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REVERSIBILITY FOR SEMIGROUP ACTIONS *

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Abstract

Let Q be a topological space and S a semigroup of local homeomorphisms of Q. The purpose of this paper is to generalize the notion of reversibility and to introduce the reversible sets. And furthermore, it is established a relation between these sets and the control sets for S and it is studied reversibility of semigroup actions on fiber bundles.

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1. Introduction

The purpose of this article is to study reversibility properties of actions of semigroups of local homeomorphisms on topological spaces and fiber bundles.

Recall that an abstract semigroup S is said to be left reversible if $aS \cap bS \neq \emptyset$ and right reversible if $Sa \cap Sb \neq \emptyset$ for every $a, b \in S$, and it is reversible if it is both left and right reversible (see e.g. Hilgert-Neeb [5] and Ruppert [8]). In this paper we consider a semigroup S, whose elements are local homeomorphisms of a topological space Q. The corresponding concept says that the action of S on Q is reversible (or S is Q-reversible, for short) if for all $x, y \in Q$ the orbits Sx and Sy have a common point.

The concept of reversibility for abstract semigroups was originally introduced in relation with the problem of embedding semigroups into groups. This concept was fully studied by Ruppert [8] for open semigroups in Lie groups and related to the connectivity properties of the semigroup, a relation which was further developed in Rocio-San Martin [7]. Reversibility properties of semigroups appear also in different situations like e.g. in the study of homotopy of semigroups in semi-simple Lie groups in San Martin-Santana [11] or, in a less explicitly way, in the study of harmonic functions on noncompact symmetric spaces in Furstenberg [4]. In view of such potential applications of the notion of reversibility, it is natural to study the generalization proposed in this paper.

We describe now the contents of the paper. In Section 2 we discuss the set up, while in Section 3 we introduce and discuss the concept of reversibility of semigroup actions. In Section 4 the we introduce the reversible sets that are, roughly speaking, the maximal subsets of the topological space in which the reversibility of the semigroup occurs. In the section 5, we relate the concept of the reversibility with that of control set for semigroup actions. Finally, in the last section we study reversibility of semigroup actions in fiber bundles in terms of reversibility on the base space and on the fiber.

2. Set up

In order to generalize the concept of reversibility to semigroups actions on topological spaces we need of a brief summary of semigroups of local homeomorphisms. We refer to San Martin [10] for more details.

Let Q be a topological space and denote by $\mathcal{C}_l(Q)$ the set of continuous maps $g : dom\,(g) \to Q$, such that the domain $dom\,(g)$ is a non-empty open subset of Q. If $g, h \in \mathcal{C}_l(Q)$ and $h^{-1}(dom(g)) \neq \emptyset$ we have a well defined map $gh : h^{-1}(dom(g)) \to Q$ which is still an element of $\mathcal{C}_l(Q)$, thus defining – partially – a product in this set. Of course, by restricting properly the domains of the maps the product is associative. In this enlarged sense we say that $\mathcal{C}_l(Q)$ is a local semigroup. More generally, a subset $S \subset \mathcal{C}_l(Q)$ is said to be local semigroup if it is closed under the multiplication in $\mathcal{C}_l(Q)$, that is, if the composition of two elements of S are still in S.

Following the commonly used terminology a local semigroup will be said to be a local monoid (or simply a monoid) if it contains the identity map $1 = 1_Q$ of Q. Any local semigroup can be turned into a monoid by adjoining the identity map.

Denote by $\mathcal{H}_l(Q) \subset \mathcal{C}_l(Q)$ the set of local homeomorphisms of Q. By definition $g : domg \to img$ belongs to $\mathcal{H}_l(Q)$ if and only if both *domg* and *img* are open sets and g is a homeomorphism between them. Of course the inverse g^{-1} of $g \in \mathcal{H}_l(Q)$ also belongs to $\mathcal{H}_l(Q)$ and maps *img* onto *domg*. Most of the theory of semigroup actions will be accomplished with the assumption that S is contained in $\mathcal{H}_l(Q)$. In this case we say that the semigroup is invertible and put

$$S^{-1} = \{g^{-1} : g \in S\}$$

for the corresponding inverse semigroup.

