Proyecciones
Vol. 24, N^o 1, pp. 49-63, May 2005.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172005000100005

SPECTRAL PROPERTIES OF A NON SELFADJOINT SYSTEM OF DIFFERENTIAL EQUATIONS WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

E. KIR **a.**, *G. BASCANBAZ-TUNCA* **b.** *C. YANIK* **c.**
a. *Gazi University, Turkey*
b. *Ankara University, Turkey*
c. *Hacettepe University, Turkey*
Received : October 2004, Accepted : April 2005.

Abstract

In this paper we investigated the spectrum of the operator $L(\lambda)$ generated in Hilbert Space of vector-valued functions $L^2(\mathbf{R}_+, \mathbf{C}_2)$ *by the system*

$$(0.1) \mathrm{i}\mathrm{y}_1' + \mathrm{q}_1(x)\,\mathrm{y}_2 = \lambda \mathrm{y}_1, \quad -\mathrm{i}\mathrm{y}_2' + \mathrm{q}_2(x)\,\mathrm{y}_1 = \lambda \mathrm{y}_2, \quad \mathrm{x} \in \mathbf{R}_+ := [0, \infty),$$

and the spectral parameter- dependent boundary condition

$$(a_1\lambda + b_1)\, y_2(0,\lambda) - (a_2\lambda + b_2)\, y_1(0,\lambda) = 0,$$

where λ is a complex parameter, $q_i,\ i = 1,2$ are complex-valued functions $a_i \neq 0,\ b_i \neq 0,\ i = 1,2$ are complex constants. Under the condition

$$\sup_{x\in R_+} \{\exp \varepsilon x\, |q_i(x)|\} < \infty, i = 1,2, \varepsilon > 0,$$

we proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities with finite multiplicities. Furthermore we show that the principal functions corresponding to eigenvalues of $L(\lambda)$ belong to the space $L^2(\mathbf{R}_+, \{\mathbf{C}_2)$ and the principal functions corresponding to spectral singularities belong to a Hilbert space containing $L^2(\mathbf{R}_+, \mathbf{C}_2)$.

AMS Subject Classification : *34L05 (34B05 47E05 34L40).*
Key Words : *Spectrum, Spectral Singularities, Non-Selfadjoint System of Differential Equations,*

1. Introduction

Let us consider the nonself-adjoint one dimensional Schrödinger operator L generated in $L^2(\mathbf{R}_+)$ by the differential expression

$$\mathrm{l}(y) = -y'' + q(x)\,y, \quad x \in \mathbf{R}_+$$

and the boundary condition $y(0) = 0$ as $Ly = ly$, where q is a complex-valued function. The spectral analysis of L has been studied by Naimark [7]. Naimark has proved that there are some poles of resolvent's kernel which are not the eigenvalues of the operator L. (Schwartz [8] named these points as spectral singularities of L).Moreover Naimark has proved that spectral singularities are on the continuous spectrum, he has also shown that L has a finite number of eigenvalues and spectral singularities with finite multiplicities if the condition

$$\int_0^\infty e^{\varepsilon x}\,|q(x)|\,dx < \infty, \quad \varepsilon > 0$$

holds. Lyance has obtained the role of the spectral singularities in the spectral expansion of the operator L in terms of principal functions[6].

The properties of the eigenvalues and vector-valued eigenfunctions of a boundary value problem for a one-dimensional Dirac system with a spectral parameter in the boundary conditions has been investigated by Kerimov [4].

We now consider the operator $L(\lambda)$ generated in
$$\mathrm{L}^2(\mathbf{R}_+, \mathbf{C}_2) := \left\{ f(x) : \; f(x) = \binom{f_1(x)}{f_2(x)}, \; \int_0^\infty \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx < \infty \right\}$$
by the system

$$iy_1' + q_1(x)\,y_2 = \lambda y_1, \tag{1.1}$$

$$-iy_2' + q_2(x)\,y_1 = \lambda y_2, \quad x \in \mathbf{R}_+$$

and the spectral parameter-dependent boundary condition

$$(a_1\lambda + b_1)\,y_2(0,\lambda) - (a_2\lambda + b_2)\,y_1(0,\lambda) = 0, \tag{1.2}$$

where $q_i, \; i = 1, 2$, are complex-valued functions, λ is the spectral parameter, a_i, b_i are complex constants, $b_i \neq 0, \; i = 1, 2$; moreover $|a_1|^2 + |a_2|^2 \neq 0$.

The spectrum of the operator generated by the system (1.1) with the boundary condition $y_2(0) - hy_1(0) = 0$, (which is the special case of (1.2) when $a_i = 0$, $b = 1$) here $h \neq 0$ is a complex constant, has been investigated in [5] and in [1].

