive measure. $\|.\|$ denotes the norm in $W_0^{1,p}(\Omega)$, $\|.\|_p$ denotes the norm in $L^p(\Omega)$.
$E(\lambda_1)$ is the subspace of $W_0^{1,p}(\Omega)$ spanned by φ_1 and $E(\lambda_1)^\perp = \{h \in W^{-1,p'}(\Omega): \ \int_\Omega h\varphi_1 = 0\}$.

Now we are ready to present the main results, let us consider the hypotheses

$$(H_1) \qquad k(x) = \limsup_{|s|\to+\infty} \frac{g(x,s)}{|s|^{p-2}s} < \lambda_2.$$

$$(H_2) \qquad \liminf_{|s|\to+\infty} \frac{g(x,s)}{|s|^{p-2}s} = 0.$$

$$(H_3) \qquad \lambda_1 \leq l_+(x) = \liminf_{|s| \to +\infty} \frac{g(x,s)}{|s|^{p-2}s}.$$

$$(H_4) \qquad \lambda_1 \lesssim L_+(x) = \liminf_{|s| \to +\infty} \frac{pG(x,s)}{|s|^p}.$$

$$(H_5) \qquad \int_\Omega G(x, t\varphi_1(x))\, dx - \frac{|t|^p}{p} \to +\infty \quad \text{as} \quad |t| \to +\infty.$$

All these limits are taken uniformly for a.e. $x \in \Omega$.

Theorem 1.1. *Assume* (H_1), (H_2), (H_3), (H_4) *and* (H_5), *then for any given* $h \in E(\lambda_1)^\perp$, *the problem* (1.1) *possesses a nontrivial solution.*

Remark 1.1. *we can replace* (H_3) *by the following condition of Landesman-Lazer type*

$$\int_{v>0} (L_+(x) - \lambda_1)|v|^p > 0; \; v \in E(\lambda_1)\backslash\{0\}.$$

In the nonlinear case $(p \neq 2)$, when the potential G satisfies $\limsup_{|s| \to +\infty} \frac{pG(x,s)}{|s|^p} < \lambda_2$, problems of nonresonance has been studied by just a few authors, a contribution in this direction is [3] where the authors studied the case when the perturbation g stays asymptotically between λ_1 and λ_2.

2. Preliminary results

From the conditions (g_0), (H_1), (H_2) and (H_3) it follows that there exists constant $a > 0$ and function $b(.) \in L^{p'}(\Omega)$ such that

$$|g(x,s)| \leq a|s|^{p-1} + b(x), \tag{1}$$

then the critical points $u \in W_0^{1,p}(\Omega)$ of the $\mathcal{C}^1$ functional

$$I(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega G(x, u(x)) - \int_\Omega hu$$

are the weak solutions of the problem (1.1).

To get a critical point of I, we will apply the following version of the Mountain-Pass theorem which is proved in [9], with condition (C).

Theorem 2.1. *Let $I \in \mathcal{C}^1(X, \mathbf{R})$ satisfying condition (PS), $\beta \in \mathbf{R}$ and let Q be a closed connected compact subset such that $\partial Q \cap (-\partial Q) \neq \emptyset$. Assume that*

1) $\forall K \in A_2$ there exists $v_k \in K$ such that $I(v_k) \geq \beta$ and $I(-v_k) \geq \beta$.

2) $\alpha = \sup I_{|\partial Q} < \beta$.

3) $\sup I_{|Q} < +\infty$.

Then I has a critical value $c \geq \beta$.

Recall that $A_2 = \{K \subset X : K$ is compact, symmetric and $\gamma(K) \geq 2\}$, $\gamma(K)$ denotes the genus of K.

