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A NEW FORM OF FUZZY β-COMPACTNESS *

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Abstract

A new form of β-compactness is introduced in L-topological spaces by means of β-open L-sets and their inequality where L is a complete de Morgan algebra. This new form doesn't rely on the structure of basis lattice L. It can also be characterized by means of β-closed L-sets and their inequality. When L is a completely distributive de Morgan algebra, its many characterizations are presented. Meanwhile countable β-compactness and the β-Lindelöf property are also researched.

Key Words and Phrases: *L-topology, compactness, β-compactness, countable β-compactness, the β-Lindelöf property*

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1. Introduction

As we know, stronger and weaker forms of compactness occupy very important places in general topology. In [5], Abd El-Monsef et al introduced the concepts of β-open sets and β-continuous functions in general topology, and Fath Alla in [1] introduced these concepts in [0,1]-topological spaces. In [2], G. Balasubramanian generalized the concept of β-compactness [6] to $[0,1]$-topological spaces along the line of Chang's compactness which is not a good extension.

In [12, 13], a new definition of fuzzy compactness is presented in L-topological spaces by means of open L-sets and their inequality where L is a complete de Morgan algebra. This new definition doesn't depend on the structure of L. When L is completely distributive, it is equivalent to the notion of fuzzy compactness in [9, 10, 15]. In this paper, following the lines of [12, 13], we shall introduce a new form of β-compactness in L-topological spaces by means of β-open L-sets and their inequality. This new form of β-compactness has many characterizations if L is completely distributive.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a complete de Morgan algebra, X a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called co-prime element if a' is a prime element [7]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [4]. In a completely distributive de Morgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b in the sense of [9, 15], in symbol $\beta(b)$. Moreover for $b \in L$, define $\beta^*(b) = \beta(b) \cap M(L)$, $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [14].

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \quad A^{(a)} = \{x \in X \mid A(x) \not\leq a\}.$$

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\mathcal{X}_\emptyset$, $\mathcal{X}_X$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Members of $\mathcal{T}$ are called open L-sets and their complements are called closed L-sets. We often don't differ a crisp subset A of X and its character function $\mathcal{X}_A$.

Definition 2.1 (**[9, 15]**) : An L-space $(X,\mathcal{T})$ is called weak induced if $\forall a \in L$, $\forall A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

Definition 2.2 ([9, 15]) **:** For a topological space (X,τ), let $\omega_L(\tau)$ denote the family of all the lower semi-continuous maps from (X,τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X, in this case, $(X,\omega_L(\tau))$ is called topologically generated by (X,τ). A topologically generated L-space is also called an induced L-space.

It is obvious that $(X,\omega_L(\tau))$ is weak induced.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamily of Φ. $2^{[\Phi]}$ denotes the set of all countable subfamily of Φ.

Definition 2.3 ([12, 13]) **:** Let $(X,\mathcal{T})$ be an L-space, $G \in L^X$ is called (countably) compact if for every (countably) family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right).$$

Definition 2.4 [**13**] : Let $(X,\mathcal{T})$ be an L-space, $G \in L^X$ is said to have the Lindelöf property if for every family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{\mathcal{A}\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right).$$

Lemma 2.5 [**13**] : Let L be a complete Heyting algebra, $f : X \to Y$ be a map, $f_L^{\rightarrow} : L^X \to L^Y$ is the extension of f, then for any family $\mathcal{P} \subseteq L^Y$, we have:

$$\bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y)\right) = \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\right).$$

The following definition is presented in [0,1]-topological spaces. Analogously we can generalize it to L-fuzzy setting as follows:

Definition 2.6 [**1**] : An L-set A in an L-space (X, δ) is said to be β-open if $A \leq cl\,(int\,(cl\,(A)))$.

Definition 2.7. A map $f : (X, \delta) \to (Y, \mu)$ is said to be fuzzy β-continuous [**1**] (resp. Mβ-continuous [**8**] if the inverse image $f_L^{\leftarrow}(B)$ of every open (resp. β-open) L-set B in Y is β-open in X.

