Proyecciones
Vol. 22, N^o 2, pp. 127-134, August 2003.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172003000200003

DIAGONALS AND EIGENVALUES OF SUMS OF HERMITIAN MATRICES. EXTREME CASES *

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Abstract

There are well known inequalities for Hermitian matrices A and B that relate the diagonal entries of $A+B$ to the eigenvalues of A and B. These inequalities are easily extended to more general inequalities in the case where the matrices A and B are perturbed through congruences of the form $UAU^* + VBV^*$, where U and V are arbitrary unitary matrices, or to sums of more than two matrices. The extremal cases where these inequalities and some generalizations become equalities are examined here.

Key words. *Hermitian matrix, eigenvalues, diagonal elements.*

AMS Subject Classification. *15A18, 15A42.*

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*Partially funded by Fondecyt 1010487

Introduction

This work is written in the same spirit of Miranda and Thompson [7] and Miranda [6] where the relationship between the diagonal elements of a product of two arbitrary matrices and the singular values of each factor was found. In this paper, every time when we mention the diagonal elements or the eigenvalues of an Hermitian matrix, we are going to assume that they are arranged in decreasing order. Schur [8] proved that the vector formed by the diagonal entries of A is majorized by the vector whose entries are the eigenvalues of A. Later on, Ky Fan [1] showed that the vector of eigenvalues of $A+B$ is majorized by the vector sum of the eigenvalues of A and B. If we combine these two results we have that:

Theorem 1. *Let A and B be Hermitian matrices in M_n with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ and $\mu_1 \geq \ldots \geq \mu_n$, respectively. Then, if $d_1 \geq \ldots \geq d_n$ denote the diagonal entries of $A+B$, we have that the vector of the diagonal elements of $A+B$ is majorized by the sum of the vectors of the eigenvalues of A and B. This means that*

$$d_1 + \ldots + d_k \leq \lambda_1 + \mu_1 + \ldots + \lambda_k + \mu_k, \quad k = 1, \ldots, n-1$$

and

$$d_1 + \ldots + d_n = \lambda_1 + \mu_1 + \ldots + \lambda_n + \mu_n.$$

Since the eigenvalues of UAU^* are the same as those of A, when U is unitary, we can easily generalize this result to a more general family of matrices.

Theorem 2. *Let A and B be Hermitian matrices in M_n. Let us denote by $d_1, d_2, \ldots, d_n$ the diagonal entries of $UAU^* + VBV^*$ where U, V are arbitrary unitary matrices. Then*

$$d_1 + \ldots + d_k \leq \lambda_1 + \mu_1 + \ldots + \lambda_k + \mu_k, \quad k = 1, \ldots, n-1$$

and

$$d_1 + \ldots + d_n = \lambda_1 + \mu_1 + \ldots + \lambda_n + \mu_n.$$

If we consider matrices of the form UAV where U and V are unitary, these are not necessarily Hermitian anymore, and their diagonal elements might be complex numbers. There is no result that relates the diagonal to the eigenvalues, but Thompson [9] found a set of inequalities between the diagonal elements and the singular values of the summands. This kind of relationship is called weak majorization.

Theorem 3. *Let A and B be Hermitian matrices in M_n. Let us denote by $d_1, d_2, \ldots, d_n$ the diagonal entries of $UAV + WBX$ arranged in decreasing order according to their absolute values, and where U, V, W, X are arbitrary unitary matrices. Then*

$$|d_1| + \ldots + |d_k| \leq \sigma_1(A) + \sigma_1(B) + \ldots + \sigma_k(A) + \sigma_k(B), \quad k = 1, \ldots, n.$$

where the sigmas denote the singular values of the matrices, also arranged in decreasing order.

Extreme Cases

Let us give now a result that shows the equality case for the partial sums of the diagonal entries and the eigenvalues of an Hermitian matrix. This result is based on Lemma 2.1 which appears in Li [3] in the context of diagonal elements and singular values of an arbitrary matrix.

Lemma 1. *Let C be an $n \times n$ Hermitian matrix. If we write C as*

$$\mathbf{C} = \begin{pmatrix} X & Z^* \\ Z & Y \end{pmatrix}$$

with $X \in M_k, Y \in M_{n-k}, Z \in M_{n-k,k}$, and if $\lambda_i(C) = \lambda_i(X)$ for $1 \leq i \leq k$, then $Z = 0$.