Remark 2.1. *The condition (C) is clearly implied by the Palais-Smale condition (PS).*

Let $(u_n) \subset W_0^{1,p}(\Omega)$ be an unbounded sequence such that

$$I'(u_n) \to 0 \text{ and } I(u_n) \text{ is bounded} \tag{2}$$

defining $v_n = \dfrac{u_n}{\|u_n\|}$ and $g_n(x) = \dfrac{g(x, u_n)}{\|u_n\|^{p-1}}$. Passing to a subsequence still denoted by (v_n) (resp (g_n)), we may assume that

$$v_n \rightharpoonup v \quad \text{weakly in } W_0^{1,p}(\Omega).$$

$$v_n(x) \to v(x) \quad a.e.\ x \in \Omega.$$

$$|v_n(x)| \leq z(x) \quad z(.) \in L^p(\Omega).$$

$$g_n \rightharpoonup \tilde{g} \quad \text{weakly in } L^{p'}(\Omega).$$

Lemma 2.1. *Assume (H_1), (H_2) and (H_3), then we have*

1) $\|v\| = 1$ and $-\Delta_p v = m(.)|v|^{p-2}v$ where $0 \leq m(.) < \lambda_2$.

2) $v(x) > 0$ p.p. $x \in \Omega$.

Proof. By (1), we have

$$I'(u_n) = -\Delta_p u_n - g(x, u_n) - h,$$

then

$$-\Delta_p v_n = \frac{I'(u_n)}{\|u_n\|^{p-1}} + g_n + \frac{h}{\|u_n\|^{p-1}}, \tag{3}$$

hence

$$\lim_{n\to+\infty} < -\Delta_p v_n, v_n - v >= 0. \tag{4}$$

Since $-\Delta_p$ is of type S^+, from (4) we conclude

$$v_n \to v \quad \text{strongly in} \quad W_0^{1,p}(\Omega),$$

so that

$$\|v\| = 1. \tag{5}$$

Passing to the limit in (3), we obtain

$$-\Delta_p v = \tilde{g}, \tag{6}$$

hence (5) and (6) give

$$\int_\Omega \tilde{g} v = 1. \tag{7}$$

Let us define

$$m(x) = \begin{cases} \frac{\tilde{g}}{|v|^{p-2}v} & \text{if } \ v \neq 0 \\ \frac{1}{2}\lambda_2 & \text{if } \ v = 0. \end{cases}$$

Combining the hypotheses (H_1), (H_2) and (H_3), we show that

$$0 \leq m(x) < \lambda_2, \tag{8}$$

and

$$\tilde{g} = 0 \ \text{ if } \ v(x) = 0. \tag{9}$$

(The results (8) and (9) are standard cf [6] e.g.)
Using (6), we have

$$-\Delta_p v = m(x)|v|^{p-2}v. \tag{10}$$

To complete the proof of Lemma 2.1, we need to show that $v > 0$ *p.p.* $x \in \Omega$.
From (7), (8) and (10) we deduce that

$$m(.) \in L^\infty(\Omega), \ 0 \precsim m(.) \tag{11}$$

and

$$1 \in \sigma(-\Delta_p, m(.)). \tag{12}$$

In view of (8) and the strict monotonicity of λ_2 (cf [4]) we get

$$\lambda_2(m(.)) > \lambda_2(\lambda_2(1)),$$

that is

$$\lambda_2(m(.)) > 1. \tag{13}$$

Combining (11), (12), (13) and the fact that
$\sigma(-\Delta_p, m(.)) \cap]\lambda_1(m), \lambda_2(m)[= \emptyset$, we conclude

$$1 = \lambda_1(m) \quad \text{and} \quad v \in E(\lambda_1(m))\backslash\{0\}, \tag{14}$$

hence v does not change sign in Ω. Assume that $v < 0$, then we have

$$u_n(x) = \|u_n\| v_n \to -\infty \quad p.p.\ x \in \Omega, \tag{15}$$

from (7) and (8), we deduce

$$\tilde{g} \lesssim 0. \tag{16}$$

On the other hand

$$\int_\Omega \frac{g(x, u_n(x))}{\|u_n\|^{p-1}} = \int_\Omega \frac{g(x, u_n(x))}{|u_n|^{p-2} u_n} |v_n|^{p-2} v_n.$$

Using (H_2) and (15), Fatou's Lemma gives

$$\liminf_{n \to +\infty} \int_\Omega \frac{g(x, u_n(x)}{\|u_n\|^{p-1}} \geq \int \liminf_{n \to +\infty} \frac{g(x, u_n(x))}{|u_n|^{p-2} u_n} |v_n|^{p-2} v_n.$$

therefore

$$\int_\Omega \tilde{g} \geq 0$$

which contradicts (16) and show that $v > 0$ $p.p.\ x \in \Omega$, then the proof of Lemma 2.1 is complete.