3. Definition and characterizations of β-compactness

Definition 3.1 : Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called (countably) β-compact if for every (countable) family $\mathcal{U}$ of β-open L-sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Definition 3.2 : Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is said to have the β-Lindelöf property (or be an β-Lindelöf L-set) if for every family $\mathcal{U}$ of β-open L-sets, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{[\mathcal{U}]}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Obviously we have the following theorem.

Theorem 3.3 : β-compactness implies countably β-compactness and the β-Lindelöf property. Moreover an L-set having the β-Lindelöf property is β-compact if and only if it is countably β-compact.

Since an open L-set must be β-open, we have the following theorem.

Theorem 3.4 : β-compactness implies compactness, countably β-compactness implies countably compactness, and the β-Lindelöf property implies the Lindelöf property.

From Definition 3.1 and Definition 3.2 we can obtain the following two theorems by simply using complement.

Theorem 3.5 : Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is (countably) β-compact if and only if for every (countable) family $\mathcal{B}$ of β-closed L-sets, it follows that

$$\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{B}} B(x)\right) \geq \bigwedge_{\mathcal{F}\in 2^{(\mathcal{B})}} \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x)\right).$$

Theorem 3.6 : Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ has the β-Lindelöf property if and only if for every family $\mathcal{B}$ of β-closed L-sets, it follows that

$$\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{B}} B(x)\right) \geq \bigwedge_{\mathcal{F}\in 2^{[\mathcal{B}]}} \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x)\right).$$

In order to present characterizations of β-compactness, countable β-compactness and the β-Lindelöf property, we generalize the notions of $a-$shading and $a-$R-neighborhood family in [12, 13] as follows:

Definition 3.7 : Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{1\}$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be

(1) An $a-$shading of G if for any $x \in X$, $\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \nleq a$.

(2) A strong $a-$shading of G if $\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \nleq a$.

(3) An $a-$remote family of G if for any $x \in X$, $\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x)\right) \ngeq a$.

(4) A strong $a-$remote family of G if $\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x)\right) \ngeq a$.

It is obvious that a strong $a-$shading of G is an $a-$shading of G, a strong $a-$remote family of G is an $a-$remote family of G, and $\mathcal{P}$ is a strong $a-$remote family of G if and only if $\mathcal{P}'$ is a strong $a'-$shading of G.

Definition 3.8 : Let $a \in L\backslash\{0\}$ and $G \in L^X$. A subfamily $\mathcal{A}$ of L^X is said to have weak $a-$nonempty intersection in G if $\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{A\in\mathcal{A}} A(x)\right) \geq a$. $\mathcal{A}$ is said to have the finite (countable) weak $a-$intersection property in G if every finite (countable) subfamily $\mathcal{F}$ of $\mathcal{A}$ has weak $a-$nonempty intersection in G.

From Definition 3.1, Definition 3.2, Theorem 3.5 and Theorem 3.6 we immediately obtain the following two results.

Theorem 3.9 : Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:

(1) G is (countably) β-compact.

(2) For any $a \in L\backslash\{1\}$, each (countable) β-open strong $a-$shading $\mathcal{U}$ of G has a finite subfamily which is a strong $a-$shading of G.

(3) For any $a \in L\backslash\{0\}$, each (countable) β-closed strong $a-$remote family $\mathcal{P}$ of G has a finite subfamily which is a strong $a-$remote family of G.

(4) For any $a \in L\backslash\{0\}$, each (countable) family of β-closed L-sets which has the finite weak $a-$intersection property in G has weak $a-$nonempty intersection in G.

Theorem 3.10 :Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:

(1) G has the β-Lindelöf property.

(2) For any $a \in L\backslash\{1\}$, each β-open strong $a-$shading $\mathcal{U}$ of G has a countable subfamily which is a strong $a-$shading of G.

(3) For any $a \in L\backslash\{0\}$, each β-closed strong $a-$remote family $\mathcal{P}$ of G has a countable subfamily which is a strong $a-$remote family of G.

(4) For any $a \in L\backslash\{0\}$, each family of β-closed L-sets which has the countable weak $a-$intersection property in G has weak $a-$nonempty intersection in G.

4. Properties of (countable) β-compactness

Theorem 4.1 : Let L be a complete Heyting algebra. If both G and H are (countably) β-compact, then $G \vee H$ is (countably) β-compact.