In this paper, we discussed the spectrum of $L(\lambda)$ defined by (1.1) and (1.2) and proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities with finite multiplicities under the conditions

$$|q_i(x)| \leq ce^{-\varepsilon x} < \infty, \quad i = 1, 2, \ \varepsilon > 0, \ c > 0$$

by using analytic continuation method ([7]). Finally we observe the properties of the principal functions corresponding to eigenvalues and spectral singularities.

In the rest of the paper, we use the following notations:

$$\begin{aligned} \mathbf{C}_+ &= \{\lambda : \lambda \in \mathbf{C}, \ \operatorname{Im}\lambda > 0\}, \ \overline{\mathbf{C}}_- = \{\lambda : \lambda \in \mathbf{C}, \ \operatorname{Im}\lambda < 0\}, \\ \overline{\mathbf{C}}_+ &= \{\lambda : \lambda \in \mathbf{C}, \ \operatorname{Im}\lambda \geq 0\}, \ \overline{\mathbf{C}}_- = \{\lambda : \lambda \in \mathbf{C}, \ \operatorname{Im}\lambda \leq 0\}, \end{aligned}$$

$\sigma_p(L(\lambda))$ denotes the eigenvalues and $\sigma_{ss}(L(\lambda))$ denotes the spectral singularities of $L(\lambda)$.

2. Preliminaries

Let us suppose that

$$|q_i(x)| \leq c(1+x)^{-(1+\varepsilon)}, \quad i = 1, 2, \quad x \in \mathbf{R}_+, \quad \varepsilon > 0 \tag{2.1}$$

holds, where $c > 0$ is a constant.

The following results were given in [1] and in the first reference there in.Under the conditions (2.1), equation (1.1) has the following vector solutions

$$e^+(x,\lambda) = \begin{pmatrix} e_1^+(x,\lambda) \\ e_2^+(x,\lambda) \end{pmatrix} = \begin{pmatrix} \int\limits_x^\infty H_{12}(x,t)\, e^{i\lambda t} dt \\ e^{i\lambda x} + \int\limits_x^\infty H_{22}(x,t)\, e^{i\lambda t} dt \end{pmatrix} \tag{2.2}$$

for $\lambda \in \overline{\mathbf{C}}_+$ and

$$e^-(x,\lambda) = \begin{pmatrix} e_1^-(x,\lambda) \\ e_2^-(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x} + \int\limits_x^\infty H_{11}(x,t)\, e^{-i\lambda t} dt \\ \int\limits_x^\infty H_{21}(x,t)\, e^{-i\lambda t} dt \end{pmatrix} \tag{2.3}$$

for $\lambda \in \overline{\mathbf{C}}_-$; moreover the kernels $H_{ij}(x,t)$, $i,j=1,2$, satisfy the inequalities

$$|H_{ij}(x,t)| \leq c \sum_{k=1}^{2} \left| q_k \left(\frac{x+t}{2} \right) \right|, \tag{2.4}$$

where $c>0$ is a constant. Therefore the functions $e_i^+(x,\lambda)$ and $e_i^-(x,\lambda)$, $i=1,2$, are analytic with respect to λ in $\mathbf{C}_+$, $\mathbf{C}_-$, and continuous on $\overline{\mathbf{C}}_+$ and $\overline{\mathbf{C}}_-$, respectively. Moreover e^+ and e^- satisfy the following asymptotic equalities ([1])

$$e^+(x,\lambda) = \begin{pmatrix} 0 \\ e^{i\lambda x} \end{pmatrix} [1+o(1)], \quad \lambda \in \overline{\mathbf{C}}_+ \quad x \to \infty \tag{2.5}$$

and

$$e^-(x,\lambda) = \begin{pmatrix} e^{-i\lambda x} \\ 0 \end{pmatrix} [1+o(1)], \quad \lambda \in \overline{\mathbf{C}}_-, \quad x \to \infty. \tag{2.6}$$

From (2.5) and (2.6) we have

$$W\{e^+, e^-\} = \lim_{x\to\infty} W\{e^+(x,\lambda), e^-(x,\lambda)\} = -1 \tag{2.7}$$

for $\lambda \in \mathbf{R}$, where $W\left\{y^{(1)}, y^{(2)}\right\}$ is the wronskian of the solutions of $y^{(1)}$ and $y^{(2)}$ which is defined as $W\left\{y^{(1)}, y^{(2)}\right\} = y_1^{(1)} y_2^{(2)} - y_1^{(2)} y_2^{(1)}$,here $y^{(i)} = \binom{y_1^{(i)}}{y_2^{(i)}}$, $i=1,2$. Therefore e^+, e^- are the fundamental system of solutions of the system (1.1) for $\lambda \in \mathbf{R}$.

