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## GENERALIZATIONS OF THE ORLICZ - PETTIS THEOREM

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### **Abstract**

*The Orlicz-Pettis Theorem for locally convex spaces asserts that a series in the space which is subseries convergent in the weak topology is actually subseries convergent in the original topology of the space. A subseries convergent series can be viewed as a multiplier convergent series where the terms of the series are multiplied by elements of the scalar sequence space  $m_0$  of sequences with finite range. In this paper we show that the conclusion of the Orlicz-Pettis Theorem holds (and can be strengthened) if the multiplier space  $m_0$  is replaced by a sequence space with the signed weak gliding hump property.*

The classical Orlicz-Pettis (OP) Theorem asserts that a series in a normed linear space (NLS) which is subseries convergent in the weak topology of the space is actually subseries convergent in the norm topology ([O], [P]). The theorem has proven to be very useful and has many applications, in particular to topics in vector measures and vector integration ([DU]). This has motivated a large number of generalizations in many directions, in particular to series in locally convex spaces (LCS) (see [K1], [DU] for a discussion and description). In this note we observe that the OP Theorem can be viewed as a result about multiplier convergent series in both NLS and LCS and give generalizations of the theorem.

Let  $X$  be a topological vector space (TVS) and  $\lambda$  a sequence space of scalar-valued sequences. A (formal) series  $\sum x_k$  in  $X$  is said to be  $\lambda$  multiplier convergent if the series  $\sum t_k x_k$  converges in  $X$  for all  $\{t_k\} \in \lambda$  ([FP], [SS]). Multiplier convergent series where the multipliers come from some of the classical sequence spaces such as  $l^1$  and  $c_0$  have been considered by various authors ([B], [FP], [LCC], [WL]); in particular,  $c_0$  multiplier convergent series have been used to characterize Banach spaces which contain no copy of  $c_0$  ([Ds]). A series  $\sum x_k$  in  $X$  is subseries convergent if the subseries  $\sum x_{n_k}$  converges in  $X$  for every subsequence  $\{x_{n_k}\}$  of  $\{x_k\}$ . If  $m_0$  is the space of all scalar sequences which have finite range, then it is easy to see that a series  $\sum x_k$  is subseries convergent iff  $\sum x_k$  is  $m_0$  multiplier convergent in  $X$ . Thus, the OP Theorem can be restated to assert that a series  $\sum x_k$  in a NLS  $X$  which is  $m_0$  multiplier convergent in the weak topology of  $X$  is  $m_0$  multiplier convergent in the norm topology.

This suggests the following question: *Are there any other sequence spaces  $\lambda$  for which every  $\lambda$  multiplier convergent series with respect to the weak topology  $\sigma(X, X')$  is also  $\lambda$  multiplier convergent with respect to the norm topology?* In this paper we show that there are a large number of sequence spaces for which the answer to this question is yes. We actually consider the more general setting of LCS and give generalizations of the OP Theorem for these spaces.

Let  $\lambda$  be a vector space of scalar valued sequences which contains  $c_{00}$ , the space of all sequences which are eventually 0. An interval in  $\mathbf{N}$  is a set of the form  $[m, n] = \{k \in \mathbf{N} : m \leq k \leq n\}$ . If  $I$  is an interval in  $\mathbf{N}$ ,  $\chi_I$  denotes the characteristic function of  $I$ , and if  $t = \{t_k\} \in \lambda$ , then  $\chi_I t$  denotes the coordinatewise product of  $\chi_I$  and  $t$ . A sequence of intervals  $\{I_k\}$  is increasing if  $\max I_k < \min I_{k+1}$ , for every  $k$ . The space  $\lambda$  has the *signed weak gliding hump property* (signed WGHP) if  $t \in \lambda$  and  $\{I_k\}$  an increasing sequence of intervals implies there exists a subsequence  $\{I_{n_k}\}$  and sequence

