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NON - AUTONOMOUS INHOMOGENEOUS BOUNDARY CAUCHY PROBLEMS AND RETARDED EQUATIONS

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Abstract

In this paper we prove the existence and the uniqueness of the classical solution of non-autonomous inhomogeneous boundary Cauchy problems, and that this solution is given by a variation of constants formula. This result is applied to show the existence of solutions of a retarded equation.

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1. Introduction

Consider the following boundary Cauchy problem

$$(IBCP)\begin{cases} \frac{d}{dt}u(t) = A(t)u(t), & 0 \le s \le t \le T, \\ L(t)u(t) = \Phi(t)u(t) + f(t), & 0 \le s \le t \le T, \\ u(s) = u_0. \end{cases}$$

This type of problems presents an abstract formulation of several natural equations such as retarded differential equations, retarded (difference) equations, dynamical population equations and neutral differential equations.

In the autonomous case ($A(t) = A, L(t) = L, \Phi(t) = \Phi$), the Cauchy problem ($IBCP$) was studied by G. Greiner [2, 3]. The author has used the perturbation of domains of infinitesimal generators to study the homogeneous boundary Cauchy problem ($f \equiv 0$). He has also showed the existence of classical solutions of ($IBCP$) via a variation of constants formula. In the non-autonomous case, Kellermann [6] and Nguyen Lan [7] have showed the existence of an evolution family $(U(t,s))_{t\ge s\ge 0}$ which provides the classical solution of the homogeneous boundary Cauchy problem. The aim of this paper is to show the well-posedness of the inhomogeneous problem (IBCP).

In Section 2, we prove the existence and the uniqueness of the classical solution of ($IBCP$). Our technique consists on transforming ($IBCP$) to an ordinary Cauchy problem (without boundary conditions) and giving an equivalence between the two problems. Moreover, the solution of ($IBCP$) is explicitly given by a variation of constants formula similar to the one given in [3] in the autonomous case. We note that the operator matrices method was also used in [4, 8, 9] for the investigation of inhomogeneous Cauchy problems without boundary conditions.

Finally, Section 3 is devoted to an application to the retarded equation

$$(RE)\begin{cases} v(t) = K(t)v_t + f(t), & t \ge s \ge 0, \\ v_s = \varphi. \end{cases}$$

We end this introduction by basic definitions which are needed for the sequel.

A family of linear (unbounded) operators $(A(t))_{0\le t\le T}$ on a Banach space X is called *stable family* if there are constants $M \ge 1$, $\omega \in \mathbf{R}$ such that $]\omega, \infty[\subset \rho(A(t))$ for all $0 \le t \le T$ and

$$\left\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \right\| \le M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and any finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$.

A family of bounded linear operators $(U(t,s))_{0 \leq s \leq t}$ on X is said to be *evolution family* if

1. $U(t,t) = I_d$ and $U(t,r)U(r,s) = U(t,s)$ for all $0 \leq s \leq r \leq t$,
2. the mapping $\{(t,s) \in \mathbf{R}_+^2 : t \geq s\} \ni (t,s) \longmapsto U(t,s)$ is strongly continuous.

For more details on evolution families and non-autonomous Cauchy problems we refer, for instance, to [1, 5, 10].

2. Well-posedness of boundary Cauchy problems

Let D, X and Y be Banach spaces, D densely and continuously embedded in X. Consider the families of operators $A(t) \in \mathcal{L}(D,X)$, $L(t) \in \mathcal{L}(D,Y)$ and $\Phi(t) \in \mathcal{L}(X,Y)$, for $0 \leq t \leq T$.

In this section, we use the operator matrices method to prove the existence of classical solution for the non-autonomous inhomogeneous boundary Cauchy problem

$$(IBCP) \begin{cases} \frac{d}{dt}u(t) = A(t)u(t), & 0 \leq s \leq t \leq T, \\ L(t)u(t) = \Phi(t)u(t) + f(t), & 0 \leq s \leq t \leq T, \\ u(s) = u_0. \end{cases}$$

To that purpose, we assume that the following hypotheses hold:

$(H1)$ $t \longmapsto A(t)x$ is continuously differentiable for all $x \in D$.