Let $\varphi(x,\lambda)$ be the solution of (1.1) satisfying the initial conditions

$$\varphi_1(0,\lambda) = a_1\lambda + b_1, \quad \varphi_2(0,\lambda) = a_2\lambda + b_2.$$

Clearly the solution $\varphi(x,\lambda)$ exists uniquely and is an entire function of λ..

3. Eigenvalues and spectral singularities

Let us define

$$a^+(\lambda) = (a_1\lambda + b_1)\, e_2^+(0,\lambda) - (a_2\lambda + b_2)\, e_1^+(0,\lambda) = 0$$

$$a^-(\lambda) = (a_1\lambda + b_1)\, e_2^-(0,\lambda) - (a_2\lambda + b_2)\, e_1^-(0,\lambda) = 0. \tag{3.1}$$

Let

$$R(x,t;\lambda)=\begin{cases}R^{+}(x,t;\lambda), & \operatorname{Im}\lambda\geq 0\\ R^{-}(x,t;\lambda), & \operatorname{Im}\lambda\leq 0\end{cases} \tag{3.2}$$

be Green's function of $L(\lambda)$ which is obtained by using classical methods, here

$$R^{+}(x,t;\lambda)=\frac{i}{a^{+}(\lambda)}\begin{cases}e^{+}(x,\lambda)\,\varphi^{*}(t,\lambda), & 0\leq t\leq x\\ \varphi(x,\lambda)\,(e^{+})^{*}(t,\lambda), & x<t\leq\infty\end{cases} \tag{3.3}$$

and

$$R^{-}(x,t;\lambda)=\frac{i}{a^{-}(\lambda)}\begin{cases}e^{-}(x,\lambda)\,\varphi^{*}(t,\lambda), & 0\leq t\leq x\\ \varphi(x,\lambda)\,(e^{-})^{*}(t,\lambda), & x<t\leq\infty\end{cases} \tag{3.4}$$

and $(e^{\pm})^{*}:=\left(e_2^{\pm},e_1^{\pm}\right),\ \varphi^{*}:=(\varphi_2,\varphi_1)$. Moreover from (2.5) and (2.6) we have

$$e^{+}(x,\lambda)\in L^{2}(\mathbf{R}_{+},\mathbf{C}_2) \tag{3.5}$$

for $\lambda\in\mathbf{C}_{+}$ and

$$e^{-}(x,\lambda)\in L^{2}(\mathbf{R}_{+},\mathbf{C}_2) \tag{3.6}$$

for $\lambda\in\mathbf{C}_{-}$.In this case we state the following

Lemma 3.1.
a) $\sigma_p(L(\lambda))=\{\lambda:\lambda\in\mathbf{C}_{+},\ a^{+}(\lambda)=0\}\cup\{\lambda:\lambda\in\mathbf{C}_{-},\ a^{-}(\lambda)=0\}$,
b) $\sigma_{ss}(L(\lambda))$
$=\{\lambda:\lambda\in\mathbf{R}\backslash\{0\},\ a^{+}(\lambda)=0\}\cup\{\lambda:\lambda\in\mathbf{R}\backslash\{0\},\ a^{-}(\lambda)=0\}$.

Proof. *a)* It is clear that

$$\{\lambda:\lambda\in\mathbf{C}_{+},\ a^{+}(\lambda)=0\}\cup\{\lambda:\lambda\in\mathbf{C}_{-},\ a^{-}(\lambda)=0\}\subset\sigma_p(L(\lambda)).$$

Now let us suppose that $\lambda_0\in\sigma_p(L(\lambda))$. If $\lambda_0\in\mathbf{C}_{+}$ then (1.1) has a nontrivial solution $y(x,\lambda_0)$ in $L^{2}(\mathbf{R}_{+},\mathbf{C}_2)$ for $\lambda=\lambda_0$ satisfying (1.2).

Since $W\{y(x,\lambda_0),\varphi(x,\lambda_0)\}=0$ then there exists a constant $c\neq 0$ such that $y(x,\lambda_0)=c\varphi(x,\lambda_0)$. Therefore

$$\begin{aligned}&W\{y(x,\lambda_0),e^{+}(x,\lambda_0)\}\\&=y_1(0,\lambda_0)\,e_2^{+}(0,\lambda_0)-y_2(0,\lambda_0)\,e_1^{+}(0,\lambda_0)=ca^{+}(\lambda_0).\end{aligned} \tag{3.7}$$

Moreover we find from (3.5) that

$$W\left\{y\left(x,\lambda_0\right),e^{+}\left(x,\lambda_0\right)\right\} \tag{3.8}$$
$$= \lim_{x\to\infty}\left\{y_1\left(x,\lambda_0\right)e_2^{+}\left(x,\lambda_0\right)-y_2\left(x,\lambda_0\right)e_1^{+}\left(x,\lambda_0\right)\right\}$$
$$=0$$

So we obtain from (3.7) and (3.8) that $a^{+}\left(\lambda_0\right)=0$.