Throughout the paper assume that

$$\bigcup_{\phi \in S} dom(\phi) = Q.$$

If $x \in Q$, we denote its orbit for the S action by

$$Sx = \{\phi(x) \text{ such that } \phi \in S\}.$$

Using the standard notation of control theory we say that a local semigroup S is accessible if $int(Sx) \neq \emptyset$ for every $x \in Q$ and we say that S is transitive on Q if $S(x) = Q$ for all $x \in Q$. From now on, and in the whole paper we assume that S is a semigroup satisfying the accessibility property.

Denote by $G(S)$, or simply G, the subgroup of $\mathcal{H}_l(Q)$ generated by S. Recall that a semigroup S of $\mathcal{H}_l(Q)$ is transitive on Q if $S(q) = Q$ for all $q \in Q$.

3. Reversibility

We begin this section by defining our generalization of reversibility.

Definition 1. *Let S be a subsemigroup of $\mathcal{H}_l(Q)$ and $A \subset Q$. We say that S is reversible on $A \subset Q$, or simply A-reversible if $S(x) \cap S(y) \neq \emptyset$ for every $x, y \in A$.*

It is clear that if S is reversible on Q, then $G(S)$ is transitive on Q. The converse it is not true as can be seen by example below

Example 1. *Take $S = Sl(n, \mathbf{R}^+)$, note that S is not reversible on $\mathbf{R}^n \backslash \{0\}$, although $G(S)$ is transitive on $\mathbf{R}^n \backslash \{0\}$.*

Proposition 1. *If S is reversible on Q then $SS^{-1}(q) \subset S^{-1}S(q)$ for all $q \in Q$. Furthermore, if $G(S)$ is transitive on Q then $SS^{-1}(q) \subset S^{-1}S(q)$ for all $q \in Q$ implies that S is Q-reversible.*

Proof: Suppose that S is reversible, take $q \in Q$ and $\phi, \psi \in S$. It follows that there exists $y \in Q$ such that

$$y \in S(\psi\phi^{-1}(q)) \cap S(q).$$

Let $\gamma, \eta \in S$ be local homeomorphisms such that $y = \gamma(\psi\phi^{-1}(q)) = \eta(q)$. Thus

$$\psi\phi^{-1}(q) = \gamma^{-1}(y) = \gamma^{-1}(\eta(q)).$$

Now, assuming that $G(S)$ is transitive take $x, y \in Q$ and suppose that $SS^{-1}(q) \subset S^{-1}S(q)$, for all $q \in Q$. Hence it is not difficult to see that $G(S)(q) = S^{-1}S(q)$, for all $q \in Q$. Then, by transitivity of $G(S)$ on Q there exist $\psi, \phi \in S$ such that $y = \phi^{-1}\psi(x)$. With this, $\psi(x) = \phi(y)$ showing that $S(x) \cap S(y) \neq \emptyset$. $\square$

The example 1 shows that the hypothesis of the converse of the above proposition is essential. The next proposition shows that the reversibility of the Definition 1 in fact generalizes the classical concept of reversibility.

Proposition 2. *If a semigroup S is right [left] reversible and if $G(S)$ is transitive on Q then S [S^{-1}] is reversible on Q.*

Proof: If S is right reversible then $SS^{-1} \subset S^{-1}S$. Therefore $SS^{-1}(q) \subset S^{-1}S(q)$ for all $q \in Q$. Since $G(S)$ is transitive, by proposition 1 it follows that S is reversible on Q. On the other hand, if S is left reversible then S^{-1} is right reversible, therefore this case is reduced to last one. □

The next example shows that the converse of this proposition is not true.