$(H2)$ the family $(A^0(t))_{0 \leq t \leq T}$, $A^0(t) := A(t)|_{kerL(t)}$, is stable, with (M_0, ω_0) constants of stability.

$(H3)$ the operator $L(t)$ is surjective for every $t \in [0,T]$ and $t \longmapsto L(t)x$ is continuously differentiable for all $x \in D$.

$(H4)$ $t \longmapsto \Phi(t)x$ is continuously differentiable for all $x \in X$.

$(H5)$ there exist constants $\gamma > 0$ and $\omega \in \mathbf{R}$ such that

$$||L(t)x||_Y \geq \gamma^{-1}(\lambda - \omega)||x||_X \text{ for } x \in ker(\lambda - A(t)), \lambda > \omega \text{ and } t \in [0,T].$$

Note that under the above hypotheses, the Cauchy problem associated to the family $A^0(\cdot)$ is well-posed and its solutions are given by an evolution family $(U(t,s))_{0 \leq s \leq t \leq T}$, see [7].

Definition 2.1. *A function* $u : [s,T] \longrightarrow X$ *is called classical solution of* $(IBCP)$ *if it is continuously differentiable,* $u(t) \in D$, $t \in [s,T]$, *and* u *satisfies* $(IBCP)$.

We recall the following results which will be used after.

Lemma 2.2. [6.7] *For* $t \in [0,T]$ *and* $\lambda \in \rho(A^0(t))$, *we have the following properties.*

i) $D = D(A^0(t)) \oplus ker(\lambda - A(t))$.

ii) $L(t)|_{ker(\lambda - A(t))}$ *is an isomorphism from* $ker(\lambda - A(t))$ *onto* Y.

iii) $t \longmapsto L_{\lambda,t} := \left(L(t)|_{ker(\lambda - A(t))}\right)^{-1}$ *is strongly continuously differentiable.*

As consequences of this lemma, we have $L(t)L_{\lambda,t} = I_{d_Y}$, $L_{\lambda,t}L(t)$ and $(I - L_{\lambda,t}L(t))$ are the projections from D onto $ker(\lambda - A(t))$ and $D(A^0(t))$ respectively.

We now introduce the Banach space $E := X \times C^1([0,T],Y) \times Y$, where $C^1([0,T],Y)$ is the space of continuously differentiable functions from $[0,T]$ into Y equipped with the norm $||g|| := ||g||_\infty + \|g'\|_\infty$ for $g \in C^1([0,T],Y)$.

Let $\mathcal{A}^\Phi(t)$ be the operator matrices defined on E by

$$\mathcal{A}^\Phi(t) := \begin{pmatrix} A(t) & 0 & 0 \\ 0 & 0 & 0 \\ L(t) - \Phi(t) & -\delta_t & 0 \end{pmatrix},$$
$$D(\mathcal{A}^\Phi(t)) := D \times C^1([0,T],Y) \times \{0\}, \quad t \in [0,T],$$

where $\delta_t : C([0,T],Y) \longrightarrow Y$ is the Dirac function concentrated at the point t.

To the family $\mathcal{A}^\Phi(\cdot)$ we associate the homogeneous Cauchy problem

$$(NCP) \begin{cases} \frac{d}{dt}\mathcal{U}(t) = \mathcal{A}^\Phi(t)\mathcal{U}(t), \quad 0 \le s \le t \le T, \\ \mathcal{U}(s) = \begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}. \end{cases}$$

In the following proposition we give the equivalence between the boundary Cauchy problem $(IBCP)$ and the Cauchy problem (NCP).

Proposition 2.3. *Let* $\begin{pmatrix} u_0 \\ f \end{pmatrix} \in D \times C^1([0,T],Y)$.

(i) If the function $t \longmapsto \mathcal{U}(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \\ 0 \end{pmatrix}$ *is a classical solution of* (NCP) *with an initial value* $\begin{pmatrix} u_0 \\ u_2(t) \\ 0 \end{pmatrix}$ *then* $t \longmapsto u_1(t)$ *is a classical solution of* $(IBCP)$ *with the initial value* u_0.