If $\lambda_0\in\mathbf{C}_{-}$ then we prove that $a^{-}\left(\lambda_0\right)=0$ similarly.

If $\lambda_0\in\mathbf{R}$, then the general solution of (1.1) is

$\mathrm{y}(x,\lambda_0)=c_1e^{+}\left(x,\lambda_0\right)+c_2e^{-}\left(x,\lambda_0\right)$
for $\lambda=\lambda_0$. From (2.5) and (2.6) we have

$$\mathrm{y}(x,\lambda_0)=\binom{c_2e^{-i\lambda_0x}}{c_1e^{i\lambda_0x}}\left(1+o\left(1\right)\right)$$

as $x\to\infty$. Therefore $y\left(x,\lambda_0\right)\notin L^2\left(\mathbf{R}_{+},\mathbf{C}_2\right)$. Hence $\sigma_p\left(L\left(\lambda\right)\right)\cap\mathbf{R}=$, so (a) follows.

$b)$ Spectral singularities which are not the eigenvalues of $L\left(\lambda\right)$, are the poles of the resolvent's kernel. From $(3.1)-(3.4)$ and (a), we can say that the spectral singularities of $L\left(\lambda\right)$ are the real zeros of a^{+} and a^{-}. So (b) follows.

Furthermore

$$\mathrm{W}\{e^{+}\left(x,\lambda\right),e^{-}\left(x,\lambda\right)\}=e_1^{+}\left(0,\lambda\right)e_2^{-}\left(0,\lambda\right)-e_2^{+}\left(0,\lambda\right)e_1^{-}\left(0,\lambda\right)=-1$$

for $\lambda\in\mathbf{R}$. Therefore we have

$$\left\{\lambda:\lambda\in\mathbf{R},\ a^{+}\left(\lambda\right)=0\right\}\cap\left\{\lambda:\lambda\in\mathbf{R},\ a^{-}\left(\lambda\right)=0\right\}=\phi \tag{3.9}$$

Now as we see from Lemma 3.1 that to investigate the properties of the eigenvalues and the spectral singularities of $L\left(\lambda\right)$, we need to investigate the properties of the zeros of a^{+} and a^{-} in $\overline{\mathbf{C}}_{+}$, $\overline{\mathbf{C}}_{-}$, respectively. For simplicity, we will consider only the zeros of a^{+} in $\overline{\mathbf{C}}_{+}$. In this point of view let us define the sets $Z_{+}=\{\lambda:\lambda\in\mathbf{C}_{+},\ a^{+}\left(\lambda\right)=0\}$,
$Z=\left\{\lambda:\lambda\in\mathbf{R},\ a^{+}\left(\lambda\right)=0\right\}$.

Lemma 3.2. (a) The set Z_{+} is bounded and has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.

(b) Z is a compact set.

Proof. From (2.2) we get that $a^+(\lambda)$ is analytic in $\mathbf{C}_+$ and satisfies

$$a^+(\lambda) = a_1\lambda + b_1 + \int_0^\infty \{(a_1\lambda + b_1) H_{22}(0,t) - (a_2\lambda + b_2) H_{12}(0,t)\} e^{i\lambda t} dt. \tag{3.10}$$

From (3.10) we get

$$a^+(\lambda) = \lambda \left(a_1 + \int_0^\infty \{a_1 H_{22}(0,t) - a_2 H_{12}(0,t)\} e^{i\lambda t} dt \right) + O(1) \tag{3.11}$$

for $\lambda \in \overline{\mathbf{C}}_+,\ |\lambda| \to \infty$. From (3.11) we find that the zeros of a^+ must lie in a bounded domain. Since a^+ is analytic in $\mathbf{C}_+$ then these zeros are at most countable numbers. From the uniqueness of analytic functions the limit points of Z_+ can lie only in a bounded subinterval of the real axis. So (a) follows. (b) is obtained from the uniqueness theorem of analytic functions [3]

From Lemma 3.1 and Lemma 3.2 we have

Theorem 3.3. If the conditions (2.1) hold, then the set of eigenvalues and spectral singularities of $L(\lambda)$ are bounded, countable and their limit points can lie only in a bounded subinterval of the real axis.

Definition 3.4. The multiplicity of a zero of a^+ (or a^-) in $\overline{\mathbf{C}}_+$ $\left(\text{or } \overline{\mathbf{C}}_-\right)$ is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of $L(\lambda)$.