Proof. For any (countable) family $\mathcal{P}$ of β-closed L-sets, by Theorem 3.5 we have that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\\
=&\left\{\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\\
\geq&\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\vee\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\\
=&\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right).
\end{aligned}$$

This shows that $G\vee H$ is (countably) β-compact.

Analogously we have the following result.

Theorem 4.2 : Let L be a complete Heyting algebra. If both G and H have the β-Lindelöf property, then $G\vee H$ has the β-Lindelöf property.

Theorem 4.3 : If G is (countably) β-compact and H is β-closed , then $G\wedge H$ is (countably) β-compact.

Proof : For any (countable) family P of β-closed L-sets, by Theorem 3.5 we have that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\wedge H)(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\\
&=\bigvee_{x\in X}\left((G\wedge H)(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\\
&\geq\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P}\cup\{H\})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\\
&=\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\wedge\\
&\qquad\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\\
&\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}
\end{aligned}$$

$$\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \left((G\wedge H)(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x) \right).$$

This shows that $G\wedge H$ is (countably) β-compact.

Analogously we have the following result.

Theorem 4.4 : If G has the β-Lindelöf property and H is β-closed , then $G\wedge H$ has the β-Lindelöf property.

Theorem 4.5 : Let L be a complete Heyting algebra and let $f : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ be $M\beta$-continuous. If G is an β-compact (respectively a countably β-compact, an β-Lindelöf) L-set in $(X,\mathcal{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y,\mathcal{T}_2)$.

Proof. We only prove that the theorem is true for β-compactness. Suppose that $\mathcal{P}$ is a family of β-closed L-sets in $(Y,\mathcal{T}_2)$, by Lemma 2.5 and β-compactness of G we have that

$$\begin{aligned}
& \bigvee_{y\in Y} \left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y) \right) \\
= & \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x) \right) \\
\geq & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X} \left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} f_L^{\leftarrow}(B)(x) \right) \\
= & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{y\in Y} \left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{F}} B(y) \right).
\end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is β-compact.

Analogously we have the following result.

Theorem 4.6 : Let L be a complete Heyting algebra and let $f : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ be β-continuous. If G is an β-compact (respectively a countably β-compact, an β-Lindelöf) L-set in $(X,\mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is a compact (countably compact, Lindelöf) L-set in $(Y,\mathcal{T}_2)$.

Definition 4.7 : Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called strongly $M\beta$-continuous if $f_L^{\leftarrow}(G)$ is open in $(X, \mathcal{T}_1)$ for every β-open L-set G in $(Y, \mathcal{T}_2)$.

It is obvious that a strongly $M\beta$-continuous map is $M\beta$-continuous.
Analogously we have the following result.

Theorem 4.8 : Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a strongly $M\beta$-continuous map. If G is a compact (respectively countably compact, Lindelöf) L-set in $(X, \mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is an β-compact (a countably β-compact, an β-Lindelöf) L-set in $(Y, \mathcal{T}_2)$.

5. Further characterizations of β-compactness and goodness

In this section, we assume that L is a completely distributive de Morgan algebra.

Now we generalize the notions of open β_a-cover and open Q_a-cover [13] as follows:

Definition 5.1 : Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta\left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\right)$. $\mathcal{U}$ is called a strong β_a-cover of G if $a \in \beta\left(\bigwedge_{x \in X}\left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)\right)\right)$.

Definition 5.2 : Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A family $\mathbf{U} \subseteq L^X$ is called a Q_a-cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \geq a$.

It is obvious that a strong β_a-cover of G must be a β_a-cover of G and a β_a-cover of G must be a Q_a-cover of G.

Analogous to the proof of Theorem 2.9 in [13] we can obtain the following theorem.

Theorem 5.3 : Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent.

(1) G is β-compact.

(2) For any $a \in L\backslash\{0\}$, each β-closed strong $a-$remote family $\mathcal{P}$ of G has a finite subfamily which is a strong $a-$remote family of G.

(3) For any $a \in L\backslash\{0\}$, each β-closed strong $a-$remote family $\mathcal{P}$ of G has a finite subfamily which is an $a-$remote family of G.

(4) For any $a \in L\backslash\{0\}$ and any β-closed strong $a-$remote family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ such that $\mathcal{F}$ is a strong $b-$remote family of G.