(ii) Let u *be a classical solution of* $(IBCP)$ *with the initial value* u_0. *Then, the function*

$t \longmapsto \mathcal{U}(t) = \begin{pmatrix} u(t) \\ f \\ 0 \end{pmatrix}$ *is a classical solution of* (NCP) *with the initial value* $\begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}$.

Proof. (i) Since $\mathcal{U}$ is a classical solution, then, from Definition 2.1, u_1 is continuously differentiable and $u_1(t) \in D$, for all $t \in [s,T]$. Moreover,

$$\begin{aligned} \mathcal{U}'(t) &= \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ 0 \end{pmatrix} \\ &= \mathcal{A}^{\Phi}(t)\mathcal{U}(t) \\ &= \begin{pmatrix} A(t)\,u_1(t) \\ 0 \\ L(t)\,u_1(t) - \Phi(t)\,u_1(t) - \delta_t u_2(t) \end{pmatrix} \end{aligned} \tag{2.1}$$

Therefore

$$u_1'(t) = A(t)u_1(t)m \text{ and } u_2'(t) = 0. \tag{2.2}$$

This implies that $u_2(t) = u_2(s) = f$ for all $t \in [s,T]$. Hence, the equation (2.1) implies

$$L(t)u_1(t) = \Phi(t)u_1(t) + f(t), \; 0 \le s \le t \le T.$$

The assertion (ii) is obvious. □

The above proposition allows us to get the aim of this section by showing the well-posedness of the Cauchy problem (NCP). To do this, we use the subsequent result due to Tanaka [11, Theorem 1.3].

Theorem 2.4. *Let $(A(t))_{0\leq t\leq T}$ be a stable family of linear operators on a Banach space X such that*

i) the domain $D := (D(A(t)), ||.||_D)$ is a Banach space independent of t,

ii) the mapping $t \longmapsto A(t)x$ is continuously differentiable in X for every $x \in D$.

Then, there is an evolution family $(U(t,s))_{0\leq s\leq t\leq T}$ on $\overline{D}$. Moreover, we have the following properties:

1. *$U(t,s)D(s) \subset D(t)$ for all $0 \leq s \leq t \leq T$, where $D(r)$ is defined by*
$$D(r) := \left\{x \in D : A(r)x \in \overline{D}\right\}, \quad 0 \leq r \leq T,$$
2. *the mapping $t \longmapsto U(t,s)x$ is continuously differentiable in X on $[s,T]$ and*
$$\frac{d}{dt}U(t,s)x = A(t)U(t,s)xm \text{ for all } x \in D(s)m \text{ and } t \in [s,T].$$

We start by stating the following lemma.

Lemma 2.5. *The family of operators $(\mathcal{A}^{\Phi}(t))_{0\leq t\leq T}$ is stable.*

Proof. For $t \in [0,T]$, we write $\mathcal{A}^{\Phi}(t)$ as

$$\mathcal{A}^{\Phi}(t) = \mathcal{A}(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Phi(t) & -\delta_t & 0 \end{pmatrix},$$

where,

$$\mathcal{A}(t) = \begin{pmatrix} A(t) & 0 & 0 \\ 0 & 0 & 0 \\ L(t) & 0 & 0 \end{pmatrix}, \quad D(\mathcal{A}(t)) = D(\mathcal{A}^{\Phi}(t)).$$

Since $\mathcal{A}^{\Phi}(t)$ is a perturbation of $\mathcal{A}(t)$ by a linear bounded operator on E, hence, in view of the perturbation result [10, Thm. 5.2.3], it is sufficient to show the stability of $(\mathcal{A}(t))_{0\leq t\leq T}$.