Let us suppose that

$$|q_i(x)| \le c e^{-\varepsilon x}, \quad c > 0, \quad \varepsilon > 0, \quad i = 1, 2 \tag{3.12}$$

hold. From (2.4) we obtain that

$$|H_{ij}(x,t)| \le c \exp\left\{ \frac{-\varepsilon}{2}(x+t) \right\}. \tag{3.13}$$

From (3.10) and (3.13), a^+ has an analytic continuation from the real axis to the half plane $\operatorname{Im}\lambda > -\frac{\varepsilon}{2}$. So the limit points of the sets Z_+ and Z cannot lie in $\mathbf{R}$ i.e. the sets Z_+ and Z have no limit points. Therefore the number of zeros of a^+ in $\overline{\mathbf{C}}_+$ are finite with finite multiplicities. Similarly we can show that a^- has a finite number of zeros with finite multiplicities in $\overline{\mathbf{C}}_-$.So we have proved the following

Theorem 3.5. The operator $L(\lambda)$ has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity if the conditions (3.12) hold.

4. Principal functions

Assume that (3.12) holds. Let $\lambda_1^+, ..., \lambda_j^+$ and $\lambda_1^-, ..., \lambda_k^-$ denote the zeros of a^+ in $\mathbf{C}_+$ and a^- in $\mathbf{C}_-$ with multiplicities $m_1^+, ..., m_j^+$ and $m_1^-, ..., m_k^-$, respectively. Similarly, let $\lambda_1, ..., \lambda_p$ and $\lambda_{p+1}, ..., \lambda_q$ denote the zeros of a^+ and a^- on the real axis with multiplicities $m_1, ..., m_p$ and $m_{p+1}, ..., m_q$, respectively. In this case we have

$$\left\{\frac{\partial^n}{\partial\lambda^n} W[\varphi(x,\lambda), e^+(x,\lambda)]\right\}_{\lambda=\lambda_i^+} = \left\{\frac{d^n}{d\lambda^n} a^+(\lambda)\right\}_{\lambda=\lambda_i^+} = 0 \tag{4.1}$$

for $n = 0, 1, ..., m_i^+ - 1,\ i = 1, 2, ..., j$, and

$$\left\{\frac{\partial^n}{\partial\lambda^n} W[\varphi(x,\lambda), e^-(x,\lambda)]\right\}_{\lambda=\lambda_l^-} = \left\{\frac{d^n}{d\lambda^n} a^-(\lambda)\right\}_{\lambda=\lambda_l^-} = 0 \tag{4.2}$$

for $n = 0, 1, ..., m_l^- - 1,\ l = 1, 2, ..., k$. Clearly we have

$$\varphi\left(x, \lambda_i^+\right) = c_0\left(\lambda_i^+\right) e^+\left(x, \lambda_i^+\right),\ i = 1, 2, ..., j, \tag{4.3}$$

$$\varphi\left(x, \lambda_l^-\right) = d_0\left(\lambda_l^-\right) e^-\left(x, \lambda_l^-\right),\ l = 1, 2, ..., k, \tag{4.4}$$

when $n = 0$. Therefore $c_0\left(\lambda_i^+\right) \neq 0,\ d_0\left(\lambda_l^-\right) \neq 0$. Therefore we can state the following lemma

Theorem 4.1. The following equalities

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i^+} = \sum_{v=0}^{n}\binom{n}{v} c_{n-v}^+\left(\lambda_i^+\right)\left\{\frac{\partial^v}{\partial\lambda^v} e^+(x,\lambda)\right\}_{\lambda=\lambda_i^+}, \tag{4.5}$$

for $n = 0, 1, ..., m_i^+ - 1,\ i = 1, 2, ..., j$ and

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_l^-} = \sum_{v=0}^{n}\binom{n}{v} d_{n-v}^-\left(\lambda_l^-\right)\left\{\frac{\partial^v}{\partial\lambda^v} e^-(x,\lambda)\right\}_{\lambda=\lambda_l^-} \tag{4.6}$$

for $n = 0, 1, ..., m_l^- - 1,\ l = 1, 2, ..., k$ hold, where the constants $c_0^+, c_1^+, ..., c_n^+$ and $d_0^-, d_1^-, ..., d_n^-$ depend on λ_i^+ and λ_l^-, respectively.

Proof. Using mathematical induction, we prove first (4.5). For $n = 0$, The proof is clear by (4.3). Now we suppose that (4.5) holds for $1 \leq n_0 \leq m_i^+ - 2$, i.e.

$$(4.7)\left\{\frac{\partial^{n_0}}{\partial\lambda^{n_0}}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i^+} = \sum_{v=0}^{n_0}\binom{n_0}{v} c_{n_0-v}^+\left(\lambda_i^+\right)\left\{\frac{\partial^v}{\partial\lambda^v}e^+(x,\lambda)\right\}_{\lambda=\lambda_i^+}.$$