Proof. Since C and X are Hermitian matrices, for $1 \leq i \leq k$ we have that $\lambda_i(C^*C) = \sigma_i^2(C)$, where σ_i denotes the singular values of a matrix, and $\lambda_i(X^*X) = \sigma_i^2(C)$.
Consequently, $\lambda_i(C) = \lambda_i(X)$ for $1 \leq i \leq k$ implies that $\sigma_i^2(C) = \sigma_i^2(X)$, and

$$\sum_{j=1}^{k} \lambda_j(C^*C) = \sum_{j=1}^{k} \sigma_j^2(C) = \sum_{j=1}^{k} \lambda_j^2(C) = \sum_{j=1}^{k} \sigma_j^2(X) = \sum_{j=1}^{k} \lambda_j^2(X) = tr(X^*X)$$

and

$$\sum_{j=1}^{k} \lambda_j(C^*C) = \sum_{i,j=1}^{k} |c_{ij}|^2 \leq \sum_{1 \leq i \leq n, 1 \leq j \leq k} |c_{ij}|^2 \leq \sum_{j=1}^{k} \lambda_j(C^*C)$$

The last inequality holds since the sum of the first diagonal entries of C^*C is not greater than the sum of its k largest eigenvalues (this is Schur's inequality). So,

$$\sum_{i,j=1}^{k} |c_{ij}|^2 = \sum_{1\leq i\leq n, 1\leq j\leq k} |c_{ij}|^2$$

Similarly,

$$\sum_{i,j=1}^{k} |c_{ij}|^2 = \sum_{1\leq i\leq k, 1\leq j\leq n} |c_{ij}|^2$$

Consequently, $c_{ij} = 0$ for $1 \leq j \leq k < i \leq n,\ 1 \leq i \leq k < j \leq n$, and $Z = 0$.
Let us note that it was also proved here that $Y = (c_{ij})$ where $k < i, j \leq n$. □

Theorem 4. *Let A and B be Hermitian matrices. Let $1 \leq k \leq n$. Then*

$$\sum_{i=1}^{k} d_i(A+B) = \sum_{i=1}^{k} \lambda_i(A+B)$$

if and only if
$A + B = (A+B)_1 \oplus (A+B)_2$ where $(A+B)_1$ has eigenvalues $\lambda_1(A+B), \cdots, \lambda_k(A+B)$.

Proof. For $1 \leq k \leq n$ let $(A+B)_{(k)}$ be the principal submatrix of $A+B$ with the diagonal entries $d_1(A+B), \cdots, d_k(A+B)$. Then

$$\sum_{i=1}^{k} d_i(A+B) = \sum_{i=1}^{k} \lambda_i((A+B)_{(k)})$$

We use now the interlacing inequalities for eigenvalues (See 4.3.15 in [2]) $\lambda_i((A+B)_{(k)}) \leq \lambda_i(A+B),\ 1 \leq i \leq k$, to have

$$\sum_{i=1}^{k} d_i(A+B) = \sum_{i=1}^{k} \lambda_i((A+B)_{(k)}) \leq \sum_{i=1}^{k} \lambda_i(A+B) = \sum_{i=1}^{k} d_i(A+B)$$

So, $\lambda_i((A+B)_{(k)}) = \lambda_i(A+B) \ \ \forall i, 1 \leq i \leq k$,

and hence by the lemma $A+B$ can be written as $A+B=(A+B)_1\oplus(A+B)_2$ where $(A+B)_1$ has eigenvalues $\lambda_1(A+B),\cdots,\lambda_k(A+B)$..
The converse is clear. $\square$

Next, we have an interesting result that produces a simultaneous congruence for A and B.

Theorem 5. *Let A and B be Hermitian matrices. Let $1\le k\le n$. Then*

$$\sum_{i=1}^{k}\lambda_i(A+B)=\sum_{i=1}^{k}\lambda_i(A)+\sum_{i=1}^{k}\lambda_i(B)$$

if and only if there exists unitary $U\in U(n)$ such that $UAU^*=A_1\oplus A_2$, $UBU^*=B_1\oplus B_2$ where A_1 has eigenvalues $\lambda_1(A),\cdots,\lambda_k(A)$ and B has eigenvalues $\lambda_1(B),\cdots,\lambda_k(B)$.

Proof. Let us assume first that $A+B$ is a diagonal matrix. Then
$A+B=diag(\lambda_1(A+B),\cdots,\lambda_n(A+B))$ and

$$\sum_{i=1}^{k}(a_{ii}+b_{ii})=\sum_{i=1}^{k}\lambda_i(A+B)=\sum_{i=1}^{k}\lambda_i(A)+\sum_{i=1}^{k}\lambda_i(B)$$

So,

$$\sum_{i=1}^{k}a_{ii}=\sum_{i=1}^{k}\lambda_i(A),\ \textit{and}\ \sum_{i=1}^{k}b_{ii}=\sum_{i=1}^{k}\lambda_i(B)$$

and

$$A=A_1\oplus A_2, B=B_1\oplus B_2$$

where A_1 and B_1 have eigenvalues $\lambda_1(A),\cdots,\lambda_k(A)$, $\lambda_1(B),\cdots,\lambda_k(B)$ respectively.
If $A+B$ is not diagonal, by the Spectral Theorem for Hermitian matrices we can find $U\in U(n)$ so that $U(A+B)U^*=diag(\lambda_1(A+B),\cdots,\lambda_n(A+B))$.
So,

$$A+B=U^*diag(\lambda_1(A+B),\cdots,\lambda_n(A+B))U=U^*[(A_1\oplus A_2)+(B_1\oplus B_2)]U$$

Consequently,
$A=U^*(A_1\oplus A_2)U,\ B=U^*(B_1\oplus B_2)U$, that is,
$UAU^*=A_1\oplus A_2, \textit{and}\ UBU^*=B_1\oplus B_2$.
The converse is immediate. $\square$

If we consider the two previous results, we can establish the relationship between the diagonal elements of $A+B$ and the eigenvalues of A and B.

