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$S_\beta-$COMPACTNESS IN L-TOPOLOGICAL SPACES *

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Abstract

In this paper, the notion of $S_\beta-$compactness is introduced in L-topological spaces by means of open β_a-cover. It is a generalization of Lowen's strong compactness, but it is different from Wang's strong compactness. Ultra-compactness implies $S_\beta-$compactness. $S_\beta-$compactness implies fuzzy compactness. But in general N-compactness and Wang's strong compactness need not imply $S_\beta-$compactness.

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1. Introduction

The concept of compactness in [0, 1]-set theory was first introduced by C.L. Chang in terms of open cover [1]. Goguen was the first to point out a deficiency in Chang's compactness theory by showing that the Tychonoff Theorem is false [5]. Since Chang's compactness has some limitations, Gantner, Steinlage and Warren introduced $\alpha-$compactness [3], Lowen introduced fuzzy compactness, strong compactness and ultra-compactness [10, 11], Liu introduced Q-compactness [8], Li introduced strong Q-compactness [7] which is equivalent to strong fuzzy compactness in [11], and Wang and Zhao introduced N-compactness [16, 18]. In 1988, fuzzy compactness, strong compactness and ultra-compactness were generalized to general L-fuzzy subset by Wang in [17] (These can also be seen in [9]).

Recently in [14] Shi introduced a new notion of fuzzy compactness by means of β_a-cover and Q_a-cover, which is called S^*-compactness. For an L-topological space, Ultra compactness implies S^*-compactness and S^*-compactness implies fuzzy compactness in the sense of [17]. When $L = [0,1]$, strong compactness implies S^*-compactness. But when $L \neq [0,1]$, we don't know whether N-compactness and strong compactness imply S^*-compactness.

In this paper, we shall present a new definition of fuzzy compactness in L-topological spaces by means of β_a-cover, which is called $S_\beta-$compactness. $S_\beta-$compactness is a generalization of strong compactness in [11], but it is different from Wang's strong compactness in [9, 17]. Ultra-compactness implies $S_\beta-$compactness. $S_\beta-$compactness implies S^*-compactness, hence it implies fuzzy compactness. But in general N-compactness and Wang's strong compactness need not imply $S_\beta-$compactness.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a completely distributive de Morgan algebra, X is a nonempty set, L^X is the set of all L-fuzzy sets on X. The smallest element and the largest element in L^X are denoted respectively by $\underline{0}$ and $\underline{1}$. An L-fuzzy set is briefly written as an L-set. We often don't differ a crisp subset A of X and its character function χ_A.

An element a in L is said to be prime if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. An element a in L is said to be co-prime if a' is prime [4]. The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The set of nonzero co-prime elements in L^X is denoted by $M(L^X)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [2]. In a completely distributive de Morgan algebra L, each member b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [9, 17], $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b, in symbol $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [13].

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \qquad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\},$$
$$A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}, \quad A^{(a)} = \{x \in X \mid A(x) \not\leq a\}.$$

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Members of $\mathcal{T}$ are called open L-sets and their complements are called closed L-sets.

Definition 2.1 ([9, 17]). *For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all lower semi-continuous maps from (X, τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X, in this case, $(X, \omega_L(\tau))$ is called topologically generated by (X, τ).*

Definition 2.2 ([9, 17]). *An L-space $(X, \mathcal{T})$ is called weak induced if $\forall a \in L$, $\forall A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.*

Lemma 2.3 ([14]). *Let $(X, \mathcal{T})$ be a weakly induced L-space, $a \in L, A \in \mathcal{T}$. Then $A_{(a)}$ is an open set in $[\mathcal{T}]$.*

Definition 2.4 ([9, 17]). *An L-space $(X, \mathcal{T})$ is called ultra-compact if $\iota_L(\mathcal{T})$ is compact, where $\iota_L(\mathcal{T})$ is the topology generated by $\{A^{(a)} \mid A \in \mathcal{T}, a \in L\}$.*

In [16], Wang introduced the notion of N-compactness in [0,1]-topological spaces by means of α-nets. Zhao [18] generalized the notion of N-compactness to L-fuzzy set theory in terms of $a-$R-neighborhood family and $a^-{-}$R-neighborhood family as follows:

Definition 2.5 ([18]). *Let $(X, \mathcal{T})$ be an L-space, $a \in M(L)$ and $G \in L^X$. A family $\mathcal{P} \subseteq \mathcal{T}'$ is called an $a-$R-neighborhood family of G if for any $x \in X$ with $G(x) \geq a$, there exists a $B \in \mathcal{P}$ such that $B(x) \not\geq a$. $\mathcal{P}$ is called an $a^{-}-$R-neighborhood family of G if there exists $b \in \beta^*(a)$ such that $\mathcal{P}$ is $b-$R-neighborhood family of G.*

It is obvious that $\mathcal{P}$ is an $a-$R-neighborhood family of G if and only if $\mathcal{P}'$ is an open $a-$Q-cover of G in [9].