of signs  $s_k = \pm 1$  such that the coordinatewise sum  $\sum_{k=1}^{\infty} s_k \chi_{I_{n_k}} t \in \lambda$  ([St1], [St2], [Sw2]). The space  $\lambda$  has the *weak gliding hump property* (WGHP) if the signs above can be chosen with  $s_k = 1$  for all  $k$  ([N], [Sw2]). For example, any monotone sequence space such as  $l^p$  ( $0 < p \leq \infty$ ),  $m_0$ , or  $c_0$  has WGHP. The space  $bs$  of bounded series ([Bo], [KG]) has signed WGHP but not WGHP ([Bo], [St1], [St2], [Sw2]). For a large class of spaces having WGHP and signed WGHP, see [BSS]. (For definitions of the sequence spaces employed see [Bo] or [KG].)

We now list some definitions and terminology which will be used in the sequel. Let  $(X, X')$  be a dual pair. The weak topology on  $X$  from the dual pairing will be denoted by  $\sigma(X, X')$ ; the Mackey topology, the topology of uniform convergence on absolutely convex  $\sigma(X', X)$  compact sets, will be denoted by  $\tau(X, X')$ ;  $\lambda(X, X')$  will denote the topology on  $X$  of uniform convergence on  $\sigma(X', X)$  compact sets and  $\gamma(X, X')$  will denote the topology on  $X$  of uniform convergence on unconditionally  $\sigma(X', X)$  sequentially compact sets (a set  $K \subset X'$  is unconditionally  $\sigma(X', X)$  sequentially compact if every sequence in  $K$  has a subsequence which is  $\sigma(X', X)$  Cauchy ([Di]). If  $X$  is a normed space, the Mackey topology is just the norm topology ([Sw1] 18.8). The topology  $\lambda(X, X')$  is obviously stronger than  $\tau(X, X')$  and can be strictly stronger ([Wi] 9.2.7). The topologies  $\lambda(X, X')$  and  $\gamma(X, X')$  are not comparable.

**Definition 1.** ([Wi] 6.1.9) *Let  $\sigma$  and  $\tau$  be vector topologies on the vector space  $X$ . We say that  $\tau$  is linked to  $\sigma$  if  $\tau$  has a neighborhood base at 0 consisting of  $\sigma$  closed sets.*

For example, any of the polar topologies defined above are linked to  $\sigma(X, X')$ .

We use the following elementary result several times below.

**Lemma 2.** ([Wi] 6.1.11) *Suppose  $\tau$  is linked to  $\sigma$ . If  $\{x_k\} \subset X$  is  $\sigma$  convergent to  $x$  and  $\{x_k\}$  is  $\tau$  Cauchy, then  $\{x_k\}$  is  $\tau$  convergent to  $x$ .*

Let  $\tau$  be a vector topology on the sequence space  $\lambda$ . For each  $k$  let  $e^k$  be the sequence with a 1 in the  $k^{th}$  coordinate and 0 in the other coordinates. The space  $(\lambda, \tau)$  is an AK space if for each  $t = \{t_k\} \in \lambda$  the series  $\sum_{k=1}^{\infty} t_k e^k$  converges to  $t$  with respect to  $\tau$ . For example, the classical sequence spaces

$l^p$  ( $0 < p < \infty$ ),  $c_0$ , and  $cs$ , the space of convergent series, are AK spaces under their natural topologies ([Bo], [KG]).

We now give a result which links the conclusion of the Orlicz-Pettis Theorem to the AK property of the multiplier space  $\lambda$ . A topology  $w(X, X')$  defined for dual pairs is said to be a Hellinger-Toeplitz topology if whenever a linear map  $T : X \rightarrow Y$  is  $\sigma(X, X') - \sigma(Y, Y')$  continuous with respect to the dual pairs  $(X, X')$  and  $(Y, Y')$ , the map is also  $w(X, X') - w(Y, Y')$  continuous ([Wi] 11.1.5; note that a Hellinger-Toeplitz topology has to be defined for all dual pairs). For example, the polar topologies  $\tau(X, X')$ ,  $\lambda(X, X')$ , and  $\gamma(X, X')$  as well as the strong topology  $\beta(X, X')$ , the topology of uniform convergence on  $\sigma(X', X)$  bounded sets ([Sw1] 17.5), are all Hellinger-Toeplitz topologies.