Example 2. *Take $S = Sl^+(n, \mathbf{R})$, the semigroup of matrices in $Sl(n, \mathbf{R})$ whose entries are non-negative real numbers. Since $Sl(n, \mathbf{R})$ is a connected semi-simple Lie group with finite center we know from Theorem 6.7 in [12] that S is not reversible. Now, let Q be the positive orthant in $\mathbf{R}^n$. Take the canonical action of S on Q given by $x \in Q \to \phi(x)$, with $\phi \in S$. Then S is reversible on Q. In fact, first take $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in Q with x_i and y_i strictly positive. Then, by putting $g = \lambda diag(y_1/x_1, \ldots, y_n/x_n)$ with $\lambda = \sqrt[n]{\frac{x_1 \cdots x_n}{y_1 \cdots y_n}}$ it follows that $g \in S$ and $gx = \lambda y, \lambda > 0$. If $\lambda = 1$ is clear that $Sx \cap Sy \neq \emptyset$. If $\lambda \neq 1$, by taking λ^{-1} if necessary, we can assume that $\lambda > 1$ In this case take $h = (a_{ij}) \in S$ given by $a_{11} = \frac{1}{\lambda^{n-1}}$, $a_{12} = (\lambda - \frac{1}{\lambda^{n-1}})\frac{y_1}{y_2}$, $a_{ii} = \lambda$ with $i = 2, 3, \ldots n$ and $a_{ij} = 0$ for the others i, j. Then $hy = \lambda y$. Therefore, as $gx = \lambda y$ we have $gx = hy$ and hence $Sx \cap Sy \neq \emptyset$. If $\lambda < 1$, a similar arguments results that for all $x, y \in intQ$. Since for all $x \in Q$ there exists $g \in S$ such that $gx \in intQ$ it follows that $Sx \cap Sy \neq \emptyset$ for all $x, y \in Q$.*

Note that S is not reversible in $\mathbf{R}^n \backslash \{0\}$. In fact, as the positive, Q, and negative orthant, Q^-, are invariant by S we have $Sx \cap Sy \neq \emptyset$ for all $x \in Q$ and $y \in Q^-$.

About transitivity we have from the above definitions that S is reversible on Q if it is transitive on Q. But the converse is not true. For example, let S be the semigroup $\mathbf{R}^+$ acting on $\mathbf{R}$ by translations. The semigroup S is reversible on $\mathbf{R}$ but it is not transitive.

We dedicate the remainder of this section to prove that under reversibility it is possible to translate compact sets into an orbit of the semigroup. An special case of this result is one of the steps used in [11] to study homotopy of semigroups.

Lemma 1. *The semigroup S is reversible on Q if and only if for every finite subset $\{x_1, \ldots, x_k\} \subset Q$ one of the following conditions holds*

1. *$S(x_1) \cap \cdots \cap S(x_k) \neq \emptyset$.*
2. *There exists $x \in Q$ such that $x_i \in S^{-1}(x)$ for $i = 1, \ldots, k$.*

Proof: It is not difficult to see that (1) implies the reversibility. Now suppose that S is reversible and also suppose, by induction , that $S(x_1) \cap \cdots \cap S(x_{k-1}) \neq \emptyset$ for $k \geq 3$. Take $x \in S(x_1) \cap \cdots \cap S(x_{k-1})$. It is clear that $S(x) \subset S(x_1) \cap \cdots \cap S(x_{k-1})$. Since S is reversible there exists $y \in S(x) \cap S(x_k)$. With this $S(y) \subset S(x) \cap S(x_k) \subset S(x_1) \cap \cdots \cap S(q_k)$. In order to show the equivalence with (2) take x in the intersection $S(x_1) \cap \cdots \cap S(x_k)$. Then $x_i \in S^{-1}(x)$ for $i = 1, \ldots, k$. On the other hand, if $x_i \in S^{-1}(x)$ for $i = 1, \ldots, k$ then $x \in (S^{-1})^{-1}(x_i)$ and thus $S(x_1) \cap \cdots \cap S(x_k) \neq \emptyset$. □

The above lemma implies the following result.

Corollary 1. *The semigroup S^{-1} is reversible on Q if and only if for every finite subset $\{x_1, \ldots, x_k\} \subset Q$ there exists $x \in Q$ such that $x_i \in S(x)$ for all $i = 1, \ldots, k$.*

It was proved in [11] that if a semigroup S is reversible then it is possible to translate compact sets of groups inside semigroups. The next proposition generalizes this result to semigroups of local homeomorphisms.