Now we will show that (4.5) also holds for $n_0 + 1$. If $U(x,\lambda)$ is a solution of the equation (1.1), then $\frac{\partial^n}{\partial\lambda^n}U(x,\lambda)$ satisfies the following equation:

$$(4.8)\qquad \left\{J\frac{d}{dx} + Q(x) - \lambda\right\}\frac{\partial^n}{\partial\lambda^n}U(x,\lambda) = n\frac{\partial^{n-1}}{\partial\lambda^{n-1}}U(x,\lambda),$$

where

$$\mathrm{J}=\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\quad Q(x) = \begin{bmatrix} 0 & q_1(x) \\ q_2(x) & 0 \end{bmatrix},\quad U(x,\lambda) = \left[\begin{smallmatrix} U_1(x,\lambda) \\ U_2(x,\lambda) \end{smallmatrix}\right].$$

Writing (4.8) for $\varphi\left(x,\lambda_i^+\right)$ and $e^+(x,\lambda_i^+)$, then using (4.7), we get

$$\left\{J\tfrac{d}{dx} + Q(x) - \lambda_i^+\right\} f_{n_0+1}\left(x,\lambda_i^+\right) = 0,$$

where

$$f_{n_0+1}\left(x,\lambda_i^+\right) = \left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i^+} - \sum_{v=1}^{n_0+1}\binom{n_0+1}{v} c_{n_0+1-v}^+\left(\lambda_i^+\right)\left\{\frac{\partial^v}{\partial\lambda^v}e^+(x,\lambda)\right\}_{\lambda=\lambda_i^+}.$$

therefore we have

$$W\left[f_{n_0+1}\left(x,\lambda_i^+\right), e^+\left(x,\lambda_i^+\right)\right] = \left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}W[\varphi(x,\lambda), e^+(x,\lambda)\right\}_{\lambda=\lambda_i^+} = 0$$

by (4.1). Hence there exists a constant $c_{n_0+1}^+\left(\lambda_i^+\right)$ such that

$$\mathrm{f}_{n_0+1}\left(x,\lambda_i^+\right)=c_{n_0+1}^+\left(\lambda_i^+\right)e^+\left(x,\lambda_i^+\right)$$

which proves the theorem. Similarly we can prove that (4.6) holds.

Now we introduce the principal functions corresponding to the eigenvalues as follows:

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi\left(x,\lambda\right)\right\}_{\lambda=\lambda_i^+},\quad n=0,1,...,m_i^+-1,\ i=1,2,...,j,$$

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi\left(x,\lambda\right)\right\}_{\lambda=\lambda_l^-},\quad n=0,1,...,m_l^--1,\ l=1,2,...,k$$

are called the principal functions corresponding to the eigenvalues $\lambda=\lambda_i^+,\ i=1,2,...,j$ and $\lambda=\lambda_l^-,\ l=1,2,...,k$ of $L\left(\lambda\right)$, respectively.

Therefore we arrive at the following result for the principal functions given above:

Theorem 4.2. The principal functions corresponding to the eigenvalues of $L\left(\lambda\right)$ are in $L^2\left(\mathbf{R}_+,\mathbf{C}_2\right)$, i.e.:

$$\begin{aligned}&\left\{\tfrac{\partial^n}{\partial\lambda^n}\varphi\left(.,\lambda\right)\right\}_{\lambda=\lambda_i^+}\in L^2\left(\mathbf{R}_+,\mathbf{C}_2\right),\\&n=0,1,...,m_i^+-1,\ i=1,2,...,j,\end{aligned}\tag{4.9}$$

$$\begin{aligned}&\left\{\tfrac{\partial^n}{\partial\lambda^n}\varphi\left(.,\lambda\right)\right\}_{\lambda=\lambda_l^-}\in L^2\left(\mathbf{R}_+,\mathbf{C}_2\right),\\&n=0,1,...,m_l^--1,\ l=1,2,...,k.\end{aligned}\tag{4.10}$$

Proof. From (3.13) and (2.2) we obtain that

$$\left|\left\{\tfrac{\partial^n}{\partial\lambda^n}e_1^+\left(x,\lambda\right)\right\}_{\lambda=\lambda_i^+}\right|\leq c_1e^{-\varepsilon x},$$

$$\left|\left\{\tfrac{\partial^n}{\partial\lambda^n}e_2^+\left(x,\lambda\right)\right\}_{\lambda=\lambda_i^+}\right|\leq x^ne^{-x\,\mathrm{Im}\,\lambda_i^+}+c_2e^{-\varepsilon x}$$

for $n=0,1,...,m_i^+-1,\ \ i=1,2,...,j$ which gives (4.9) by using (4.5). Equation (4.10) may be derived, by using (4.6), analogously.

Definition 4.3.