Recall that the  $\beta$ -dual of  $\lambda$  is

$$\lambda^\beta = \left\{ s = \{s_k\} : \sum s_k t_k \text{ converges for every } t = \{t_k\} \in \lambda \right\}.$$

If  $s \in \lambda^\beta$  and  $t \in \lambda$ , we write  $s \cdot t = \sum_{k=1}^\infty s_k t_k$  and note that if  $\lambda$  contains  $c_{00}$ , the space of sequences which are eventually 0, then  $(\lambda, \lambda^\beta)$  form a dual pair under the bilinear map  $s \cdot t$ .

Let  $w$  be a Hellinger-Toeplitz topology on dual pairs.

**Theorem 3.** *The following are equivalent:*

(1) for every dual pair  $(X, X')$  a series  $\sum x_k$  in  $X$  which is  $\lambda$  multiplier convergent with respect to the weak topology  $\sigma(X, X')$  is  $\lambda$  multiplier convergent with respect to  $w(X, X')$ .

(2)  $(\lambda, w(\lambda, \lambda^\beta))$  is an AK-space.

**Proof:** Assume (1). Then  $\sum e^k$  is  $\lambda$  multiplier convergent with respect to  $\sigma(\lambda, \lambda^\beta)$  so by (1)  $\sum e^k$  is  $\lambda$  multiplier convergent with respect to  $w(\lambda, \lambda^\beta)$ . But this means that  $t = \sum_{k=1}^\infty t_k e^k$  where the series is  $w(\lambda, \lambda^\beta)$  convergent, so (2) holds.

Assume (2). Let  $\sum x_k$  be  $\lambda$  multiplier convergent with respect to  $\sigma(X, X')$ . Define a linear map  $T : \lambda \rightarrow X$  by  $Tt = \sum_{k=1}^\infty t_k x_k$  ( $\sigma(X, X')$  sum). Let  $x' \in X'$ . Then  $\langle x', Tt \rangle = \sum_{k=1}^\infty t_k \langle x', x_k \rangle$  so  $\{\langle x', x_k \rangle\} \in \lambda^\beta$  and  $\langle x', Tt \rangle = \{\langle x', x_k \rangle\} \cdot t$ . This implies that  $T$  is  $\sigma(\lambda, \lambda^\beta) - \sigma(X, X')$  continuous and, therefore,  $w(\lambda, \lambda^\beta) - w(X, X')$  continuous. If  $t \in \lambda$ , then  $t =$

$w(\lambda, \lambda^\beta) - \lim_n \sum_{k=1}^n t_k e^k$  so  $Tt = w(X, X') - \lim_n \sum_{k=1}^n t_k x_k = \sum_{k=1}^\infty t_k x_k$  and (1) holds.

The implication (2)  $\Rightarrow$  (1) was used in [SS] 2.3.

Condition (1) of Theorem 3 is, of course, just the conclusion of an Orlicz-Pettis type theorem for the Hellinger-Toeplitz topology  $w(X, X')$ . The usual conclusion of the Orlicz-Pettis theorem for locally convex spaces involves the Mackey topology and this includes the normed linear case since the Mackey topology and the norm topology coincide ([Mc]). Bennett and Kalton have given a generalization of the locally convex version of the Orlicz-Pettis theorem by employing the polar topology  $\lambda(X, X')$  defined above ([BK], see also [D]). We now use Theorem 3 to give a generalization of the Bennett-Kalton result for  $\lambda$  multiplier convergent series when  $\lambda$  has the signed WGHP. We also consider the polar topology  $\gamma(X, X')$  defined above.

The proof of the following theorem uses the notion of a signed K-matrix. See [Sw2] 2.2 for the definition and many related results.