(5) For any $a \in L\backslash\{0\}$ and any β-closed strong $a-$remote family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ such that $\mathcal{F}$ is a $b-$remote family of G.

(6) For any $a \in L\backslash\{1\}$, each β-open strong $a-$shading $\mathcal{U}$ of G has a finite subfamily which is a strong $a-$shading of G.

(7) For any $a \in L\backslash\{1\}$, each β-open strong $a-$shading $\mathcal{U}$ of G has a finite subfamily which is an $a-$shading of G.

(8) For any $a \in L\backslash\{1\}$ and any β-open strong $a-$shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ such that $\mathcal{V}$ is a strong $b-$shading of G.

(9) For any $a \in L\backslash\{1\}$ and any β-open strong $a-$shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ such that $\mathcal{V}$ is a $b-$shading of G.

(10) For any $a \in L\backslash\{0\}$, each β-open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily which is a strong β_a-cover of G.

(11) For any $a \in L\backslash\{0\}$, each β-open strong β_a-cover $\mathcal{U}$ of G has a finite subfamily which is a β_a-cover of G.

(12) For any $a \in L\backslash\{0\}$ and any β-open strong β_a-cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a strong β_b-cover of G.

(13) For any $a \in L\backslash\{0\}$ and any β-open strong β_a-cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ with $a \in \beta(b)$ such that $\mathcal{V}$ is a β_b-cover of G.

(14) For any $a \in L\backslash\{0\}$ and any $b \in \beta(a)\backslash\{0\}$, each β-open Q_a-cover of G has a finite subfamily which is a Q_b-cover of G.

(15) For any $a \in L\backslash\{0\}$ and any $b \in \beta(a)\backslash\{0\}$, each β-open Q_a-cover of G has a finite subfamily which is a β_b-cover of G.

(16) For any $a \in L\backslash\{0\}$ and any $b \in \beta(a)\backslash\{0\}$, each β-open Q_a-cover of G has a finite subfamily which is a strong β_b-cover of G.

Remark 5.4 : In Theorem 5.3, $a \in L \backslash \{0\}$ and $b \in \beta(a)$ can be replaced by $a \in M(L)$ and $b \in \beta^*(a)$ respectively, $a \in L \backslash \{1\}$ and $b \in \alpha(a)$ can be replaced by $a \in P(L)$ and $b \in \alpha^*(a)$

Now we consider the goodness of β-compactness.

For $a \in L$ and a crisp subset $D \subset X$, we define $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \qquad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ a, & x \notin D. \end{cases}$$

Theorem 5.5 ([**14**]) **:**For an L-set $A \in L^X$, the following facts are true.
(1) $A = \bigvee_{a \in L} (a \wedge A_{(a)}) = \bigvee_{a \in L} (a \wedge A_{[a]})$.
(2) $A = \bigwedge_{a \in L} (a \vee A^{(a)}) = \bigwedge_{a \in L} (a \vee A^{[a]})$.

Theorem 5.6 ([**15**])Let $(X, \omega_L(\tau))$ be the L-space topologically generated by (X, τ) and $A \in L^X$. Then the following facts hold.
(1) $cl(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^-) = \bigvee_{a \in L} (a \wedge (A_{[a]})^-)$;
(2) $cl(A)_{(a)} \subset (A_{(a)})^-) \subset (A_{[a]})^-) \subset cl(A)_{[a]}$;
(3) $cl(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^-) = \bigwedge_{a \in L} (a \vee (A^{[a]})^-)$;
(4) $cl(A)^{(a)} \subset (A^{(a)})^-) \subset (A^{[a]})^-) \subset cl(A)^{[a]}$;
(5) $int(A) = \bigvee_{a \in L} (a \wedge (A_{(a)})^\circ) = \bigvee_{a \in L} (a \wedge (A_{[a]})^\circ)$;
(6) $int(A)_{(a)} \subset (A_{(a)})^\circ) \subset (A_{[a]})^\circ) \subset int(A)_{[a]}$;
(7) $int(A) = \bigwedge_{a \in L} (a \vee (A^{(a)})^\circ) = \bigwedge_{a \in L} (a \vee (A^{[a]})^\circ)$;
(8) $int(A)^{(a)} \subset (A^{(a)})^\circ) \subset (A^{[a]})^\circ) \subset int(A)^{[a]}$,
where $(A_{(a)})^-$ and $(A_{(a)})^\circ$ denote respectively the closure and the interior of $A_{(a)}$ in (X, τ), and $cl(A)$ and $int(A)$ denote respectively the closure and the interior of A in $(X, \omega_L(\tau))$.

