Proyecciones
Vol. 24, N^o 1, pp. 13-20, May 2005.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172005000100002

A NOTE ON POLYNOMIAL CHARACTERIZATIONS OF ASPLUND SPACES

GERALDO BOTELHO *
Universidade Federal de Uberlândia, Brasil.
and
DANIEL M. PELLEGRINO
Universidade Federal de Campina Grande, Brasil

Received : June 2004. Accepted : December 2004

Abstract

In this note we obtain several characterizations of Asplund spaces by means of ideals of Pietsch integral and nuclear polynomials, extending previous results of R. Alencar and R. Cilia-J. Gutiérrez.

2000 Mathematics Subject Classification. *Primary: 46G25; Secondary: 47B10.*

*The authors were partially supported by Instituto do Milênio, IMPA. The second named author was partially supported by CNPq and Fundação de Amparo à Pesquisa-FAPESQ.

Introduction

A Banach space E is an Asplund space if every separable subspace of E has a separable dual. Let $\mathcal{P}_{PI}(^nE;F)$ (resp. $\mathcal{P}_N(^nE;F)$) denote the space of Pietsch integral (resp. nuclear) n-homogeneous polynomials from E to F (see definitions below). For linear operators ($n=1$) we write $PI(E;F)$ and $N(E;F)$. The inclusion $\mathcal{P}_N(^nE;F) \subseteq \mathcal{P}_{PI}(^nE;F)$ holds true for every E, F and n. The following results are due to R. Alencar:

Theorem 1. [1, Theorem 1.3] A Banach space E is an Asplund space if and only if $PI(E;F) = N(E;F)$ for every Banach space F.

Theorem 2. [2, Proposition 1] Let E be a Banach space and $n \in \mathbf{N}$. If E is an Asplund space, then $\mathcal{P}_{PI}(^nE;F) = \mathcal{P}_N(^nE;F)$ for every Banach space F.

Improvements of Theorem 2 were proved by C. Boyd-R. Ryan [4] and D. Carando-V. Dimant [5]. Recently, R. Cilia-J. Gutiérrez [6, Theorem 6] proved the converse of Theorem 2. This note has a twofold purpose: to give a simpler non-tensorial proof of this result of [6] and to extend this characterization of Asplund spaces to other ideals of polynomials which are related to Pietsch integral and nuclear operators.

Preliminaries

Throughout this note $E, E_1, \ldots, E_n$ and F are real or complex Banach spaces, B_E denotes the closed unit ball of E and $\mathbf{N}$ denotes the set of natural numbers. The Banach spaces of continuous n-linear mappings from $E_1 \times \cdots \times E_n$ into F and of continuous n-homogeneous polynomials from E into F with the sup norm will be denoted by $\mathcal{L}(E_1, \ldots, E_n;F)$ ($\mathcal{L}(^nE;F)$ if $E_1 = \cdots = E_n = E$) and $\mathcal{P}(^nE;F)$, respectively. If $A \in \mathcal{L}(^nE;F)$ and P is the polynomial generated by A we write $P = \hat{A}$. Conversely, we write $\check{P}$ for the (unique) symmetric n-linear mapping associated to the polynomial P. For the general theory of multilinear mappings and homogeneous polynomials the reader is referred to S. Dineen [8].

A polynomial $P \in \mathcal{P}(^nE;F)$ is *nuclear*, resp. *Pietsch integral*, if it can be written as

$$P(x) = \sum_{i=1}^{\infty} \varphi_i(x)^n y_i \text{ for every } x \in E,$$

where $(\varphi_i) \subset E'$ and $(y_i) \subset F$ are such that $\sum_{i=1}^{\infty} \|\varphi_i\|^n \|y_i\| < \infty$,

$$\text{resp. } P(x) = \int_{B_{E'}} \varphi(x)^n d\mu(\varphi), \text{ for every } x \in E,$$

where μ is an F-valued regular countably additive Borel measure of bounded variation on $B_{E'}$ with the weak-star topology. The notation for the spaces of such polynomials was established in the introduction. *Mutatis mutandis*, one defines Pietsch integral and nuclear n-linear mappings.