**Theorem 4.** *Assume that  $\lambda$  is a sequence space which contains  $c_{00}$  and has the signed WGHP. Then*

- (a)  $(\lambda, \gamma(\lambda, \lambda^\beta))$  has AK.
- (b)  $(\lambda, \lambda(\lambda, \lambda^\beta))$  has AK.

**Proof:** (a) It suffices to show that the series  $\sum t_k e^k$  is  $\gamma(\lambda, \lambda^\beta)$  Cauchy for all  $t = \{t_k\} \in \lambda$  since  $\gamma(\lambda, \lambda^\beta)$  is linked to  $\sigma(\lambda, \lambda^\beta)$  (Lemma 2). Suppose there exists  $t \in \lambda$  such that  $\sum t_k e^k$  is not  $\gamma(\lambda, \lambda^\beta)$  Cauchy. Then there exists  $\varepsilon > 0$ ,  $K \subset \lambda^\beta$  which is  $\sigma(\lambda^\beta, \lambda)$  unconditionally sequentially compact, and increasing intervals  $\{I_k\}$  such that  $\sup_{u \in K} |u \cdot \sum_{j \in I_k} t_j e^j| > \varepsilon$  for all  $k$ . For each  $k$  pick  $u_k \in K$  such that

$$(\clubsuit) \quad \left| u_k \cdot \sum_{j \in I_k} t_j e^j \right| > \varepsilon.$$

There exists  $n_k$  increasing such that  $\{u_{n_k}\}$  is  $\sigma(\lambda^\beta, \lambda)$  Cauchy. Set

$$M = \left[ u_{n_i} \cdot \sum_{l \in I_{n_j}} t_l e^l \right] = [m_{ij}].$$

We claim that  $M$  is a signed K-matrix (see [Sw2] 2.2). First, the columns of  $M$  converge since  $\{u_{n_i}\}$  is  $\sigma(\lambda^\beta, \lambda)$  Cauchy. Given  $p_j$  increasing there exists a subsequence  $\{q_j\}$  of  $\{p_j\}$  and signs  $s_j = \pm 1$  such that  $\tilde{t} = \sum_{j=1}^{\infty} s_j \sum_{l \in I_{n_{q_j}}} t_l e^l \in \lambda$ . Then  $\sum_{j=1}^{\infty} s_j m_{iq_j} = u_{n_i} \cdot \tilde{t}$  and  $\lim_i \sum_{j=1}^{\infty} s_j m_{iq_j}$  exists. Therefore,  $M$  is a signed K-matrix and the diagonal of  $M$  converges to 0 by Theorem 2.2.2 of [Sw2]. But this contradicts ( $\clubsuit$ ).

(b) Consider  $\lambda$  with the Mackey topology so the dual of  $\lambda$  is  $\lambda^\beta$ . We claim that  $(\lambda, \tau(\lambda, \lambda^\beta))$  is  $\tau(\lambda, \lambda^\beta)$  separable. This follows since  $(\lambda, \sigma(\lambda, \lambda^\beta))$  is an AK space so that the  $\sigma(\lambda, \lambda^\beta)$  closure of  $S = \text{span}\{e^k : k \in \mathbf{N}\}$  is  $\sigma(\lambda, \lambda^\beta)$  dense in  $\lambda$ . But  $S$  has the same closure in  $\sigma(\lambda, \lambda^\beta)$  and  $\tau(\lambda, \lambda^\beta)$  so  $S$  is  $\tau(\lambda, \lambda^\beta)$  dense in  $\lambda$  and  $(\lambda, \tau(\lambda, \lambda^\beta))$  is separable. This implies that  $\sigma(\lambda, \lambda^\beta)$  compact sets are sequentially compact ([Wi] 9.5.3). Now the proof of part (a) may be repeated using a  $\sigma(\lambda^\beta, \lambda)$  compact (sequentially compact) set  $K$ .

Part (b) was established in [St1] by other means; see also [Bo] 11.2.11.

From Theorems 3 and 4 we obtain the following Orlicz-Pettis theorems for  $\lambda$  multiplier convergent series.