Lemma 2.2. *Assume* (H_1), (H_2) *and* (H_3), *then*

$$m(.) = \lambda_1 \ p.p.\ x \in \Omega.$$

Proof. Let $A_0 = \{x \in \Omega : \ m(x) < \lambda_1\}$, combining (H_1) and (H_3) we get

$$\begin{array}{rcl} \dfrac{g(x,u_n(x))}{\|u_n\|^{p-1}} & \geq & (1+sign(u_n))(\lambda_1-\varepsilon)|v_n|^{p-2}v_n \\ & + & (1-sign(u_n))(\lambda_2+\varepsilon)|v_n|^{p-2}v_n + 0(n). \end{array}$$

Then

$$\begin{array}{rcl} \int_\Omega g_n\chi_{A_0} & \geq & (1+sign(v_n))(\lambda_1-\varepsilon)|v_n|^{p-2}v_n\chi_{A_0} \\ & + & (1-sign(u_n))(\lambda_2+\varepsilon)|v_n|^{p-2}v_n\chi_{A_0} + 0(n),. \end{array}$$

passing to the limit we conclude

$$\int_{A_0} \tilde{g} \geq (\lambda_1-\varepsilon)\int_{A_0} |v|^{p-2}v,$$

hence

$$\int_{A_0} (m(x)-\lambda_1)|v|^{p-2}v \geq 0.$$

Since $v > 0$, then necessarily $mes(A_0) = 0$, so it follows that

$$m(x) \geq \lambda_1 \quad p.p. \ x \in \Omega. \tag{17}$$

If $m(.) \gtrsim \lambda_1$, then by the strict monotonicity of λ_1, we have

$$\lambda_1(m) < 1$$

which contradicts (14), hence $m(.) = \lambda_1 \ p.p \ x \in \Omega$.

Lemma 2.3. *Assume (H_1), (H_2), (H_3) and (H_4), then the functional I satisfies the Palais-Smale condition (PS), that is whenever $(u_n) \subset W_0^{1,p}(\Omega)$ is a sequence such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ then (u_n) possesses a convergent subsequence.*

Proof. Remark that, using (1) any bounded sequence (u_n) such that $I'(u_n) \to 0$ and $I(u_n)$ is bounded possesses a convergent subsequence, so we will show that (u_n) is bounded.

Suppose by contradiction that $\|u_n\| \to +\infty$. Then, as we observed in the previous Lemmas, a subsequence of (v_n) ($v_n = \dfrac{u_n}{\|u_n\|}$) still denoted by (v_n) is such that

$$v_n \to v \quad \text{strongly in} \ \ W_0^{1,p}(\Omega),$$

$$\|v\| = \lambda_1 \int_\Omega |v|^p = 1 \text{ and } \; v > 0 \; p.p. \; x \in \Omega. \tag{18}$$

In view of (H_2) and (H_3), we obtain

$$G(x, u_n(x)) \geq \tfrac{1}{2p}(1 + sign(u_n))(L_+(x) - \varepsilon)|u_n|^p + \tfrac{1}{2p}(1 - sign(u_n))(-\varepsilon)|u_n|^p + B_\varepsilon(x). (19)$$

Since $I(u_n)$ is bounded below, we have

$$\frac{1}{p} - \int_\Omega \frac{G(x, u_n(x))}{\|u\|^p} - \int_\Omega \frac{hv_n}{\|u_n\|^{p-1}} \geq \frac{M}{\|u_n\|^p} \quad (M \in \mathbf{R}). \tag{20}$$

Combining (19) and (20) and passing to the limit we get

$$1 - \int_\Omega L_+(x)|v|^p \geq 0,$$

hence, by (18) we deduce

$$\int_\Omega (\lambda_1 - L_+(x))|v|^p \geq 0, \tag{21}$$

as $v > 0$ $p.p.$ $x \in \Omega$ and $L_+(x) \gtrsim \lambda_1$, (21) can not occur, then I satisfies the condition (PS). The proof is now complete.