According to [3], an n-linear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is said to be *semi-integral* if there exist $C \geq 0$ and a regular probability measure ν on the Borel sets of $B_{E_1{}'} \times \cdots \times B_{E_n{}'}$ endowed with the weak-star topologies $\sigma(E_j{}', E_j)$, $j = 1, \ldots, n$, such that

$$\|A(x_1, \ldots, x_n)\| \leq C \Big(\int_{B_{E_1{}'} \times \cdots \times B_{E_n{}'}} |\varphi_1(x_1) \cdots \varphi_n(x_n)| d\nu(\varphi_1, \ldots, \varphi_n) \Big),$$

for every $x_j \in E_j$, $j = 1, \ldots, n$.

Now we describe two methods, introduced by A. Pietsch [9], for the generation of ideals of polynomials from a given operator ideal. For $i = 1, \ldots, n$, let $\Psi_i^{(n)} : \mathcal{L}(E_1, \ldots, E_n; F) \to \mathcal{L}(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F)$ represent the canonical isometric isomorphism defined by $\Psi_i^{(n)}(A)(x_i)(x_1, \overset{[i]}{\ldots}, x_n) := A(x_1, \ldots, x_n)$, where the notation $\overset{[i]}{\ldots}$ means that the i-th coordinate is not involved. Let $\mathcal{I}$ be an arbitrary operator ideal.

• The factorization method: a polynomial $P \in \mathcal{P}(^nE; F)$ is of *type* $\mathcal{P}_{\mathcal{L}(\mathcal{I})}$ - $P \in \mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE; F)$ - if there exist a Banach space G, a linear operator $u \in \mathcal{I}(E; G)$ and a polynomial $Q \in \mathcal{P}(^nG; F)$ such that $P = Q \circ u$.

• The linearization method: a multilinear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is of *type* $[\mathcal{I}]$ if $\Psi_i^{(n)}(A) \in \mathcal{I}(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n; F))$ for every $i = 1, \ldots, n$. A polynomial $P \in \mathcal{P}(^nE; F)$ is of *type* $[\mathcal{I}]$ - $P \in \mathcal{P}_{[\mathcal{I}]}(^nE; F)$ - if $\check{P}$ is of type $[\mathcal{I}]$.

Results

First we give an alternative simpler proof of [6, Theorem 6].

Theorem 3. Let E be a Banach space. If $\mathcal{P}_{PI}(^nE;F) = \mathcal{P}_N(^nE;F)$ for every Banach space F and some $n \in \mathbf{N}$, then E is an Asplund space.

Proof. In view of Theorem 1 it suffices to show that $PI(E;F) \subseteq N(E;F)$ for every F. Let $u \in PI(E;F)$. By [7, Theorem VI.3.11] there exist a regular Borel measure μ on $B_{E'}$ with the weak-star topology and a linear operator $v : L_1(\mu) \to F$ such that $u = v \circ j \circ i$, where $i : E \to C(B_{E'})$ is the canonical injection and $j : C(B_{E'}) \to L_1(\mu)$ is the formal inclusion. Fix $0 \neq a \in E$ and choose a linear functional φ on $C(B_{E'})$ such that $\varphi(i(a)) = 1$. Define $R \in \mathcal{L}(^nC(B_{E'}); L_1(\mu))$ by

$$R(f_1, \ldots, f_n) := \frac{1}{n} \sum_{k=1}^{n} \Big(j(f_k) \prod_{m=1, m\neq k}^{n} \varphi(f_m) \Big).$$