**Corollary 5.** *Assume  $\lambda$  has the signed WGHP. Then condition (1) holds for the polar topologies  $\gamma(X, X')$ ,  $\lambda(X, X')$ , and  $\tau(X, X')$ .*

Since  $m_0$  has WGHP and  $m_0$  multiplier convergence is equivalent to subseries convergence, Corollary 5 gives a generalization of the usual versions of the Orlicz-Pettis Theorem for locally convex spaces ([Mc]). The version of the Orlicz-Pettis theorem for the topology  $\lambda(X, X')$  was given by Bennett and Kalton ([BK]); the version for  $\gamma(X, X')$  is contained in [D]. We give an example of a situation covered by Corollary 5 but not the classical version of the Orlicz-Pettis theorem.

**Example 6.** Let  $cs$  be the space of all convergent series and  $bv_0$  the space of all null sequences of bounded variation ([Bo], [KG]). Consider the series  $\sum \frac{1}{k}e^k$  in  $cs$ . This series is obviously not subseries convergent in  $cs$ . However, since  $bv_0$  is the  $\beta$ -dual of  $bs$ , if  $t = \{t_k\} \in bs$  then the sequence  $\left\{\frac{t_k}{k}\right\} \in cs$ , because  $\left\{\frac{1}{k}\right\} \in bv_0$ . Therefore,  $\sum \frac{t_k}{k}e^k$  converges in  $cs$ . That is,  $\sum \frac{1}{k}e^k$  is  $bs$  multiplier convergent in  $cs$  but not subseries convergent in  $cs$ .

Without some assumption on the multiplier space  $\lambda$  in Corollary 5 the conclusion of the result may fail.

**Example 7.** Let  $\mu = c_{00} \oplus \text{span}(1, 1, 1, \dots)$ , the space of all scalar sequences which are eventually constant. Then the series  $\sum x_k$  is  $\mu$  multiplier convergent (in any topology) iff the series is convergent. The series  $\sum (e^{k+1} - e^k)$  is  $\sigma(c_0, l^1)$  convergent in  $c_0$  (to  $-e^1$ ) and, therefore,  $\mu$  multiplier convergent with respect to  $\sigma(c_0, l^1)$ , but is not  $\mu$  multiplier convergent with respect to the norm topology of  $c_0$ .

It should be noted that the conclusion in Corollary 5 cannot be strengthened to assert that a series which is  $\lambda$  multiplier convergent with respect to  $\sigma(X, X')$  is also  $\lambda$  multiplier convergent with respect to the strong topology  $\beta(X, X')$ .

**Example 8.** The series  $\sum e^k$  is  $\sigma(l^\infty, l^1)$  subseries convergent in  $l^\infty$  but is not subseries convergent with respect to  $\beta(l^\infty, l^1) = \|\cdot\|_\infty$ .

It is the case, however, that if stronger conditions are imposed on the multiplier space  $\lambda$ , then a series which is  $\lambda$  multiplier convergent with respect to  $\sigma(X, X')$  is also  $\lambda$  multiplier convergent with respect to  $\beta(X, X')$ . Such a condition, called the infinite gliding hump property, was defined in [Sw3] and an Orlicz-Pettis result with this conclusion was established. It follows from Theorem 3 that such spaces are AK spaces under the strong topology. See also [LCC] and [WL] for the case when  $\lambda$  is either  $c_0$  or  $l^p$  ( $0 < p < \infty$ ).

We next consider multiplier Orlicz-Pettis theorems for series of continuous linear operators which are multiplier convergent in the strong operator topology. We seek stronger topologies for which series that are convergent with respect to the strong operator topology are multiplier convergent in a stronger topology. There seem to be few results in this direction even for subseries convergent series, except for the case of compact operators (see Kalton's theorem in [K2], [Sw2] 10.5.6).