Proposition 3. *Suppose that S^{-1} is reversible and accessible on Q. Take the compact subset $K \subset Q$. Then there exists $x \in Q$ such that $K \subset S(x)$.*

Proof: As S^{-1} is reversible we have $G(y) = SS^{-1}(y)$ for all $y \in Q$. Consequently, by transitivity of G we have

$$K \subset \bigcup_{\phi \in S} S(\phi^{-1}(y)) \text{ for all } y \in Q.$$

Since S is accessible and K is compact there exist $\phi_1, \ldots \phi_k \in S$ such that

$$K \subset \bigcup_{i=1}^{k} S(\phi_i^{-1}(y)).$$

Now by Lemma 1 there exists $x \in Q$ such that $\phi_i^{-1}(y) \in S(x)$ for $i = 1, \ldots, k$. With this $S(\phi_i^{-1}(y)) \subset S(x)$ for $i = 1, \ldots, k$ and hence $K \subset S(x)$. □

4. Reversible set

In this section we look at the regions of the space Q where reversibility occurs. The notion of restricted reversibility is formalized in the next definition. We assume here that S is a semigroup of $\mathcal{H}_l\,(Q)$ and $G(S)$ transitive on Q.

Definition 2. *We say that a non-empty subset $A \subset Q$ is S-reversible on A if $S(x) \cap S(y) \neq \emptyset$ for all $x, y \in A$.*

Note that the existence of subsets where occurs this reversibility is obvious.

Note that if $A \subset Q$ is S- reversible on A and $S(x) \cap A \neq \emptyset$ then $A \cup \{x\}$ is also S-reversible on $A \cup \{x\}$. In fact, take $y \in A$ and $\phi \in S$ such that $\phi(x) \in A$. Then $S(\phi(x)) \cap S(y) \neq \emptyset$, so that $S(x) \cap S(y) \neq \emptyset$.

The next result guarantees the existence of a kind of maximal reversible sets, defined as follow. A subset $R \subset Q$ is a maximal S-reversible on R if it is S-reversible on R and if $x \notin R$ then $R \cup \{x\}$ it is not a S-reversible on $R \cup \{x\}$. By the above remark this is equivalent to $S(x) \cap R = \emptyset$.

Proposition 4. *If $A \subset Q$ is a S-reversible on A then there exists a maximal S-reversible containing A.*

Proof: Consider $\mathcal{F}$ the family of all subsets of Q that are S-reversible on Q and that contain A. If $T \subset \mathcal{F}$ is a chain then $\mathcal{U} = \cup_{C \in T} C$ is an upper bound of T in $\mathcal{F}$. Clearly, S is $\mathcal{U}$-reversible. Therefore $\mathcal{F}$ has at least one maximal element. □

Definition 3. *A non empty subset $R \subset Q$ is called reversible set for the S-action if:*

1. *R is S-reversible on R,*
2. *R is maximal with this property.*

The next example shows that the intersection of reversible sets can be non-empty.

Example 3. *Consider the natural action of the semigroup $S = SL^{+}(2, \mathbf{R})$ on $\mathbf{R}^2$ and denote by Q_i the ith quadrant. The reversible sets for this action are $R_1 = \{(x, y) : 0 \leq y\} \cup \{(x, y) : 0 \leq x\}$ and $R_2 = \{(x, y) : y \leq 0\} \cup \{(x, y) : x \leq 0\}$. In fact, in case of R_1, as $S(x) \cap S(y) \neq \emptyset$ for all $x, y \in Q_1$ (see Example 2) it is enough to prove that for all $x = (x_1, x_2) \in Q_4$, there exists $g \in S$ such that $gx \in Q_1$. Hence, taking, e.g., $g = (a_{ij}) \in S$ with $a_{12} = 0$, $a_{22} = 1/a_{11}$ and a_{21} satisfying $a_{21}x_1 + (1/a_{11})x_2 > 0$ we have $gx \in Q_1$ and therefore R_1 is S-reversible on R_1. The maximality is consequence of Q_3 be invariant for S-action (see [9]). In case of R_2, the computations are similar.*

Example 4. *Let $S = \mathbf{R}^{+} \backslash \{0\}$ and consider its action on $\mathbf{R}^2$ defined by $t(x, y) = (x, ty)$. Then the reversible sets for this action are the sets $\{(x, 0)\}$, $\{(x, y) : y > 0\}$ and $\{(x, y) : y < 0\}$.*

The next result shows that the reversible sets can be characterized by intersection of $G(S)$-orbits of its elements.