It is easy to see that R is semi-integral (use, e.g., the fact that j is absolutely summing). From a result due to R. Alencar-M. Matos [3, Theorem 5.6], it follows that R is Pietsch integral. By [2, Proposition 2], $\hat{R}$ is Pietsch integral, and consequently nuclear, by hypothesis. Now define a polynomial $P := (v \circ \hat{R} \circ i) \in \mathcal{P}(^nE;F)$ and a linear operator $S : E \to F$ by $S(x) = \check{P}(x, a, \ldots, a)$. Then S is nuclear ($\hat{R}$ nuclear $\Rightarrow$ P nuclear $\Rightarrow$ S nuclear). From

$$\check{P}(x_1, \ldots, x_n) = \frac{1}{n} \sum_{k=1}^{n} \Big(u(x_k) \prod_{m=1, m\neq k}^{n} \varphi(i(x_m)) \Big),$$

for every $x_1, \ldots, x_n \in E$, we obtain

$$S(x) = \frac{1}{n} u(x) + \frac{n-1}{n} (\varphi \circ i)(x) u(a),$$

for every $x \in E$. But S is nuclear and $(\varphi \circ i)(\cdot) u(a)$ is a finite rank operator, so we conclude that u is nuclear, what completes the proof. $\Box$

Theorem 4. For a Banach space E and operator ideals $\mathcal{I}_1$ and $\mathcal{I}_2$, the following assertions are equivalent:

(i) $\mathcal{I}_1(E;F) \subseteq \mathcal{I}_2(E;F)$ for every Banach space F.

(ii) $\mathcal{P}_{\mathcal{L}(\mathcal{I}_1)}(^nE;F) \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{I}_2)}(^nE;F)$ for every F and every $n \in \mathbf{N}$.

(iii) $\mathcal{P}_{\mathcal{L}(\mathcal{I}_1)}(^nE;F) \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{I}_2)}(^nE;F)$ for every F and some $n \in \mathbf{N}$.

(iv) $\mathcal{P}_{[\mathcal{I}_1]}(^nE;F) \subseteq \mathcal{P}_{[\mathcal{I}_2]}(^nE;F)$ for every F and every $n \in \mathbf{N}$.

(v) $\mathcal{P}_{[\mathcal{I}_1]}(^nE;F) \subseteq \mathcal{P}_{[\mathcal{I}_2]}(^nE;F)$ for every F and some $n \in \mathbf{N}$.

Proof. (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are obvious.

(i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iv): Let $P \in \mathcal{P}_{\mathcal{L}(\mathcal{I}_1)}(^nE;F)$ (resp. $P \in \mathcal{P}_{[\mathcal{I}_1]}(^nE;F)$). Then $P = Q \circ u$ with $u \in \mathcal{I}_1(E;G) \subseteq \mathcal{I}_2(E;G)$ (resp. $\Psi_i^{(n)}(\check{P}) \in \mathcal{I}_1(E;\mathcal{L}(^{n-1}E;F)) \subseteq \mathcal{I}_2(E;\mathcal{L}(^{n-1}E;F))$), hence $P \in \mathcal{P}_{\mathcal{L}(\mathcal{I}_2)}(^nE;F)$ (resp. $P \in \mathcal{P}_{[\mathcal{I}_2]}(^nE;F)$).

(iii) $\Rightarrow$ (i): Assume that $\mathcal{P}_{\mathcal{L}(\mathcal{I}_1)}(^nE;F) \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{I}_2)}(^nE;F)$ for every F and let $u \in \mathcal{I}_1(E;F)$, $u \neq 0$. Choosing $\varphi \in F'$, $\varphi \neq 0$, $a \in E$ such that $u(a) \neq 0$ and $\varphi(u(a)) = 1$, and defining $P \in \mathcal{P}(^nE;F)$, $Q \in \mathcal{P}(^nF;F)$ by

$$P(x) := \varphi(u(x))^{n-1}u(x);\ Q(y) := \varphi(y)^{n-1}y,$$

we have that $P = Q \circ u$. Therefore $P \in \mathcal{P}_{\mathcal{L}(\mathcal{I}_1)}(^nE;F) \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{I}_2)}(^nE;F)$. Thus there exist a Banach space G, $R \in \mathcal{P}(^nG;F)$ and $v \in \mathcal{I}_2(E;G)$ so that $P = R \circ v$. For every $x \in E$, $\check{P}(x,a,\ldots,a) = (\check{R}(\cdot,v(a),\ldots,v(a)) \circ v)(x)$, hence $\check{P}(\cdot,a,\ldots,a) \in \mathcal{I}_2(E;F)$. From $\check{P} = \check{Q} \circ (u,\ldots,u)$ we have that

$$\check{P}(\cdot,a,\ldots,a) = \frac{1}{n}u(\cdot) + \frac{n-1}{n}\varphi(u(\cdot))u(a).$$