Definition 2.6 ([18]). *Let $(X, \mathcal{T})$ be an L-space, $A \in L^X$. A is called N-compact if for every $a \in M(L)$, every $a-$R-neighborhood family of G has a finite subfamily which is an $a^{-}-$R-neighborhood family of G.*

Definition 2.7 ([15]). *A net S with index set D is also denoted by $\{S(n) \mid n \in D\}$ or $\{S(n)\}_{n \in D}$. For $G \in L^X$, a net S is said to quasi-coincide with G if $\forall n \in D, S(n) \not\leq G'$.*

Definition 2.8 ([9, 17]). *Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. G is called strongly compact if for every $a \in M(L)$, every constant $a-$net in G has a cluster point in G with height a.*

Definition 2.9. *Let $(X, \mathcal{T})$ be an L-space, $a \in L \backslash \{1\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is said to be an $a-$shading of G if for any $x \in X$ with $G(x) \geq a'$, there exists an $A \in \mathcal{U}$ such that $A(x) \not\leq a$.*

The notion of $a-$shading in Definition 2.9 is a generalization of the corresponding notion in [6, 17].

Theorem 2.10 ([17]). *Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. Then G is strongly compact if and only if for every $a \in P(L)$, every open $a-$shading of G has a finite subfamily which is an $a-$shading of G.*

Definition 2.11 ([14]). *Let $(X, \mathcal{T})$ be an L-space, $a \in M(L)$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is called a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta\left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\right)$.*

Definition 2.12. *Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is called a Q_a-cover of G if for any $x \in X$ with $G(x) \not\leq a'$, it follows that $\bigvee_{A \in \mathcal{U}} A(x) \geq a$.*

It is obvious that for $a \in M(L)$, the notion of Q_a-cover in Definition 2.12 is a generalization of Q_a-open cover in [15].

Definition 2.13 ([9, 17]). *Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. G is called fuzzy compact if for any $a \in M(L)$ and for any $b \in \beta^*(a)$, every constant $a-$net in G has a cluster point in G with height b.*

Theorem 2.14 ([15]). *Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. Then G is fuzzy compact if and only if for all $a \in M(L)$, for all $b \in \beta^*(a)$, each open Q_a-cover Φ of G has a finite subfamily $\mathcal{B}$ such that $\mathcal{B}$ is an open Q_b-cover of G.*

Definition 2.15 ([14]). *Let $(X, \mathcal{T})$ be an L-space, $G \in L^X$. Then G is S^*-compact if and only if for all $a \in M(L)$, each open β_a-cover Φ of G has a finite subfamily $\mathcal{B}$ such that $\mathcal{B}$ is an open Q_a-cover of G.*

3. $S_\beta-$compactness

Definition 3.1. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is called $S_\beta-$compact if for each $a \in M(L)$, each open β_a-cover of G has a finite subfamily which is still an open β_a-cover of G. $(X, \mathcal{T})$ is called $S_\beta-$compact if $\underline{1}$ is $S_\beta-$compact.*

When $L = [0, 1]$, $\mathcal{U}$ is an open β_a-cover of X if and only if $\mathcal{U}$ is an open $a-$shading of X in the sense of [3]. Therefore $S_\beta-$compactness is a generalization of strong compactness in [11].

The following two theorems are obvious.

Theorem 3.2. *An L-set with finite support is $S_\beta-$compact.*

Theorem 3.3. *In an L-space $(X, \mathcal{T})$ with a finite L-topology $\mathcal{T}$, each L-set is $S_\beta-$compact.*

Theorem 3.4. *If G is $S_\beta-$compact and H is closed, then $G \wedge H$ is $S_\beta-$compact.*

Proof. Suppose that $\mathcal{U}$ is an open β_a-cover of $G\wedge H$. Then $\mathcal{U}\cup\{H'\}$ is an open β_a-cover of G. By $S_\beta-$compactness of G we know that $\mathcal{U}\cup\{H'\}$ has a finite subfamily $\mathcal{V}$ which is an open β_a-cover of G. Take $\mathcal{W}=\mathcal{V}\backslash\{H'\}$. Then $\mathcal{W}$ is a finite open β_a-cover of $G \wedge H$. This shows that $G \wedge H$ is $S_\beta-$compact.