Obviously we also have

$$\left\{\frac{\partial^n}{\partial\lambda^n}W[\varphi\left(x,\lambda\right),e^+\left(x,\lambda\right)\right\}_{\lambda=\lambda_i}=\left\{\frac{d^n}{d\lambda^n}a^+\left(\lambda\right)\right\}_{\lambda=\lambda_i}=0$$

for $n=0,1,...,m_i-1,\ i=1,2,...,p$ and

$$\left\{\tfrac{\partial^n}{\partial\lambda^n}W[\varphi(x,\lambda), e^-(x,\lambda)\right\}_{\lambda=\lambda_l} = \left\{\tfrac{d^n}{d\lambda^n}a^-(\lambda)\right\}_{\lambda=\lambda_l} = 0$$

for $n = 0,1,...,m_l - 1,\ l = p+1, p+2, ..., q$. Using the last two formulas given above, in a similar way to Theorem 4.1 we get that

Remark 4.3. The formulas

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i} = \sum_{v=0}^{n}\binom{n}{v}c_{n-v}(\lambda_i)\left\{\frac{\partial^v}{\partial\lambda^v}e^+(x,\lambda)\right\}_{\lambda=\lambda_i}, \tag{4.11}$$

for $n = 0,1,...,m_i - 1,\ i = 1,2,...,p$, and

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_l} = \sum_{v=0}^{n}\binom{n}{v}d_{n-v}(\lambda_l)\left\{\frac{\partial^v}{\partial\lambda^v}e^-(x,\lambda)\right\}_{\lambda=\lambda_l}, \tag{4.12}$$

for $n = 0,1,...,m_l - 1,\ l = p+1, p+2, ..., q$ hold, where the constants $c_0, c_1, ..., c_n$
and $d_0, d_1, ..., d_n$ depend on λ_i and λ_l, respectively.

Now we introduce the principal functions corresponding to the spectral singularities as follows:

$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_i}$, $\quad n = 0,1,...,m_i - 1,\ i = 1,2,...,p,$

$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(x,\lambda)\right\}_{\lambda=\lambda_l}$, $\quad n = 0,1,...,m_l - 1,\ l = p+1, p+2, ..., q$

are called the principal functions corresponding to the spectral singularities $\lambda = \lambda_i\ i = 1,2,...,p$ and $\lambda = \lambda_l\ l = p+1, p+2, ..., q$ of $L(\lambda)$, respectively. Therefore we arrive at the following

Lemma 4.4 The principal functions for the spectral singularities do not belong to the space $L^2(\mathbf{R}_+, \mathbf{C}_2)$, i.e: $\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(.,\lambda)\right\}_{\lambda=\lambda_i} \notin L^2(\mathbf{R}_+, \mathbf{C}_2)$, for $n = 0,1,...,m_i - 1,\ i = 1,2,...,p$, $\left\{\frac{\partial^n}{\partial\lambda^n}\varphi(.,\lambda)\right\}_{\lambda=\lambda_l} \notin L^2(\mathbf{R}_+, \mathbf{C}_2)$, for $n = 0,1,...,m_l - 1,\ l = p+1, p+2, ..., q$.

The proof of the lemma is obtained from (2.2), (2.3), (4.11) and (4.12).

Now let us introduce the following Hilbert spaces [2]

$$\begin{aligned} H(\mathbf{R}_+, \mathbf{C}_2, m) \ &:= \left\{ f(x) : f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \right. \\ &\left. \int_0^\infty (1+x)^{2m}\left\{|f_1(x)|^2 + |f_2(x)|^2\right\}dx < \infty \right\}, \end{aligned}$$

$m = 0, 1, \ldots$ with norm

$$\|f\|^2_{H(\mathbf{R}_+,\mathbf{C}_2,m)} = \int_0^\infty (1+x)^{2m} \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx$$

and

$$H(\mathbf{R}_+, \mathbf{C}_2, -m) \; : \; = \left\{ g(x) : g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \right.$$
$$\left. \int_0^\infty (1+x)^{-2m} \left\{ |g_1(x)|^2 + |g_2(x)|^2 \right\} dx < \infty \right\},$$

$m = 0, 1, \ldots$ with norm

$$\|g\|^2_{H(\mathbf{R}_+,\mathbf{C}_2,-m)} = \int_0^\infty (1+x)^{-2m} \left\{ |g_1(x)|^2 + |g_2(x)|^2 \right\} dx.$$

Clearly $H(\mathbf{R}_+, \mathbf{C}_2, 0) = L^2(\mathbf{R}_+, \mathbf{C}_2)$ and

$$\mathrm{H}(\mathbf{R}_+, \mathbf{C}_2, m)\, L^2(\mathbf{R}_+, \mathbf{C}_2,)\, H(\mathbf{R}_+, \mathbf{C}_2, -m).$$

Therefore we reach to the following

Theorem 4.5.