Proposition 5. *If R is a reversible set then $R = \cap_{x \in R} S^{-1}S(x)$.*

Proof: If $x, y \in R$ then there exist $\phi, \gamma \in S$ such that $\phi(x) = \gamma(y)$. Thus $y = \gamma^{-1}\phi(x)$ and therefore $R \subset \cap_{x \in R} S^{-1}S(x)$. On the other hand, if $y \in \cap_{x \in R} S^{-1}S(x)$ then $y \in S^{-1}S(x)$ for all $x \in R$. Consequently $S(x) \cap S(y) \neq \emptyset$ for all $x \in R$. Hence, by maximality of R it follows that $y \in R$. □

5. Reversible sets and control sets

In this section it is established relations between the reversible sets and the control sets. To obtain these relations we recall some basic facts about action of semigroups on topological spaces.

A control set of S is a subset $D \subset Q$ with non-empty interior such that $D \subset cl(Sx)$ for every $x \in D$ and is maximal with these properties. If in addiction D is invariant under the action of S it is called an invariant control set (see San Martin [9] and San Martin-Tonelli [12] for a more detailed study).

Let
$$D_0 = \{x \in D : x \in int(Sx) \cap int(S^{-1}x)\}$$
be the set D_0 is called the set of transitivity (or core) of the control set D. The control set D is said to be effective control set if $D_0 \neq \emptyset$.

The domain of attraction of a subset $D \subset Q$ is defined to be
$$\mathcal{A}(D) = \{x \in Q : Sx \cap D \neq \emptyset\}.$$
The control sets for the S-action on Q are ordered by putting $D_1 \leq D_2$ if there are $x \in D_1$ and $g \in S$ such that $gx \in D_2$, that is, $D_1 \cap \mathcal{A}(D_2) \neq \emptyset$.

Proposition 6. *Suppose that S is accessible. Then any control set $D \subset Q$ is S-reversible on D.*

Proof: We begin by claiming that $intD \cap Sx \neq \emptyset$, for all $x \in D$. In fact, consider $y \in intD \subset D \subset cl(Sx)$. Then as $intD$ is open, it follows that $int(D) \cap Sx \neq \emptyset$. Now take $x, y \in D$, as $intD \cap Sy \neq \emptyset$ there exists $a \in S$ such that $ay \in intD$. Then $ay \in intD \subset cl(Sx)$. If $b \in S$ we have $b(ay) \subset b(cl(Sx)) \subset cl(bSx) \subset cl(Sx)$ implying that $S(ay) \subset cl(Sx)$. With this and recalling that $S(ay)$ is open it follows that there exists $c \in S$ such that $cay \in Sx$. Therefore, $Sx \cap Sy \neq \emptyset$. □

Corollary 2. *Suppose that S is accessible. Then any control set $D \subset Q$ is S^{-1}-reversible on D.*

Proof: Take $x, y \in D_0$, then there exists $g, h \in int(S)$ such that $gx = y = h^{-1}x$. Thus $g^{-1}y = x = g^{-1}h^{-1}x$ and hence $S^{-1}x \cap S^{-1}y \neq \emptyset$. Now by accessibility and density arguments it is easy to conclude that $S^{-1}x \cap S^{-1}y \neq \emptyset$ for all $x, y \in D$. □

Note that if D is a control set for S then D is S-reversible on D and hence there exists a reversible set $R(D)$ containing D.

The next result shows that the domain of attraction $\mathcal{A}(C)$ of a S-invariant control set C in Q is equal to the reversible set for S-action.