Let $\lambda > \omega_0, t \in [0,T]$ and $\begin{pmatrix} x \\ f \\ y \end{pmatrix} \in E$. We have

$$\begin{aligned}
&(\lambda - \mathcal{A}(t)) \begin{pmatrix} R\left(\lambda, A^0\left(t\right)\right) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ f \\ y \end{pmatrix} \\
&= \begin{pmatrix} \left(\lambda - A\left(t\right)\right) R\left(\lambda, A^0\left(t\right)\right) x - \left(\lambda - A\left(t\right)\right) L_{\lambda,t} y \\ f \\ -L\left(t\right) R\left(\lambda, A^0\left(t\right)\right) x + L\left(t\right) L_{\lambda,t} y \end{pmatrix}.
\end{aligned}$$

Since $R(\lambda, A^0(t))x \in D(A^0(t)) = ker(L(t))$, $L_{\lambda,t}y \in ker(\lambda - A(t))$ and $L(t)L_{\lambda,t} = I_{d_Y}$, we obtain

$$(\lambda - \mathcal{A}(t)) \begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} = Id_E. \tag{2.3}$$

On the other hand, for $\begin{pmatrix} x \\ f \\ 0 \end{pmatrix} \in D(\mathcal{A}(t))$, we have

$$\begin{aligned}
&\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} (\lambda - \mathcal{A}(t)) \begin{pmatrix} x \\ f \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} R\left(\lambda, A^0\left(t\right)\right) \left(\lambda - A\left(t\right)\right) x + L_{\lambda,t} L\left(t\right) x \\ f \\ 0 \end{pmatrix}.
\end{aligned} \tag{2.4}$$

From Lemma 2.2, let $x_1 \in D(A^0(t))$ and $x_2 \in ker(\lambda - A(t))$ such that $x = x_1 + x_2$. Then,

$R\left(\lambda, A^0\left(t\right)\right) \left(\lambda - A\left(t\right)\right) x + L_{\lambda,t} L\left(t\right) x$
$= R\left(\lambda, A^0\left(t\right)\right) \left(\lambda, A\left(t\right)\right) \left(x_1 + x_2\right) + L_{\lambda,t} L\left(t\right) \left(x_1 + x_2\right)$
$= R\left(\lambda, A^0\left(t\right)\right) \left(\lambda - A\left(t\right)\right) x_1 + L_{\lambda,t} L\left(t\right) x_2$
$= x_1 + x_2$
$= x.$

Therefore,

$$\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} (\lambda - \mathcal{A}(t)) \begin{pmatrix} x \\ f \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ f \\ 0 \end{pmatrix}. \tag{2.5}$$

From (2.3) and (2.5), we obtain that the resolvent of $\mathcal{A}(t)$ is given by

$$R(\lambda, \mathcal{A}(t)) = \begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By a direct computation we can obtain

$$\prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) = \begin{pmatrix} \prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) & 0 & \prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) L_{\lambda,t_1} \\ 0 & \frac{1}{\lambda^k} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for a finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$.

From the hypothesis $(H5)$, we conclude that $||L_{\lambda,t}|| \leq \gamma(\lambda - \omega)^{-1}$ for all $t \in [0, T]$ and $\lambda > \omega$. Define $\omega_1 = max(0, \omega_0, \omega)$. Therefore, by using $(H2)$, we obtain for $\begin{pmatrix} x \\ f \\ 0 \end{pmatrix} \in E$

$$\begin{aligned} &\left\| \prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) \begin{pmatrix} x \\ f \\ y \end{pmatrix} \right\| \\ &= \left\| \prod_{i=1}^{k} R(\lambda, A^0(t_i))x - \prod_{i=1}^{k} R(\lambda, A^0(t_i)) L_{\lambda,t_1 y} \right\| + \frac{1}{\lambda^k} \|f\| \\ &\leq M (\lambda - \omega_1)^{-k} \|x\| + M (\lambda - \omega_1)^{-k-1} \gamma (\lambda - \omega_1)^{-1} \|y\| + (\lambda - \omega_1)^{-k} \|f\| \\ &\leq M' (\lambda - \omega_1)^{-k} \left\| \begin{pmatrix} x \\ f \\ y \end{pmatrix} \right\|, \end{aligned}$$

where $M' := max(M, M\gamma)$. This achieves the proof of the lemma. □

Let $(U(t,s))_{t \geq s \geq 0}$ be the evolution family generated by $A^0(\cdot)$ and $f(t, u(t)) := \Phi(t)u(t) + f(t), \quad 0 \leq t \leq T$. We are now ready to state our main result.