Since $\check{P}(\cdot,a,\ldots,a) \in \mathcal{I}_2(E;F)$ and $\varphi(u(\cdot))u(a)$ is a finite rank operator, we conclude that $u \in \mathcal{I}_2(E;F)$.

(v) $\Rightarrow$ (i): Assume that $\mathcal{P}_{[\mathcal{I}_1]}(^nE;F) \subseteq \mathcal{P}_{[\mathcal{I}_2]}(^nE;F)$ for every F and let $u \in \mathcal{I}_1(E;F)$, $u \neq 0$. Fixing $0 \neq a \in E$, choosing $\varphi \in E'$ such that $\varphi(a) = 1$ and defining $P \in \mathcal{P}(^nE;F)$ by $P(x) := \varphi(x)^{n-1}u(x)$ we have that

$$n\Psi_1^{(n)}(\check{P})(x_1)(x_2,\ldots,x_n) = \varphi(x_2)\cdots\varphi(x_n)u(x_1)+\cdots+\varphi(x_1)\cdots\varphi(x_{n-1})u(x_n),$$

for every $x_1,\ldots,x_n \in E$. Defining $R : E \to \mathcal{L}(^{n-1}E;F)$, $S : F \to \mathcal{L}(^{n-1}E;F)$ by

$$R(x_1)(x_2,\ldots,x_n) := \frac{1}{n}\varphi(x_2)\cdots\varphi(x_n)u(x_1),$$

$$S(y_1)(x_2,\ldots,x_n) := \frac{1}{n}\varphi(x_2)\cdots\varphi(x_n)y_1,$$

and $T : E \to \mathcal{L}(^{n-1}E;F)$ by $T(x_1)(x_2,\ldots,x_n) :=$

$$\frac{1}{n}\varphi(x_1)\varphi(x_3)\cdots\varphi(x_n)u(x_2) + \cdots + \frac{1}{n}\varphi(x_1)\varphi(x_2)\cdots\varphi(x_{n-1})u(x_n),$$

it follows that $R = S \circ u$, hence $R \in \mathcal{I}_1(E;\mathcal{L}(^{n-1}E;F))$, and that T is a finite rank operator. Since $\Psi_1^{(n)}(\check{P}) = R + T$ we have that $\Psi_1^{(n)}(\check{P})$ belongs to $\mathcal{I}_1$. So, $P \in \mathcal{P}_{[\mathcal{I}_1]}({}^nE;F) \subseteq \mathcal{P}_{[\mathcal{I}_2]}({}^nE;F)$, and therefore $\Psi_1^{(n)}(\check{P})$ belongs to $\mathcal{I}_2$. Now let us define $J : \mathcal{L}({}^{n-1}E;F)) \to F$ by $J(A) := A(a,\ldots,a)$ to obtain

$$\left(J \circ \Psi_1^{(n)}(\check{P})\right)(x) = \frac{1}{n}u(x) + \frac{n-1}{n}\varphi(x)u(a) \text{ for every } x \in E.$$

But $J \circ \Psi_1^{(n)}(\check{P}) \in \mathcal{I}_2(E;F)$ and $\varphi(\cdot)u(a)$ has finite rank, so $u \in \mathcal{I}_2(E;F)$. $\square$

Combining Theorems 1, 2, 3 and 4 we obtain the announced characterizations of Asplund spaces.

Theorem 5. For a Banach space E, the following assertions are equivalent:

(i) E is an Asplund space.