Proposition 7. *If C is an invariant control set for S then $\mathcal{A}(C) = R(C)$.*

Proof: Take $x \in R$ and $y \in C^0$. As $C^0 \subset R$ then $Sx \cap Sy \neq \emptyset$. But $Sy = C^0$ (see [9] Proposition 2.1), then $Sx \cap C^0 \neq \emptyset$. Hence there exists $\phi \in S$ such that $\phi(x) \in C^0$. So $x \in \mathcal{A}(C)$. Therefore, $R \subset \mathcal{A}(C)$. Conversely, take $x, y \in \mathcal{A}(C)$. Then there exist $\phi, \varphi \in S$ such that $\phi(x), \varphi(y) \in C$. Thus, as S is reversible on C we have $S\phi(x) \cap S\varphi(y) \neq \emptyset$ and on the other hand $S\phi = S\varphi = S$. Then $S(x) \cap S(y) \neq \emptyset$. Hence S is reversible on $\mathcal{A}(C)$ and by maximality, $\mathcal{A}(C) \subset R$. Therefore, $\mathcal{A}(C) = R$. □

About the relation between uniqueness of invariant control set and reversibility we have

Proposition 8. *If S is accessible and reversible on Q then there exists at most one invariant control set for S in Q.*

Proof: Suppose that there are two invariant control sets, C_1 and C_2, for S in Q. Take $x \in C_1$ and $y \in C_2$. By invariance of C_1 and C_2 and by accessibility of S it follows that $cl(Sx) = C_1$ and $cl(Sy) = C_2$. Now using the reversibility we have $Sx \cap Sy \neq \emptyset$, hence $Sy \cap C_1 \neq \emptyset$. This shows that $C_1 \leq C_2$. Similarly, it shows that $C_2 \leq C_1$. Therefore $C_1 = C_2$. □

A sufficient condition for the existence of just one invariant control set for S in Q is $C = \cap_{x \in Q} cl(Sx) \neq \emptyset$. On the other hand, considering $C = \cap_{x \in Q} cl(Sx) \neq \emptyset$ and taking $x \in Q$ we have $Sx \cap C \neq \emptyset$ and thus $x \in \mathcal{A}(C)$. In this case, the domain of attraction is whole space Q. Hence we have:

Proposition 9. *If S is accessible on Q and $C = \cap_{x \in Q} cl(Sx)$ is not empty then S is reversible on Q.*

A kind of converse of this result is:

Proposition 10. *Suppose that semigroup S is reversible on Q. Then there is at most one invariant control set for S in Q.*

Proof: Suppose that there are two invariant control sets for S in Q, C_1 and C_2. Take $x \in C_1$ and $y \in C_2$. By invariance of C_1 and C_2 it follows that $Sx \subset C_1$ and $Sy \subset C_2$. Since the control sets do not overlap we conclude that $Sx \cap Sy = \emptyset$. Now by Definition 1 it follows that S is not reversible, contradicting the hypothesis. □

Corollary 3. *Let Q be a compact space and assume that the semigroup S is accessible and reversible. Then there exists exactly one invariant control set for S in Q.*

Proof: This follows immediately from the above proposition and from the fact that in compact spaces, under the hypothesis of accessibility, invariant control sets always exist (see Lemma 3.1 in [1] and Chapter 3 in [3]). □

6. Reversibility on fiber bundles

We begin by supplying some basic facts about fiber bundles and the action of semigroups on them. We refer to Kobayashi-Nomizu [6] for the theory of fiber bundles and to Braga Barros-San Martin [2] for the concepts of semigroups of local diffeomorphisms of fiber bundles.

Let $Q(M, G)$ be a principal bundle with base space M, total space Q and structure group G. Thus G acts freely on the right on Q and its orbits are the fibers $Q_x = \pi^{-1}\{x\}$, $x \in M$, where $\pi : Q \to M$ is the canonical projection. Each fiber is homeomorphic to G. We are assuming here that $Q \to M$ is locally trivial, often a local trivialization is realized through a local cross section $\chi : U \to Q$, $U \subset M$.

Recall that $\mathcal{H}_l(Q)$ is the set of local homeomorphisms of Q. We say that an element $\phi \in \mathcal{H}_l(Q)$ is right invariant if $\phi(q \cdot g) = \phi(q) \cdot g$, for every $g \in G$. A semigroup $S \subset \mathcal{H}_l(Q)$ is right invariant if ϕ is right invariant for every $\phi \in S$. In this section it is assumed that the semigroup S of local homeomorphisms of Q is right invariant. With this hypothesis we define

$$b : S \to \mathcal{H}_l(M)$$

by $b(\phi)(\pi(q)) = \pi(\phi(q))$ if $\phi \in S$ and $q \in Q$. Since it is reasonable and cause no confusion we identify $b(\phi)$ with ϕ.