Theorem 2.6. *Let f be continuously differentiable function on $[0, T]$ onto Y. Assume that $(H1) - (H5)$ are satisfied. Then, for every initial value $u_0 \in D$, such that $L(s)u_0 = \Phi(s)u_0 + f(s)$, the boundary Cauchy problem*

$(IBCP)$ has a unique classical solution u. Moreover, u is given by the variation of constants formula

$$\begin{aligned} u(t) &= U(t,s)(I - L_{\lambda,s}L(s))u_0 + L_{\lambda,t}f(t,u(t)) \\ &+ \int_s^t U(t,r)[\lambda L_{\lambda,r}f(r,u(r)) - (L_{\lambda,r}f(r,u(r)))'] \, dr \end{aligned} \tag{2.6}$$

for $t \geq s$.

Proof. By Lemma 2.5 and the assumptions $(H1)$, $(H3)$ and $(H4)$, the family $(\mathcal{A}^{\Phi}(t))_{0\leq t\leq T}$ satisfies all conditions of Theorem 2.4, then there exists an evolution family $(\mathcal{U}^{\Phi}(t,s))_{s\leq t\leq T}$ such that, for all initial value $\begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix} \in D(\mathcal{A}^{\Phi}(s))$, the function $\begin{pmatrix} u(t) \\ v(t) \\ 0 \end{pmatrix} := \mathcal{U}^{\Phi}(t,s) \begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}$ is the classical solution of (NCP). Therefore, from (i) of Proposition 2.3, u is a classical solution of $(IBCP)$. The uniqueness of u comes from the uniqueness of the solution of (NCP) and Proposition 2.3.

Let now u be a classical solution of $(IBCP)$. At first, assume that $\Phi(t) = 0$. Then, the functions

$$\begin{aligned} u_2(t) &:= L_{\lambda,t}L(t)u(t) \\ &= L_{\lambda,t}f(t), \quad 0 \leq t \leq T, \end{aligned}$$

and $u_1(t) := (I - L_{\lambda,t}L(t))u(t),\ 0 \leq t \leq T$, are differentiable and

$$\begin{aligned} u_1'(t) &= u'(t) - u_2'(t) \\ &= A(t)(u_1(t) + u_2(t)) - (L_{\lambda,t}f(t))' \\ &= A^0(t)u_1(t) + \lambda L_{\lambda,t}f(t) - (L_{\lambda,t}f(t))'. \end{aligned}$$

If we define $\tilde{f}(t) := \lambda L_{\lambda,t}f(t) - (L_{\lambda,t}f(t))'$, we have

$$u_1(t) = U(t,s)u_1(s) + \int_s^t U(t,r)\tilde{f}(r) \, dr, \qquad 0 \leq s \leq t \leq T.$$

By replacing $u_1(s)$ by $(I - L_{\lambda,s}L(s))u_0$, we obtain

$$u_1(t) = U(t,s)(I - L_{\lambda,s}L(s))u_0 + \int_s^t U(t,r)\tilde{f}(r) \, dr.$$

It follows that

$$(2.7)\qquad \begin{array}{l} u(t) = U(t,s)(I - L_{\lambda,s}L(s))u_0 + L_{\lambda,t}f(t) \\ + \int_s^t U(t,r)\left[\lambda L_{\lambda,r}f(r) - (L_{\lambda,r}f(r))'\right] dr. \end{array}$$

In the case $\Phi \neq 0$, since $f(\cdot, u(\cdot))$ is continuously differentiable, similar arguments are used to obtain the formula (2.7) for $f(\cdot) := f(\cdot, u(\cdot))$, and consequently (2.6) is showed. □

3. Retarded equation

Let E be a Banach space and $K(t) \in \mathcal{L}(C([-r,0],E),E), \quad 0 \le t \le T$. We consider the retarded equation

$$(RE)\begin{cases} v(t) = K(t)v_t + f(t), & 0 \le s \le t \le T, \\ v_s = \varphi, \end{cases}$$

where $v_t(\tau) := v(t+\tau)$, for $\tau \in [-r,0]$, and $f : [0,T] \longrightarrow E$.