Let  $E$  and  $F$  be Hausdorff locally convex spaces and  $L(E, F)$  the space of continuous linear operators from  $E$  into  $F$ . Let  $L_s(E, F)$  be  $L(E, F)$  with the strong operator topology, i.e., the topology of pointwise convergence on  $E$ . Let

$$\mathcal{B} = \{B \subset E : \text{if } \{x_k\} \subset B, \text{ then } \lim Tx_k \text{ exists for every } T \in L(E, F)\}$$

and let  $L_{\mathcal{B}}(E, F)$  be  $L(E, F)$  with the topology of uniform convergence on elements of  $\mathcal{B}$ . From the definition of the semi-norms defining the topologies, it is clear that the topology of  $L_{\mathcal{B}}(E, F)$  is linked to the topology of  $L_s(E, F)$  ([GDS], III.II.1).

**Theorem 9.** *Assume  $\lambda$  has signed WGHP. If  $\sum T_k$  is  $\lambda$  multiplier convergent in  $L_s(E, F)$  then  $\sum T_k$  is  $\lambda$  multiplier convergent in  $L_{\mathcal{B}}(E, F)$ .*

**Proof:** Suppose  $\sum T_k$  is  $\lambda$  multiplier convergent in  $L_s(E, F)$ . By Lemma 2 it suffices to show that  $\sum t_k T_k$  is Cauchy in  $L_{\mathcal{B}}(E, F)$  for every  $t \in \lambda$ . Suppose this is not the case. Then there exists  $\varepsilon > 0$ ,  $B \in \mathcal{B}$  a continuous semi-norm  $p$  on  $F$  and an increasing sequence of intervals  $\{I_k\}$  such that

$$\sup \left\{ p \left( \sum_{j \in I_k} t_j T_j x \right) : x \in B \right\} > \varepsilon.$$

For each  $k$  pick  $x_k \in B$  such that

$$(\clubsuit) \quad p \left( \sum_{j \in I_k} t_j T_j x_k \right) > \varepsilon.$$

Put  $M = [\sum_{l \in I_j} t_l T_l x_i] = [m_{ij}]$ . As in Theorem 4 we show that  $M$  is a signed  $K$ -matrix so the diagonal of  $M$  should converge to 0 ([Sw2], 2.2.2) contradicting  $(\clubsuit)$ . First, the columns of  $M$  converge by the definition of  $\mathcal{B}$ . Next, given an increasing sequence  $\{p_j\}$  there is a subsequence  $\{q_j\}$  of  $\{p_j\}$  and signs  $s_j = \pm 1$  such that  $\tilde{t} = \sum_{j=1}^{\infty} s_j \sum_{l \in I_{q_j}} t_l e^l \in \lambda$ . Then  $\sum_{j=1}^{\infty} s_j m_{iq_j} = \sum_{k=1}^{\infty} \tilde{t}_k T_k(x_i)$ , where  $\sum_{k=1}^{\infty} \tilde{t}_k T_k$  is the sum in  $L_s(E, F)$ . Hence,  $\lim_i \sum_{j=1}^{\infty} s_j m_{iq_j}$  exists by the definition of  $\mathcal{B}$  and  $M$  is a signed  $K$ -matrix.

In [W], Wang obtains a stronger conclusion than Theorem 9 using stronger conditions on  $\lambda$ . See also [Sw3].

We now consider two of the most common topologies on  $L(E, F)$ . Let  $\xi = \{\{x_k\} : x_k \rightarrow 0 \text{ in } E\}$  and, in the notation of [GDS], let  $L_{\rightarrow 0}(E, F)$  be



$L(E, F)$  with the topology of uniform convergence on the elements of  $\xi$ . Let  $L_{pc}(E, F)$  be  $L(E, F)$  with the topology of uniform convergence on precompact subsets of  $E$  ([GDS], III.II.2).

**Corollary 10.** *Assume that  $\lambda$  has signed WGHP. If  $\sum T_k$  is  $\lambda$  multiplier convergent in  $L_s(E, F)$ , then  $\sum T_k$  is  $\lambda$  multiplier convergent in  $L_{\rightarrow 0}(E, F)$ .*

In [GDS] III.II.19(b), conditions on the space  $E$  are given which guarantee that the spaces  $L_{\rightarrow 0}(E, F)$  and  $L_{pc}(E, F)$  coincide. Using this result and Corollary 10, we have.