Lemma 5.7 : Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). If A is an β-open L-set in (X, τ), Then $\mathcal{X}_A$ is an β-open set in $(X, \omega_L(\tau))$. If B is an β-open L-set in $(X, \omega_L(\tau))$, Then $B_{(a)}$ is an β-open set in (X, τ) for every $a \in L$.

Proof. If A is a β-open set in (X,τ), then $A \subseteq ((A^-)^\circ)^-$. Thus we have that

$$\mathcal{X}_A \leq \mathcal{X}_{((A^-)^\circ)^-} = cl(\mathcal{X}_{(A^-)^\circ}) = cl(int(\mathcal{X}_{A^-})) = cl(int(cl(_A))).$$

This shows that $_A$ is β-open in $(X,\omega_L(\tau))$.

If B is a β-open L-set in $(X,\omega_L(\tau))$, then $B \leq cl(int(cl(B)))$. From Theorem 4.2 we have that

$$B_{(a)} \subseteq cl(int(cl(B)))_{(a)} \subseteq (int(cl(B))_{(a)})^- \subseteq ((cl(B)_{(a)})^\circ)^- \subseteq (((B_{(a)})^-)^\circ)^-.$$

This shows that $B_{(a)}$ is a β-open set in (X,τ).

The following two theorems show that β-compactness, countable β-compactness and the β-Lindelöf property are good extensions.

Theorem 5.8 : Let $(X,\omega_L(\tau))$ be generated topologically by (X,τ). Then $(X,\omega_L(\tau))$ is (countably) β-compact if and only if (X,τ) is (countably) β-compact.

Proof : (Necessity) Let $\mathcal{A}$ be an (a countable) β-open cover of (X,τ). Then $\{_A|\ A \in \mathcal{A}\}$ is a family of β-open L-sets in $(X,\omega_L(\tau))$ with $\bigwedge_{x\in X}\left(\bigvee_{A\in\mathbf{U}} {}_A(x)\right) = 1$. From (countable) β-compactness of $(X,\omega_L(\tau))$ we know that

$$1 \geq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} {}_A(x)\right) \geq \bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{U}} {}_A(x)\right) = 1.$$

This implies that there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} {}_A(x)\right) = 1$. Hence $\mathcal{V}$ is a cover of (X,τ). Therefore (X,τ) is (countably) β-compact.

(Sufficiency) Let $\mathcal{U}$ be a (countable) family of β-open L-sets in $(X,\omega_L(\tau))$ and let $\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right) = a$. If $a = 0$, then obviously we have that

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}} B(x)\right).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a)\backslash\{0\}$ we have that

$$b \in \beta\left(\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right)\right) \subseteq \bigcap_{x\in X} \beta\left(\bigvee_{B\in\mathcal{U}} B(x)\right) = \bigcap_{x\in X}\bigcup_{B\in\mathcal{U}} \beta\left(B(x)\right).$$

From Lemma 5.7 this implies that $\{B_{(b)} \mid B \in \mathcal{U}\}$ is a β-open cover of (X,τ). From (countable) β-compactness of (X,τ) we know that there exists $\mathcal{V} \in 2^{(\mathcal{U})}$ such that $\{B_{(b)} \mid B \in \mathcal{V}\}$ is a cover of (X,τ). Hence $b \leq \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} B(x)\right)$. Further we have that

$$b \leq \bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} B(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} B(x)\right).$$

This implies that

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}} B(x)\right) = a = \bigvee\{b \mid b \in \beta(a)\} \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}} B(x)\right).$$

Therefore $(X, \omega_L(\tau))$ is (countably) β-compact.

Analogously we have the following result.

Theorem 5.1. *Let $(X, \omega_L(\tau))$ be generated topologically by (X,τ). Then $(X, \omega_L(\tau))$ has the β-Lindelöf property if and only if (X,τ) has the β-Lindelöf property.*

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