Furthermore, the semigroup S is called reversible on base M if the semigroup $b(S)$ is reversible on M. Analogously, S is called reversible on fibers if for all $q_1, q_2 \in Q$ such that $\pi(q_1) = \pi(q_2)$ we have $S(q_1) \cap S(q_2) \neq \emptyset$. The relation between reversibility on principal bundle and on fiber is given by the next statement.

Proposition 11. *The semigroup S is reversible on Q if and only if is reversible on fibers and on the base space.*

Proof: Suppose that S is reversible on base space and on fibers. Given $q_1, q_2 \in Q$, by reversibility on base there exist $\phi, \varphi \in S$ such that $\phi(\pi(q_1)) = \varphi(\pi(q_2))$, that is, $\pi(\phi(q_1)) = \pi(\varphi(q_2))$. Now by reversibility on fibers there exist $\gamma, \xi \in S$ such that $\gamma(\phi(q_1)) = \xi(\varphi(q_2))$. This implies that $S(q_1) \cap S(q_2) \neq \emptyset$. The converse is immediate from the definitions. □

Given $q \in Q$ we put $x = \pi(q)$ and define the subset

$$S_q = \{g \in G : \exists \phi \in S, \phi(q) = q \cdot g\} \subset G.$$

It is easy to check that S_q is a subsemigroup of G if $S_q \neq \emptyset$ (cf. [2]).

In Proposition 3 we showed that if S^{-1} is reversible on a topological space then we can translate compact sets into the orbits. In case of bundles we can to obtain the same property under the assumption that S_q is left reversible on G and that S^{-1} is

reversible on base M. The proof of this will be divided into 4 steps given by the next results.

Lemma 2. *Suppose that S_q is a left reversible semigroup and take $x = \pi(q)$. If $q_1, q_2 \in \pi^{-1}(x)$ then there exists $q_3 \in \pi^{-1}(x)$ such that $q_1, q_2 \in S(q_3)$.*

Proof: We have $q_1 = q \cdot g_1$ and $q_2 = q \cdot g_2$ for some $g_1, g_2 \in G$. As S_q is left reversible on G there exist $a_1, a_2 \in S_q$ satisfying $g_1^{-1} a_1 = g_2^{-1} a_2$. Thus $a_1^{-1} g_1 = a_2^{-1} g_2$. Now take $y = q \cdot (a_1^{-1} g_1) = q \cdot (a_2^{-1} g_2)$ then $y \in \pi^{-1}(x)$ and

$$q_1 = q \cdot g_1 = q \cdot (a_1 a_1^{-1} g_1) = (q \cdot a_1) \cdot a_1^{-1} g_1) =$$

$$= \phi_{a_1}(q) \cdot (a_1^{-1} g_1) = \phi_{a_1}(q \cdot (a_1^{-1} g_1)) = \phi_{a_1}(y).$$

Analogously it follows that $q_2 \in \phi_{a_2}(y)$. □

Lemma 3. *With the same hypothesis of the above lemma take $q_1, q_2, \ldots, q_k \in \pi^{-1}(x)$. Then there exists $z \in \pi^{-1}(x)$ such that $q_1, q_2, \ldots, q_k \in S(z)$.*

Proof: Suppose that $q_1, q_2, \ldots, q_{k-1} \in S(y)$ for some $y \in \pi^{-1}(x)$. By the previous lemma there exists $z \in \pi^{-1}(x)$ with $x_k, y \in S(z)$. Then $S(y) \subset S(z)$ and hence $q_1, q_2, \ldots, q_k \in S(z)$. $\square$

Lemma 4. *Suppose S_q left reversible for all q and S^{-1} is reversible on base M. If $x = \pi(q)$ and $q_1, q_2, \ldots, q_k \in \pi^{-1}(x)$ then there exists $y \in \pi^{-1}(x)$ such that $q_1, q_2, \ldots, q_k \in S(y)$.*