Assume that:

(A) $t \longmapsto K(t)\varphi$ is continuously differentiable for all $\varphi \in C([-r,0],E)$ and $\sup_{0\le t\le T} \|K(t)\| < 1$.

Definition 3.1. *A function $v : [s-r,T] \longrightarrow E$ is said to be solution of (RE), if it is continuous and v satisfies (RE).*

In this section we are interested in the study of the retarded equation (RE) by applying the abstract result obtained in the previous section. To be more specific, we write (RE) as the boundary Cauchy problem

$$(IBCP)'\begin{cases} \frac{d}{dt}u(t) = A(t)u(t), & 0 \le s \le t \le T, \\ L(t)u(t) = \Phi(t)u(t) + f(t), & \\ u(s) = \varphi, & \end{cases}$$

where, for each $0 \le t \le T$, $A(t)$ is defined on the Banach space $X := C([-r,0],E)$ by

$$\begin{cases} A(t)u = u' \\ D := D(A(t)) = C^1([-r,0],E), \end{cases}$$

$L(t) : D \longrightarrow Y = E : L(t)\varphi = \varphi(0)$ and $\Phi(t) : X \longrightarrow E : \Phi(t)\varphi = K(t)\varphi$.

It is known that the operator A^0 defined by

$$\begin{cases} A^0\varphi = \varphi' \\ D(A^0) = \{\varphi \in C^1([-r,0],E)\,;\, \varphi(0) = 0\} \end{cases}$$

generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X given by

$$(T(t)\varphi)(\tau) = \begin{cases} \varphi(t+\tau), & -r \leq t+\tau \leq 0 \\ \varphi(0), & t+\tau \geq 0 \end{cases} \tag{3.1}$$

We first show that $(IBCP)'$ has a classical solution. To do this, in view of Theorem 2.6, we have to verify the hypotheses $(H1) - (H5)$ for this problem. The hypotheses $(H1)$, $(H3)$ are obvious and $(H4)$ can be obtained from assumption (A).

For $(H2)$, let $\varphi \in D(A^0(t)) = \{\varphi \in C^1([-r,0],E)\,;\, \varphi(0) = 0\}$ and $f \in C([-r,0],E)$ such that $(\lambda - A^0(t))\varphi = f$. Then,

$$\varphi(\tau) = e^{\lambda\tau}\varphi(0) + \int_\tau^0 e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma, \quad \tau \in [-r,0].$$

Since $\varphi(0) = 0$, we get

$$(R(\lambda, A^0(t))f)(\tau) = \int_\tau^0 e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma.$$

Hence, for $\lambda > 0$,

$$\left\| R(\lambda, A^0(t)) \right\| \leq \frac{1}{\lambda}.$$

This proves the stability of $(A^0(t))_{t\in[0,T]}$.

On the other hand, if $\varphi \in ker(\lambda - A(t))$, then $\varphi(\tau) = e^{\lambda\tau}\varphi(0)$ for $\tau \in [-r,0]$, and

$$\begin{array}{rcl} L\,(t)\,\varphi & = & \|\varphi\,(0)\| \\ & 0 & \left\| e^{-\lambda\tau}\varphi\,(\tau) \right\|. \end{array}$$

Since $\lim_{\lambda\to+\infty} \dfrac{e^{-\lambda\cdot}}{\lambda} = +\infty$, in C_E, we take $c > 0$ such that $\dfrac{e^{-\lambda\cdot}}{\lambda} \geq c$. Therefore,

$$\|L(t)\varphi\| \geq c\lambda||\varphi||, \quad t \in [0,T],$$

and so $(H5)$ also holds. □

We obtain the following proposition.