3. Proof of theorem 1.1

Let $A = \left\{u \in W_0^{1,p}(\Omega) : \lambda_2(k(x)) \int_\Omega k(x)|u|^p \leq \int_\Omega |\nabla u|^p\right\}$, where $k(x) = \limsup\limits_{|s| \to +\infty} \dfrac{g(x,s)}{|s|^{p-2}s}$. Recall that $\limsup\limits_{|s| \to +\infty} \dfrac{pG(x,s)}{|s|^p} \leq k(x)$.
It is easy to see that A is nonempty and symmetric set. For $u \in A$ we have

$$\begin{aligned} I(u) &\geq \tfrac{1}{p}\|u\|^p - \tfrac{1}{p}\int_\Omega (k(x) + \varepsilon)|u|^p - \|u\|_p \|h\|_{p'} - \|B_\varepsilon\|_1 \\ &\geq \tfrac{1}{p}\mu\|u\|^p - \|u\|_p\|h\|_{p'} - \|B_\varepsilon\|_1, \end{aligned}$$

since $\lambda_2(k(x)) > \lambda_2(\lambda_2(1)) = 1, \quad \mu = \left(1 - \dfrac{}{\lambda_2(k(x))} - \dfrac{varepsilon}{\lambda_1}\right) > 0,$
then

$$\lim_{\|u\| \to +\infty, \; u \in A} I(u) = +\infty,$$

hence

$$I_{|A} \geq \beta \quad \text{for some } \; \beta \in \mathbf{R}. \tag{22}$$

Let $K \subset W_0^{1,p}(\Omega)$ compact, symmetric and $\gamma(K) \geq 2$, we will show that

$$K \cap A \neq \emptyset. \tag{23}$$

Indeed, if $0 \in K$, then (23) is proved by setting $v = 0$. if $0 \notin K$, we consider $\tilde{K} = \left\{ \frac{}{\|u\|},\ u \in K \right\}$. It is easy to see that $\gamma(\tilde{K}) \geq 2$, hence by the variational characterization of $\lambda_2(k(x))$:

$$\frac{1}{\lambda_2(k(x))} = \sup_{K \in A_2} \min_{u \in K} \int_\Omega k(x)|u|^p,$$

we have

$$\min_{u \in \tilde{K}} \int_\Omega k(x)|u|^p \leq \frac{1}{\lambda_2(k(x))}.$$

Since $\tilde{K}$ is compact, there exists $\tilde{v}_0 \in \tilde{K}$ such that

$$\int_\Omega k(x)|\tilde{v}_0|^p \leq \frac{1}{\lambda_2(k(x))}.$$

(recall that $\tilde{v}_0 = \frac{v_0}{\|v_0\|},\ v_0 \in K$),
then

$$\lambda_2(k(x)) \int_\Omega k(x)|v_0|^p \leq \int_\Omega |\nabla v_0|^p,$$

hence

$$v_0 \in A \cap K. \tag{24}$$

On the other hand, by the hypothesis (H_5), we can easily see that

$$\lim_{|t| \to +\infty} I(t\varphi_1) = -\infty. \tag{25}$$

From this, there exists $R_1 > 0$ such that

$$I(t\varphi_1) < \beta \quad \text{for} \quad |t| \geq R_1 \tag{26}$$

where φ_1 is a normalized, λ_1-eigenfunction.