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_i} \in H(\mathbf{R}_+, \mathbf{C}_2, -(n+1)), \tag{4.13}$$

for $n = 0, 1, \ldots, m_i - 1,\; i = 1, 2, \ldots, p$, and

$$\left\{ \frac{\partial^n}{\partial \lambda^n} \varphi(., \lambda) \right\}_{\lambda=\lambda_l} \in H(\mathbf{R}_+, \mathbf{C}_2, -(n+1)) \tag{4.14}$$

for $n = 0, 1, \ldots, m_l - 1,\; l = p+1, p+2, \ldots, q$.

Proof. From (2.2), we obtain that

$$\left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_1^+(x, \lambda) \right\}_{\lambda=\lambda_i} \right| \leq \int_x^\infty t^n \, |H_{12}(x,t)| \, dt \tag{4.15}$$

and

$$\left| \left\{ \frac{\partial^n}{\partial \lambda^n} e_2^+ (x, \lambda) \right\}_{\lambda = \lambda_i} \right| \leq x^n + \int_x^\infty t^n \left| H_{22} (x, t) \right| dt. \tag{4.16}$$

for $n = 0, 1, ..., m_i - 1,\ i = 1, 2, ..., p$. By the definition of
$H (\mathbf{R}_+, \mathbf{C}_2, -(n+1))$
and using (4.15) and (4.16) we arrive at (4.13). In a similar way, we can show that (4.14) also holds.

Now let us choose n_0 so that

$$\mathrm{n}_0 = \max \{m_1, ..., m_p, m_{p+1}, ..., m_q\}.$$

Then

$$H (\mathbf{R}_+, \mathbf{C}_2, n_0) \underset{\neq}{\subset} L^2 (\mathbf{R}_+, \mathbf{C}_2,) \underset{\neq}{\subset} H (\mathbf{R}_+, \mathbf{C}_2, -n_0).$$

From Theorem 4.5, we finally reach to the following

Conclusion 4.6. The principal functions for the spectral singularities of the operator $L(\lambda)$ belong to the space $H (\mathbf{R}_+, \mathbf{C}_2, -n_0)$, i.e.:

$$\left\{ \tfrac{\partial^n}{\partial \lambda^n} \varphi (., \lambda) \right\}_{\lambda = \lambda_i} \in H (\mathbf{R}_+, \mathbf{C}_2, -n_0),$$

for $n = 0, 1, ..., m_i - 1,\ i = 1, 2, ..., p$ and

$$\left\{ \tfrac{\partial^n}{\partial \lambda^n} \varphi (., \lambda) \right\}_{\lambda = \lambda_l} \in H (\mathbf{R}_+, \mathbf{C}_2, -n_0)$$

for $n = 0, 1, ..., m_l - 1,\ l = p+1, p+2, ..., q$.

References

[1] Ö. Akın and E. Bairamov, On the structure of discrete spectrum of the non-selfadjoint system of differential equations in the first order, J. Korean Math. Soc. No 3, **32**, pp. 401-413, (1995).

[2] Yu. M. Berezanski, Expansion in Eigenfunctions of Selfadjoint Operators, Amer. Math. Soc., Providence R. I. (1968).

[3] E. P. Dolzhenko, Boundary Value Uniqueness Theorems for Analytic Functions, Math. Notes **25** No 6, pp. 437-442, (1979).

[4] N. B. Kerimov, A Boundary Value Problem for the Dirac System with a Spectral Parameter in the Boundary Conditions, Differential Equations, Vol. **38,** No 2, pp. 164-174, (2002).

[5] E. Kır, Spectral Properties of Non-Selfadjoint System of Differential Equations, Commun. Fac. Sci. Univ. Ank. Series A1, Vol.**49** (2000),111-116.

[6] V.E. Lyance, A differential Operator with Spectral Singularities, I,II, Amer. Math. Soc.Trans. Ser. 2, Vol. **60**, pp. 185-225, pp. 227-283, (1967).

[7] M.A.Naimark, Investigation of the Spectrum and the Expansion in Eigenfunctions of a Non-selfadjoint Operator of Second Order on a Semi-axis, Amer. Math. Soc. Trans. Ser. 2, Vol **16,** pp.103-193, Amer. Math. Soc., Providence, (1960).

[8] J.T.Schwartz, Some non-selfadjoint operators, Comm. Pure and Appl.Math. **13**, pp. 609-639, (1960).

Esra Kir
Faculty of Art and Science
06500, Teknikokullar
Ankara
Turkey
e-mail : esrakir@gazi.edu.tr

Gülen Başcanbaz-Tunca
Department of mathematics
Faculty of science
Ankara University
06100, Tandogan, ankara
Turkey
e-mail: tunca@science.ankara.edu.tr

and

Canan Yanik
Department of Mathematics Education
Hacettepe University
06532 Beytepe
Ankara
Turkey
e-mail : canany@hacettepe.edu.tr