Theorem 3.5. *Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ be two L-spaces, $f : X \to Y$ be a set map and G be $S_\beta-$compact in $(X, \mathcal{T}_1)$. If $f_L^{\rightarrow} : L^X \to L^Y$ is continuous and for any $y \in Y$, there exists $x \in f^{-1}(y)$ such that $f_L^{\rightarrow}(G)(y) = G(x)$, then $f_L^{\rightarrow}(G)$ is $S_\beta-$compact in $(Y, \mathcal{T}_2)$.*

Proof. Let $\mathcal{U} \subseteq \mathcal{T}_2$ be an open β_a-cover of $f_L^{\rightarrow}(G)$. Then for any $y \in Y$, we have that $a \in \beta\left(f_L^{\rightarrow}(G)'(y) \vee \bigvee_{A\in\mathcal{U}} A(y)\right)$. Hence for any $x \in X$, $a \in \beta\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} f_L^{\leftarrow}(A)(x)\right)$. This shows that $f_L^{\leftarrow}(\mathcal{U}) = \{f_L^{\leftarrow}(A) \mid A \in \mathcal{U}\}$ is an open β_a-cover of G. By $S_\beta-$compactness of G we know that $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ such that $f_L^{\leftarrow}(\mathcal{V})$ is an open β_a-cover of G. For any $y \in Y$, take $x \in f^{-1}(y)$ such that $f_L^{\rightarrow}(G)(y) = G(x)$. We have that

$$\begin{aligned} a &\in \beta\left(G'(x) \vee \left(\bigvee_{A\in\mathcal{V}} f_L^{\leftarrow}(A)(x)\right)\right) = \beta\left(G'(x) \vee \left(\bigvee_{A\in\mathcal{V}} A(f(x))\right)\right) \\ &= \beta\left(f_L^{\rightarrow}(G)'(y) \vee \left(\bigvee_{A\in\mathcal{V}} A(y)\right)\right) \end{aligned}$$

This implies that $\mathcal{V}$ is an open β_a-cover of $f_L^{\rightarrow}(G)$. Therefore $f_L^{\rightarrow}(G)$ is $S_\beta-$compact.

Theorem 3.6. *If $(X, \mathcal{T})$ is a weakly induced L-space, then $(X, [\mathcal{T}])$ is compact if and only if $(X, \mathcal{T})$ is $S_\beta-$compact.*

Proof. Necessity. Let $(X, [\mathcal{T}])$ be compact. For $a \in M(L)$, let $\mathcal{U}$ be an open β_a-cover of $\underline{1}$ in $(X, \mathcal{T})$. Then by Lemma 2.3 we know that $\{A_{(a)} \mid A \in \mathcal{U}\}$ is an open cover of $(X, [\mathcal{T}])$. By compactness of $(X, [\mathcal{T}])$, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}_{(a)} = \{A_{(a)} \mid A \in \mathcal{V}\}$ is an open cover of $(X,[\mathcal{T}])$. Obviously $\mathcal{V}$ is an open β_a-cover of $\underline{1}$ in $(X,\mathcal{T})$. This shows that $(X,\mathcal{T})$ is $S_\beta-$compact.

Sufficiency. Let $(X,\mathcal{T})$ be $S_\beta-$compact and $\mathcal{U}$ be an open cover of X in $(X,[\mathcal{T}])$. Then for any $a\in\beta^*(1)$, $\mathcal{U}$ is an open β_a-cover of $\underline{1}$ in $(X,\mathcal{T})$. By $S_\beta-$compactness of $(X,\mathcal{T})$, $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ which is an open β_a-cover. Obviously $\mathcal{V}$ is an open cover of $(X,[\mathcal{T}])$. This shows that $(X,[\mathcal{T}])$ is compact.