Theorem 6.

$$\sum_{i=1}^{k} d_i(A+B) = \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B)$$

if and only if there exists $U \in U(n)$ such that

$$A+B = (A+B)_1 \oplus (A+B)_2 = U^*(A_1+B_1)U \oplus U^*(A_2+B_2)U$$

where $(A+B)_1$ has eigenvalues $\lambda_1(A+B), \cdots, \lambda_k(A+B)$, A_1 has eigenvalues $\lambda_1(A), \cdots, \lambda_k(A)$, and B_1 has eigenvalues $\lambda_1(B), \cdots, \lambda_k(B)$.

Let us notice that if we write U as $U = (U_1|U_2)$ where U_1 is the submatrix of U which consists of its first k columns, then we can write the relationship among the three $k \times k$ diagonal blocks of A, B and $A+B$:

$$U_1^*(A_1+B_1)U_1 = (A+B)_1$$

Comments

These comments show how to try to generalize some of the results given here, and also, some different points of view are considered.

Remark 1. *It is interesting to notice that the same proof of the previous theorem works in the case when we have more than two matrices. We prefered to give here the two matrices case instead of the general case to see better the beauty of the results. In this context, Theorem 6 takes the form:*

Theorem 7. *Let $A_1, \cdots, A_m$ be Hermitian matrices with eigenvalues*

Theorem 8. *$\lambda_1(A_j), \cdots, \lambda_n(A_j)$, where $1 \leq j \leq m$, and where the eigenvalues are arranged in decreasing order. Furthermore, if $d_i(A_1 + \cdots + A_m)$ are the diagonal entries, also arranged in decreasing order; and if $1 \leq k \leq n$, then*

$$\sum_{i=1}^{k} d_i(A_1+\cdots+A_m) = \sum_{i=1}^{k} \lambda_i(A_1) + \cdots + \sum_{i=1}^{k} \lambda_i(A_m)$$

if and only if there exists $U \in U(n)$ such that

$$A_1+\cdots+A_m = (A_1+\cdots+A_m)_1 \oplus (A_1+\cdots+A_m)_2, UA_jU^* = (A_j)_1$$

and $(A_1+\cdots+A_m)_1$ has eigenvalues $\lambda_1(A_1+\cdots+A_m), \cdots, \lambda_k(A_1+\cdots+A_m)$ and $(A_j)_1$ has eigenvalues $\lambda_1(A_j), \cdots, \lambda_k(A_j)$.

Remark 2. *If we replace the matrix B by $-B$ in the inequality*

$$\sum_{i=1}^{k} \lambda_i(A+B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{1}^{k} \lambda_i(B),$$

and we use the facts that the eigenvalues of $-B$ are the negatives of the eigenvalues of B, and that $\lambda_i(-B) = -\lambda_{n-i+1}(B)$, we can write the above inequality as

$$\sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_{n-i+1}(B) \leq \sum_{i=1}^{k} \lambda_i(A+B)$$

which is a lower bound for the partial sums of the eigenvalues of $A+B$. By using the same proof of Theorem 6 we obtain equality here if and only if there exists unitary U such that $UAU^* = A_1 \oplus A_2$ and $UBU^* = B_1 \oplus B_2$, where A_1 has eigenvalues $\lambda_1(A), \cdots, \lambda_k(A)$ and B_1 has eigenvalues $\lambda_{n-k+1}(B), \cdots, \lambda_n(B)$.

Remark 3. *There are papers where the additive case for arbitrary instead of Hermitian matrices is considered. To see some extremal results for the diagonal elements and the singular values of the matrices involved, where certain group actions are used, see [4].*

Remark 4. *In the case of product of matrices, Miranda [5] proved an extremal result for the trace of the product of two arbitrary matrices in terms of the singular values of the factors which produced also a simultaneous singular decomposition in the same spirit of Theorem 6. It is known that*

$$\sum_{i=1}^{k} |d_i(UAU^*VBV^*)| \leq \sum_{i=1}^{k} \lambda_i(A)\lambda_i(B),$$

but this result cannot be extended to more than two matrices. For instance, consider A, B Hermitians arbitrary, $C = -I_n$, and $U = V = W = I_n$. Then

$$\sum_{i=1}^{k} d_i(ABC) = -\sum_{i=1}^{k} d_i(AB) \geq -\sum_{i=1}^{k} \lambda_i(A)\lambda_i(B) = \sum_{i=1}^{k} \lambda_i(A)\lambda_i(B)\lambda_i(C)$$

Acknowledgement

I would like to thank Roger Horn for his valuable comments in Theorem 5 and the referee for important suggestions.

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Received : January 2003.

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