Proposition 3.2. *Let $f \in C^1([0,T],E)$. Then, for every $\varphi \in C^1([-r,0],E)$ such that $\varphi(0) = K(s)\varphi + f(s)$, the boundary Cauchy problem $(IBCP)'$ has a classical solution u satisfying the variation of constants formula*

$$\begin{aligned} u(t) = T(t-s)(\varphi - e^{\lambda\cdot}\varphi(0)) + e^{\lambda\cdot} f(t,u(t)) \\ + \textstyle\int_s^t T(t-\sigma)e^{\lambda\cdot}\left[\lambda f(\sigma,u(\sigma)) - (f(\sigma,u(\sigma)))'\right] d\sigma, \end{aligned} \tag{3.2}$$

where $f(t,u(t)) := K(t)u(t) + f(t), \quad 0 \le t \le T$.

Moreover, u satisfies the translation property

$$u(t)(\tau) := \begin{cases} u\,(t+\tau)\,(0)\,, & s \le t+\tau \le T \\ \varphi\,(t+\tau-s)\,, & -r+s \le t+\tau \end{cases} \tag{3.3}$$

Proof. First, one can see that for $\lambda > 0$ and $x \in E$, $L_{\lambda,t} = e^{\lambda\cdot}x$. On the other hand, the evolution family generated by $A^0(\cdot)$ is given by $U(t,s) = T(t-s), \quad t \ge s$.

Since the hypotheses $(H1) - (H5)$ are fulfilled, then the existence of a classical solution for $(IBCP)'$ and the formula (3.2) are a direct consequences of Theorem 2.6. It remains to show the translation property (3.3).

Let $\tau \in [-r,0]$ such that $t+\tau \ge s$. By using (3.1) and integration by parts, we obtain

$$\begin{aligned} u(t)(\tau) &= (T(t-s)\varphi)(\tau) - (T(t-s)e^{\lambda\cdot}\varphi(0))(\tau) + e^{\lambda\tau} f(t,u(t)) \\ &\quad + \int_s^{t+\tau} T(t-\sigma)e^{\lambda\cdot}\left[\lambda f(\sigma,u(\sigma)) - (f(\sigma,u(\sigma)))'\right](\tau)\,d\sigma \\ &\quad + \int_{t+\tau}^{t} T(t-\sigma)e^{\lambda\cdot}\left[\lambda f(\sigma,u(\sigma)) - (f(\sigma,u(\sigma)))'\right](\tau)\,d\sigma \\ &= e^{\lambda\tau} f(t,u(t)) + \int_s^{t+\tau} \lambda f(\sigma,u(\sigma)) - (f(\sigma,u(\sigma)))'\,d\sigma \\ &\quad + \int_{t+\tau}^{t} e^{\lambda(t+\tau-\sigma)}\lambda f(\sigma,u(\sigma))\,d\sigma - \int_{t+\tau}^{t} e^{\lambda(t+\tau-\sigma)}(f(\sigma,u(\sigma)))'\,d\sigma \\ &= f(s,u(s)) + \int_s^{t+\tau} \lambda f(\sigma,u(\sigma))\,d\sigma. \end{aligned}$$

If $-r+s \le t+\tau \le s$, we have

$$u(t)(\tau) = (T(t-s)\varphi)(\tau) - (T(t-s)e^{\lambda\cdot}\varphi(0))(\tau) + e^{\lambda\tau} f(t,u(t))$$

$$
\begin{aligned}
& + \int_s^t T(t-\sigma)e^{\lambda\cdot}\left[\lambda f(\sigma,u(\sigma)) - (f(\sigma,u(\sigma)))'\right](\tau)\,d\sigma \\
= \;& \varphi(t+\tau-s) - e^{\lambda(t+\tau-s)}\varphi(0) + e^{\lambda\tau} f(t,u(t)) \\
& + \int_s^t e^{\lambda(t+\tau-s)}\left[\lambda f(\sigma,u(\sigma)) - (f(\sigma,u(\sigma)))'\right]\,d\sigma \\
= \;& \varphi(t+\tau-s) - e^{\lambda(t+\tau-s)}\varphi(0) + e^{\lambda\tau} f(t,u(t)) \\
& + \int_s^t \lambda e^{\lambda(t+\tau-s)} f(\sigma,u(\sigma))\,d\sigma - \int_s^t e^{\lambda(t+\tau-s)}(f(\sigma,u(\sigma)))'\,d\sigma \\
= \;& \varphi(t+\tau-s) - e^{\lambda(t+\tau-s)}\varphi(0) + e^{\lambda(t+\tau-s)} f(s,u(s)).
\end{aligned}
$$