Corollary 3.7. *For a topological space (X,τ), $(X,\omega_L(\tau))$ is $S_\beta-$compact if and only if (X,τ) is compact.*

4. The Tychonoff Theorem

Lemma 4.1. *Let $(X,\mathcal{T})$ be an L-space and for any $b,c\in L$, $\beta(b\wedge c)=\beta(b)\cap\beta(c)$. Then for each $a\in L$, $(X,\mathcal{T}_{(a)})$ is a topological space, where $\mathcal{T}_{(a)}=\{A_{(a)}\mid A\in\mathcal{T}\}$.*

Proof. This can be proved from the following fact.

$$(A\wedge B)_{(a)}=A_{(a)}\cap B_{(a)},\ \left(\bigvee_{i\in\Omega}A_i\right)_{(a)}=\bigcup_{i\in\Omega}(A_i)_{(a)}\,.$$

Theorem 4.2. *Let $(X,\mathcal{T})$ be an L-space, $G\in L^X$ and for any $b,c\in L$, $\beta(b\wedge c)=\beta(b)\cap\beta(c)$. Then G is $S_\beta-$compact if and only if for each $a\in M(L)$, $G^{[a']}$ is compact in $(X,\mathcal{T}_{(a)})$.*

Proof. This can be shown from the following fact.

A subfamily $\mathcal{U}$ of $\mathcal{T}$ is an open β_a-cover of G if and only if for any $x\in X$, it follows that $a\in\beta\left(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\right)$ if and only if for each $a\in M(L)$, $a\notin\beta(G'(x))$ implies $a\in\beta\left(\bigvee_{A\in\mathcal{U}}A(x)\right)$ if and only if for each $a\in M(L)$, $a'\notin\alpha(G(x))$ implies $a\in\bigcup_{A\in\mathcal{U}}\beta\left(A(x)\right)$ if and only if for each $a\in M(L)$, $x\in G^{[a']}$ implies $x\in\bigcup_{A\in\mathcal{U}}A_{(a)}$ if and only if for each $a\in M(L)$, $G^{[a']}\subset\bigcup_{A\in\mathcal{U}}A_{(a)}$. $\square$

The proof of the following two lemmas is easy.

Lemma 4.3. *Suppose that for any* $a, b \in L$, $\beta(a \wedge b) = \beta(a) \wedge \beta(b)$. *If* $(X, \mathcal{T})$ *is the product of* $\{(X_i, \mathcal{T}_i)\}_{i\in\Omega}$, *then for each* $a \in L$, $\mathcal{T}_{(a)} = \prod\limits_{i\in\Omega} (\mathcal{T}_i)_{(a)}$.

Lemma 4.4. *Let* $X = \prod\limits_{i\in\Omega} X_i, G = \prod\limits_{i\in\Omega} G_i$, *where* $G_i \in L^{X_i}$. *Then for each* $a \in L$, $G^{[a]} = \prod\limits_{i\in\Omega} (G_i)^{[a]}$.

By Theorem 4.2, Lemma 4.3 and Lemma 4.4 we can obtain the following theorem.

Theorem 4.5. *Suppose that for any* $a, b \in L$, $\beta(a \wedge b) = \beta(a) \wedge \beta(b)$. *Let* $(X, \mathcal{T})$ *be the product of* $\{(X_i, \mathcal{T}_i)\}_{i\in\Omega}$. *If for each* $i \in \Omega$, G_i *is* $S_\beta-$*compact in* $(X_i, \mathcal{T}_i)$, *then* $G = \prod\limits_{i\in\Omega} G_i$ *is* $S_\beta-$*compact in* $(X, \mathcal{T})$.

The following example shows that $\beta(b \wedge c) = \beta(b) \wedge \beta(c)$ cannot be omitted in Theorem 4.5.

Example 4.6. *Let* $X = Y$ *be the set of all natural numbers and let* $L = [0, 1/3] \cup \{a, b\} \cup [2/3, 1]$, *where* a, b *are incomparable and* $1/3 = a \wedge b, 2/3 = a \vee b$. *For each* $e \in L$ *with* $e \neq a, b$, *define* $e' = 1 - e$, *and* $a' = b, b' = a$. *Then* L *is a completely distributive de Morgan algebra, and*

$$M(L) = (0, 1/3] \cup \{a, b\} \cup (2/3, 1],$$

$$\beta(a \wedge b)) = \beta(1/3) = [0, 1/3) \neq [0, 1/3] = \beta(a) \cap \beta(b).$$

For each $n \in X$, *define* $S_{2n}, S_{2n+1} \in L^X$ *as follows:*

$$S_{2n}(y) = \begin{cases} a, & y = 2n; \\ b, & y \neq 2n, \end{cases} \qquad S_{2n+1}(y) = \begin{cases} b, & y = 2n+1; \\ a, & y \neq 2n+1. \end{cases}$$