Letting $Q = \{t\varphi_1 : \ |t| \leq R_1\}$.
We have

$$\sup I_{|Q} < +\infty \tag{27}$$

and from (26), we conclude

$$\sup I_{|\partial Q} < \beta. \tag{28}$$

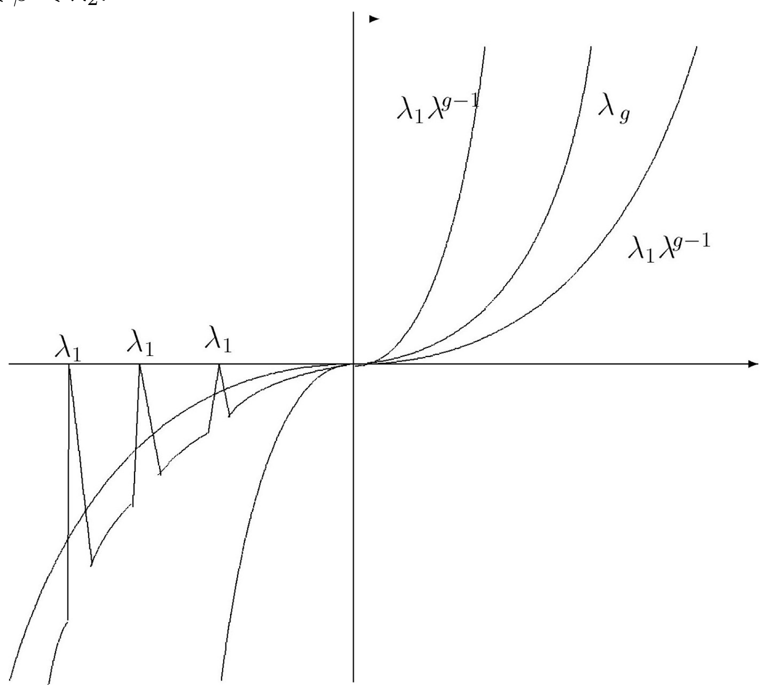
In view of Lemma 2.3, (22), (24), (27) and (28) we may apply Theorem 2.1, to conclude the existence of a critical point $u_0 \in W_0^{1,p}(\Omega)$ of I. ∎

4. Exemple

Let g be a continuous function given by

$$g(s)=\begin{cases}\beta s^{p-1} & \text{if } \; s\geq 0\\ -\beta|s|^{p-1} & \text{if } \; 0\geq s\geq -1+\frac{1}{e}\\ -\beta e^{n}(n-\frac{1}{e^{n}})^{p-1}(s+n) & \text{if } \; s\in[-n,-n+\frac{1}{e^{n}}] \; (n\in\mathbf{N}^{*})\\ \beta e^{n}(n+\frac{1}{e^{n}})^{p-1}(s+n) & \text{if } \; s\in[-n-\frac{1}{e^{n}},-n]\\ -\beta|s|^{p-1} & \text{if } \; s\in[-(n+1)+\frac{1}{e^{n+1}},-n-\frac{1}{e^{n}}]\end{cases}$$

where $\lambda_1<\beta<\lambda_2$.



It is not difficult to see that

$$k(x)=\limsup_{|s|\to+\infty}\frac{g(x,s)}{|s|^{p-2}s}=\beta<\lambda_2. \tag{29}$$

$$\liminf_{|s|\to+\infty}\frac{g(x,s)}{|s|^{p-2}s}=0. \tag{30}$$

$$\lambda_1\leq\liminf_{|s|\to+\infty}\frac{g(x,s)}{|s|^{p-2}s}. \tag{31}$$

$$\lambda_1<\liminf_{|s|\to+\infty}\frac{pG(x,s)}{|s|^{p}}. \tag{32}$$

and

$$\begin{aligned}\int_\Omega G(x,t\varphi_1(x))\,dx-\frac{|t|^p}{p} &\geq \frac{\beta}{p\lambda_1}|t|^p-\frac{1}{p}|t|^p-\sum_{n\geq 1}2\beta\left(n+\frac{1}{e^n}\right)^{p-1}\frac{1}{e^n}\\ &\geq \frac{1}{p}|t|^p\left(\frac{\beta}{\lambda_1}-1\right)-I,\end{aligned}$$

where $I=\sum_{n\geq 1}2\beta\left(n+\frac{}{e^n}\right)^{p-1}\frac{}{e^n}\in\mathbf{R}$.
So the hypotheses of Theorem 1.1 are satisfied.

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