As, by hypothesis, $\varphi(0) = f(s,u(s))$, we conclude

$$
u(t)(\tau) = \begin{cases} \varphi(0) + \int_s^{t+\tau} \lambda f(\sigma,u(\sigma))\,do, & t+\tau \geq s \\ \varphi(t+\tau-s) & -r+s \leq t+\tau \leq s. \end{cases}
$$

The translation property (3.3) is obtained. □

We end this section by the following equivalence result between (RE) and $(IBCP)'$.

Theorem 3.3. *(i) Let u be the classical solution of $(IBCP)'$. Then, the function v defined by*

$$
v(t) := \begin{cases} u(t)(0), & s \leq t \leq T, \\ \varphi(t-s), & -r+s \leq t \leq s, \end{cases}
$$

is a solution of (RE).

(ii) If v is a solution of (RE), then $t \longmapsto u(t) := v_t$ satisfies $(IBCP)'$.

Proof. (i) Let u be the classical solution of $(IBCP)'$. Then the function v is continuous. On the other hand, (3.3) implies that $v_t = u(t), \quad s \leq t \leq T$. Therefore, we have

$$
\begin{aligned}
v(t) &= u(t)(0) \\
&= L(t)u(t)(\cdot) \\
&= K(t)u(t)(\cdot) + f(t) \\
&= K(t)v_t(\cdot) + f(t), \quad t \geq s.
\end{aligned}
$$

Consequently, v satisfies (RE).

The assumption $\sup_{0\leq t\leq T}\|K(t)\| < 1$ assure the uniqueness of the solution of (RE). Hence, (ii) follows easily by (i) and this uniqueness. □

References

[1] K. J. Engel and R. Nagel: One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics **194**, Springer-Verlag 2000.

[2] G. Greiner: Perturbing the boundary conditions of a generator, Houston J. Math. **13**, pp. 213-229, (1987).

[3] G. Greiner: Semilinear boundary conditions for evolution equations of hyperbolic type, Lecture Notes in Pure and Appl. Math. **116** Dekker, New York, pp. 127-153, (1989).

[4] R. Grimmer and E. Sinestrari: Maximum norm in one-dimensional hyperbolic problems, Diff. Integ. Equat. **5**, pp. 421-432, (1992).

[5] T. Kato: Linear evolution equations of "hyperbolic" type, J. Fac. Sci. Tokyo, **17**, pp. 241-258, (1970).

[6] H. Kellermann: Linear evolution equations with time dependent domain, Semesterbericht Funktionalanalysis Tübingen, Wintersemester, pp. 15-44, (1985/1986).

[7] N. T. Lan: On nonautonomous Functional Differential Equations, J. Math. Anal. Appl. **239,** pp. 158-174, (1999).

[8] N. T. Lan and G. Nickel: Time-Dependent operator matrices and inhomogeneous Cauchy Problems, Rend. Circ. Mat. Palermo XLVII, pp. 5-24, (1998).

[9] R. Nagel and E. Sinestrari: Inhomogeneous Volterra integrodifferential equations for Hille-Yosida operators, Marcel Dekker, Lecture Notes Pure Appl. Math. **150**, pp. 51-70, (1994).

[10] A. Pazy: *Semigroups of Linear Operators and Applications to Partial Differential Equations*, New York Springer (1983).

[11] N. Tanaka: Quasilinear evolution equations with non-densely defined operators, Diff. Int. Eq. **9**, pp. 1067-1106, (1996).

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