Let $\mathcal{T}_1$ *be the* L*-topology on* X *generated by* $\mathcal{A} = \{S_n \mid n \in X\}$, *and let* $\mathcal{T}_2$ *be the* L*-topology on* Y *generated by* $\{C_b, C_a\}$, *where* C_a *and* C_b *are respectively the constant* L*-sets on* Y *with value* a *and* b. *It is easy to prove that each* β_a-*open cover of* X *consisting of members of* $\mathcal{A}$ *has a finite subfamily which is an open* β_a-*cover of* X *and each* β_b-*open cover of* X *consisting of members of* $\mathcal{A}$ *has a finite subfamily which is a* β_b-*open cover of* X. *Moreover it is easy to prove that for all* $e \in [0, 1/3]$, *each* β_e-*open* *cover of X consisting of members of $\mathcal{A}$ has a finite subfamily which is a β_e-open cover of X. This implies that $(X, \mathcal{T}_1)$ is $S_\beta-$compact. Obviously $(Y, \mathcal{T}_2)$ is also $S_\beta-$compact. But $(X \times Y, \mathcal{T}_1 \times \mathcal{T}_2)$ is not $S_\beta-$compact. In fact, it is easy to see that*

$$\{P_2^{\leftarrow}(C_a), P_2^{\leftarrow}(C_b)\} \cup \{P_1^{\leftarrow}(S_n) \mid n \in X\}$$

is a base of $\mathcal{T}_1 \times \mathcal{T}_2$ and

$$\{P_2^{\leftarrow}(C_a) \wedge P_1^{\leftarrow}(S_{2n}) \mid n \in X\} \cup \{P_2^{\leftarrow}(C_b) \wedge P_1^{\leftarrow}(S_{2n+1}) \mid n \in X\}$$

is a $\beta_{1/3}-$open cover of $X \times Y$, but it has no finite subfamily which is a $\beta_{1/3}-$open cover of $X \times Y$.

Corollary 4.7. *Suppose that for any $a, b \in L$, $\beta(a \wedge b) = \beta(a) \wedge \beta(b)$. Then the product of $\{(X_i, \mathcal{T}_i)\}_{i \in \Omega}$ is $S_\beta-$compact if and only if for each $i \in \Omega$, $(X_i, \mathcal{T}_i)$ is $S_\beta-$compact.*

Proof can be obtain from Theorem 3.5 and Theorem 4.5.

5. $S_\beta-$compactness characterized by nets

Definition 5.1. *Let $\{S(n) \mid n \in D\}$ be a net in $(X, \mathcal{T})$ and $e \in M(L^X)$. e is called a $\beta-$cluster point of S, if for all $U \in \mathcal{T}$ with $e \in \beta(U)$, S is frequently in $\beta(U)$. e is a $\beta-$limit point of S, if for all $U \in \mathcal{T}$ with $e \in \beta(U)$, S is eventually in $\beta(U)$, in this case we also say that S $\beta-$converges to e, denoted by $S \xrightarrow{\beta} e$.*

Theorem 5.2. *An L-set G is $S_\beta-$compact in $(X, \mathcal{T})$ if and only if $\forall a \in M(L)$, each constant $a-$net $\{S(n)\}_{n \in D}$ which is not in $\beta^*(G')$ has a $\beta-$cluster point $x_a \notin \beta^*(G')$.*

Proof. Suppose that G is $S_\beta-$compact. For $a \in M(L)$, let $\{S(n) \mid n \in D\}$ be a constant $a-$net which is not in $\beta^*(G')$. Suppose that S has no $\beta-$cluster point x_a which is not in $\beta^*(G')$. Then for each $x_a \notin \beta^*(G')$, there exist an open L-set U_x with $x_a \in \beta^*(U_x)$ and $n_x \in D$ such that $\forall n \geq n_x, S(n) \notin \beta^*(U_x)$. Take $\Phi = \{U_x \mid x_a \notin \beta^*(G')\}$, then Φ is an open β_a-cover of G. Since G is $S_\beta-$compact, Φ has a finite subfamily

$\Psi = \{U_{x^i} \mid i = 1, 2, \cdots, k\}$ which is a β_a-open cover of G. Since D is a directed set, there exists $n_0 \in D$ such that $n_0 \geq n_{x^i}$ for each $i \leq k$. Thus we can obtain that $\forall n \geq n_0, S(n) \notin \beta\left(\bigvee_{i=1}^{k} U_{x^i}\right)$. This contradicts that Ψ is an open β_a-cover of G. Therefore S has at least a $\beta-$cluster point $x_a \notin \beta^*(G')$.