(ii) For all $n \in \mathbf{N}$ and every F, we have $\mathcal{P}_{\mathcal{L}(PI)}({}^nE;F) = \mathcal{P}_{\mathcal{L}(N)}({}^nE;F)$.

(iii) There is $n \in \mathbf{N}$ such that $\mathcal{P}_{\mathcal{L}(PI)}({}^nE;F) = \mathcal{P}_{\mathcal{L}(N)}({}^nE;F)$ for every F.

(iv) For all $n \in \mathbf{N}$ and every F, we have $\mathcal{P}_{[PI]}({}^nE;F) = \mathcal{P}_{[N]}({}^nE;F)$.

(v) There is $n \in \mathbf{N}$ such that $\mathcal{P}_{[PI]}({}^nE;F) = \mathcal{P}_{[N]}({}^nE;F)$ for every F.

(vi) For all $n \in \mathbf{N}$ and every F, we have $\mathcal{P}_{PI}({}^nE;F) = \mathcal{P}_{N}({}^nE;F)$.

(vii) There is $n \in \mathbf{N}$ such that $\mathcal{P}_{PI}({}^nE;F) = \mathcal{P}_{N}({}^nE;F)$ for every F.

The same techniques can be used to prove the following additional characterizations:

Theorem 6. For a Banach space E, the following assertions are equivalent:

(i) E is an Asplund space.

(ii) For all $n \in \mathbf{N}$ and every F, we have $\mathcal{P}_{\mathcal{L}(PI)}({}^nE;F) \subseteq \mathcal{P}_{[N]}({}^nE;F)$.

(iii) There is $n \in \mathbf{N}$ such that $\mathcal{P}_{\mathcal{L}(PI)}({}^nE;F) \subseteq \mathcal{P}_{[N]}({}^nE;F)$ for every F.

(iv) For all $n \in \mathbf{N}$ and every F, we have $\mathcal{P}_{\mathcal{L}(PI)}({}^nE;F) \subseteq \mathcal{P}_{N}({}^nE;F)$.

(v) There is $n \in \mathbf{N}$ such that $\mathcal{P}_{\mathcal{L}(PI)}({}^nE;F) \subseteq \mathcal{P}_{N}({}^nE;F)$ for every F.

Acknowledgements. The authors thank R. Alencar and M. Matos for helpful conversations on the subject of this paper.

References

[1] R. Alencar. *Multilinear mappings of nuclear and integral type*, Proc. Amer. Math. Soc. **94** (1985), 33-38.

[2] R. Alencar. *On reflexivity and basis for $\mathcal{P}({}^mE)$*, Proc. Roy. Irish Acad. Sect. A **85** (1985), 131-138.

[3] R. Alencar and M. Matos. *Some classes of multilinear mappings between Banach spaces*, Publ. Dep. Analisis Mat. Univ. Complut. Madrid **12** (1989).

[4] C. Boyd and R. Ryan. *Geometric theory of spaces of integral polynomials and symmetric tensor products*, J. Funct. Anal. **179** (2001), 18-42.

[5] D. Carando and V. Dimant. *Duality in spaces of nuclear and integral polynomials*, J. Math. Anal. Appl. **241** (2000), 107-121.

[6] R. Cilia and J. Gutiérrez. *Polynomial characterization of Asplund spaces*, to appear in Bull. Belg. Math. Soc. Simon Stevin.

[7] J. Diestel and J. J. Uhl. *Vector Measures*, Amer. Math. Soc. Math. Surveys **15**, Providence, 1979.

[8] S. Dineen. *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag, London, 1999.

[9] A. Pietsch. *Ideals of multilinear functionals*, Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, 185-199, Teubner-Texte, Leipzig, 1983.

Geraldo Botelho
Faculdade de Matemática
Univ. Federal de Uberlândia
38.400-902 Uberlândia
Brazil
e-mail: botelho@ufu.br

and

Daniel M. Pellegrino
Departamento de Matemática e Estatística
Univ. Federal de Campina Grande
58.109-970 Campina Grande
Brazil
e-mail: dmp@dme.ufcg.edu.br