Proof: Consider the projections $\pi(q_1), \ldots, \pi(q_k)$ into M. By assumptions there exist $\phi_1, \ldots, \phi_k \in S$ such that $\phi_1^{-1}(\pi(q_1)) = \cdots = \phi_k^{-1}(\pi(q_k))$, or rather, $\pi(\phi_1^{-1}(q_1)) = \cdots = \pi(\phi_k^{-1}(q_k))$. By the last lemma there exists $y \in Q$ satisfying $\phi_1^{-1}(q_1), \ldots, \phi_k^{-1}(q_k) \in S(y)$. Therefore $q_i \in \phi_i(S(y)) \subset S(y)$ for $i = 1, \ldots, k$. $\square$

Now we are in conditions to prove the property that allow us to put a compact set of Q inside a given S-orbit.

Proposition 12. *Keep assuming that S_q is left reversible for all q and that S^{-1} is reversible on the base M. If K is a compact set in Q then there exists $y \in Q$ such that $K \subset S(y)$.*

Proof: By Proposition 3 it is enough to prove that S^{-1} is reversible on Q. Take $q_1, q_2 \in Q$ and consider the projection maps $\pi(q_1), \pi(q_2)$ into M. By assumption of reversibility of S^{-1} on M there exist $\phi_1, \phi_2 \in S$ such that $\phi_1^{-1}(\pi(q_1)) = \phi_2^{-1}(\pi(q_2))$ and by hypothesis that S acts on M it follows that $\pi(\phi_1^{-1}(q_1)) = \pi(\phi_2^{-1}(q_2))$. Thus $\phi_1^{-1}(q_1)$ and $\phi_2^{-1}(q_2)$ are in the same fiber $q \cdot G$, that is, there exist $g_1, g_2 \in G$ such that $\phi_1^{-1}(q_1) = q \cdot g_1$ and $\phi_2^{-1}(q_2) = q \cdot g_2$. Now by left reversibility of S_q on G there exist $a_1, a_2 \in S_q$ satisfying $g_1^{-1}a_1 = g_2^{-1}a_2$, i.e. , $a_1^{-1}g_1 = a_2^{-1}g_2$. Then $q\cdot(a_1^{-1}g_1) = q\cdot(a_2^{-1}g_2)$ and hence $q\cdot(a_i^{-1}g_i) = (q\cdot a_i^{-1})\cdot g_i = \phi_{a_i}^{-1}(q)\cdot g_i = \phi_{a_i}^{-1}(q\cdot g_i) = \phi_{a_i}^{-1}(\phi_i^{-1}(q_i))$, for $i = 1, 2$. This implies that $\phi_{a_1}^{-1}(\phi_1^{-1}(q_1)) = \phi_{a_2}^{-1}(\phi_2^{-1}(q_2))$ showing that $S^{-1}(q_1) \cap S^{-1}(q_2) \neq \emptyset$ and therefore S^{-1} is reversible on Q. $\square$

Corollary 4. *With the same assumptions and notations of the previous proposition suppose that S acts transitively on M. Then for every $p \in Q$ there exists $g \in G$ such that $K \cdot g \subset S(p)$.*

Proof: By previous proposition, $K \subset S(y)$ for some $y \in Q$. Now, by transitivity of the action of S on M, there exists $\phi_0 \in S$ such that $\phi_0(\pi(p)) = \pi(y)$, that is, $\pi(\phi_0(p)) = \pi(y)$. Then there exists $g_0 \in G$ such that $y = \phi_0(p) \cdot g_0 = \phi_0(p \cdot g_0)$. Take $\alpha \in K$, since $K \subset S(y)$ there exists $\phi \in S(y)$ such that $\alpha = \phi(y)$. Thus $\alpha = \phi(y) = \phi(\phi_0(p \cdot g_0)) = \psi(p) \cdot g_0$, where ψ denote the composition $\phi \circ \phi_0 \in S$. Then $\alpha \cdot g_0^{-1} = \psi(p) \in S(p)$. Therefore, $K \cdot g_0^{-1} \subset S(p)$. □

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