Conversely suppose that $\forall a \in M(L)$, each constant $a-$net which is not in $\beta^*(G')$ has a $\beta-$cluster point $x_a \notin \beta^*(G')$. We now prove that G is $S_\beta-$compact. Let Φ be an open β_a-cover of G. If none of finite subfamilies of Φ is an open β_a-cover of G , then for each finite subfamily Ψ of Φ, there exists $S(\Psi) \in M(L^X)$ with height a such that $S(\Psi) \notin \beta\left(G' \vee \bigvee \Psi\right)$.

Take

$$S = \{S(\Psi) \mid \Psi \text{ is a finite subfamily of } \Phi\}.$$

S is a constant $a-$net which is not in $\beta^*(G')$. Let x_a be a $\beta-$cluster point of S and $x_a \notin \beta^*(G')$. Then for each finite subfamily Ψ of Φ we have that $x_a \notin \beta\left(\bigvee \Psi\right)$, in particular, $x_a \notin \beta^*(B)$ for each $B \in \Phi$. But since Φ is an open β_a-cover of G , we know that there exists $B \in \Phi$ such that $x_a \in \beta(B)$, this is a contradiction. So G is $S_\beta-$compact.

Corollary 5.3. *$(X, \mathcal{T})$ is $S_\beta-$compact if and only if $\forall a \in M(L)$, each constant $a-$net has a $\beta-$cluster point x_a with height a.*

6. A comparison of different compactness

Theorem 6.1. *If $(X, \mathcal{T})$ is an ultra-compact L-space, then it is $S_\beta-$compact.*

Proof. By ultra-compactness of $(X, \mathcal{T})$ we know that $(X, \iota(\mathcal{T}))$ is compact. This implies $(X, \omega_L \circ \iota_L(\mathcal{T}))$ is $S_\beta-$compact from Corollary 3.7. Further from $\omega_L \circ \iota_L(\mathcal{T}) \supseteq \mathcal{T}$ we can obtain the proof.

Theorem 6.2. *$S_\beta-$compactness implies S^*-compactness, hence fuzzy compactness.*

Proof. Let G be $S_\beta-$compact in $(X, \mathcal{T})$ and $\mathcal{U}$ be an open β_a-cover of G. Then $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ which forms an open β_b- cover of G, of course $\mathcal{V}$ is also an open Q_b-cover of G. Therefore G is S^*-compact.

The following example shows that N-compactness in [17, 18] need not imply $S_\beta-$compactness, hence strong compactness in [9, 17] need not imply $S_\beta-$compactness.

Example 6.3. *In Example 4.6, we have proved that $X \times Y$ is not $S_\beta-$compact. To prove that it is N-compact, we only need to prove that X, Y are N-compact.*

(i) For $a \in M(L)$, let $\mathcal{F} \subseteq \mathcal{A}'$ and $\mathcal{F}$ be a closed $a-R$-neighborhood family of X. Then for each $x \in X$, there exists $A \in \mathcal{F}$ such that $A(x) \not\geq a$. In particular, for $2, 4 \in X$, there exists $A, B \in \mathcal{F}$ such that $A(2) \not\geq a, B(4) \not\geq a$. In this case, we have that $A(2) = b$ and $B(4) = b$. This implies that $\{A, B\}$ is an $a^- - R$-neighborhood family of X. Analogously we can prove that each closed $b-R$-neighborhood family of X has a finite subfamily which is a $b^- - R$-neighborhood family of X.

(ii) Let $e \in M(L)$ and $e \neq a, b$. We need only consider $e > \frac{2}{3}$. Let $\mathcal{F} \subseteq \mathcal{A}'$ and $\mathcal{F}$ be a closed $e-R$-neighborhood family of X. Then for $1, 2 \in X$, there exists $A, B \in \mathcal{F}$ such that $A(1) \not\geq e, B(2) \not\geq e$. In this case, $\{A, B\}$ is an $e^- - R$-neighborhood family of X.

By (i), (ii) and the Alexander Subbase Theorem for N-compactness, we know that $(X, \mathcal{T}_1)$ is N-compact. N-compactness of $(Y, \mathcal{T}_2)$ is obvious. Therefore $X \times Y$ is N-compact.

When $L = [0, 1]$, since $S_\beta-$compactness is equivalent to strong compactness, we know that $S_\beta-$compactness need not imply N-compactness and S^*-compactness need not imply $S_\beta-$compactness (see Example 6.4 in [14]). Moreover we don't know whether $S_\beta-$compactness implies strong compactness. We leave it as an open problem.

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