**Corollary 11.** *Assume that  $\lambda$  has signed WGHP and that  $E$  is either metrizable or the hyper-strict inductive limit of such spaces. If  $\sum T_k$  is  $\lambda$  multiplier convergent in  $L_s(E, F)$ , then  $\sum T_k$  is  $\lambda$  multiplier convergent in  $L_{pc}(E, F)$ .*

Again without some condition on the multiplier space  $\lambda$ , the conclusions of the results above may fail.

**Example 12.** *Let  $\mu$  be as in Example 7. Let  $E = (c_0, \sigma(c_0, l^1))$  and  $F = (c_0, \|\cdot\|_\infty)$ . Define  $S_k : E \rightarrow F$  by  $S_k x = \langle e^k, x \rangle e^k$ . Then  $S_k \rightarrow 0$  in  $L_s(E, F)$  but  $S_k \not\rightarrow 0$  in  $L_{\rightarrow 0}(E, F)$  [ $e^k \rightarrow 0$  in  $E$  but  $S_k e^k = e^k \not\rightarrow 0$  in  $F$ ]. If  $S_0 = 0$ , the series  $\sum_{k=0}^\infty (S_{k+1} - S_k)$  is  $\mu$  multiplier convergent in  $L_s(E, F)$  but not in  $L_{\rightarrow 0}(E, F)$ .*

**Problem:** It would be interesting to know if the topologies above can be replaced by stronger topologies.

For series of continuous linear operators there is also a vector-valued version of multipliers. Let  $\Lambda$  be a vector space of  $E$  valued sequences. If  $\tau$  is any topology on  $F$ , then a series  $\sum T_k$  in  $L(E, F)$  is said to be  $\Lambda$  multiplier convergent with respect to  $\tau$  if the series  $\sum T_k x_k$  is  $\tau$  convergent for every  $x = \{x_k\} \in \Lambda$  (Thorp refers to this as  $\Lambda$  evaluation convergence [T]). For example, if  $\Lambda = l^\infty(E)$  is the space of all  $E$  valued bounded sequences, then  $\Lambda$  multiplier convergence is called bounded multiplier convergence and this notion has applications to vector valued measures (see [Ba], [T]).

The definition of the signed WGHP is easily extended to vector sequence spaces and we have.

**Theorem 13.** Assume that  $\Lambda$  has signed WGHP. If  $\sum T_k$  is  $\Lambda$  multiplier convergent with respect to  $(F, \sigma(F, F'))$ , then  $\sum T_k$  is  $\Lambda$  multiplier convergent with respect to  $(F, \gamma(F, F'))$ .

**Proof:** Let  $\{x_j\} \in \Lambda$  and set  $y_k = \sum_{j=1}^k T_j x_j$ . By Lemma 2 it suffices to show that  $\{y_k\}$  is  $\gamma(F, F')$  Cauchy. If this is not the case, we may proceed as in the proof of Theorem 4 to obtain a contradiction.

Again without some condition on the multiplier space  $\Lambda$  the conclusion of Theorem 13 may fail.

**Example 14.** Let  $E = l^\infty$ ,  $F = c_0$  and let  $\Lambda$  be all  $E$  valued sequences which are eventually constant. Define  $S_k : l^\infty \rightarrow c_0$  by  $S_k x = \langle e^k, x \rangle e^k$ . Set  $S_0 = 0$ ,  $e^0 = 0$ , and  $T_k = S_{k+1} - S_k$ . Then for every  $x = \{x_k\} \in \Lambda$ ,  $\sum_{k=0}^\infty T_k x_k$  is  $\sigma(c_0, l^1)$  convergent but if  $e = \{1, 1, 1, \dots\}$ , then  $\sum_{k=0}^\infty T_k e = \sum_{k=1}^\infty (e^{k+1} - e^k)$  is not  $\|\cdot\|_\infty$  convergent in  $c_0$ .

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