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TOPOLOGICAL CLASSIFICATION OF COMPACT SURFACES WITH NODES OF GENUS 2

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Abstract

We associate to each Riemann or Klein surface with nodes a graph that classifies it up homeomorphism. We obtain that, for surfaces of genus two, there are 7 topological types of stable Riemann surfaces, 33 topological types of stable Klein surfaces and 35 topological types of symmetric stable Riemann surfaces (this last type of surfaces corresponds to the new surfaces appearing in the compactification of the Moduli space of real algebraic curves, see [Se] and [Si]).

Riemann surfaces with nodes appear as compactification points of the Deligne-Mumford compactification of the moduli space of compact and connected smooth complex curves of given genus [D-M]. These compactifications are a useful tool in recent important progress in mathematics. Concretely, the proof of Witten's conjec- ture given by M. Kontsevich (see [K], [L-Z] and [Z]) uses, in an essential way, the cell decomposition of the spaces $\overline{\mathcal{M}}_{g,n} \times \mathbf{R}^n_+$, where $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford compactification by Riemann surfaces with nodes of the moduli space of compact and connected smooth complex curves of genus g with n marked points.

In this paper we assign topological invariants to each Riemann and Klein surface with nodes in order to classify them topologically. Moreover, using such classification, we compute how many stable surfaces of genus 2 (Riemann surfaces, Klein surfaces and symmetric Riemann surfaces) there exist.

We assign a pseudomultidigraph with weights in its vertices to each Riemann surface with nodes. The vertices of the graph correspond to the parts of the surface and the edges of the graph correspond to the nodes of the surface. Such a graph will be called the graph of the Riemann surface with nodes. Similarly, we assign a pseudomultidigraph with weights in its vertices and edges to each Klein surface with nodes and such graph determines completely its topological type. In this case, the vertices of the graph correspond to the parts and inessential nodes of the surface and the edges of the graph correspond to the boundary and conic nodes. Using this classification we conclude that there are 7 topological types of stable Riemann surfaces of genus 2.

Finally, studying the different involutions of these stable Riemann surfaces, we prove that there exist 33 topological types of stable Klein surfaces and 35 topological types of symmetric stable Riemann surfaces of genus 2. We only study stable surfaces and no other surfaces with nodes because these surfaces appear in the Deligne-Mumford compactification of the moduli space of complex algebraic curves [D-M].

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1. Riemann and Klein surfaces with nodes

We shall define the main concepts and hence we fix the notation. All the definitions and properties we deal with in this section are exhaustively defined in [G2].

A surface with nodes is a pair $S = (\Sigma, \mathcal{D})$ where Σ is a topological Haussdorff space, $\mathcal{D} \subset \Sigma$ is a discrete set of distinguished points of Σ, and each point z of Σ has a chart (U_i, φ_i) where U_i is a neighbourhood of z and φ_i is a homeomorphism from U_i to one of the following sets:

1) An open subset of $\mathbf{C}$,

2) An open subset of $\mathbf{C}^+ = \{z \in C \mid \quad \mathrm{Im}\,(z) \geq 0\}$,

3) $\mathcal{M} = \{(z, w) \in \mathbf{C}^2 \mid \quad z \cdot w = 0, \quad |z| < 1, \quad |w| < 1\}$,

4) $\mathcal{M}^+ = \left\{(z, w) \in (\mathbf{C}^+)^2 \mid \quad z \cdot w = 0, \quad |z| < 1, \quad |w| < 1\right\}$.

Moreover, if $z \in \mathcal{D}$, then U_i is homeomorphic to an open set of $\mathbf{C}$.

If $\{(U_i, \varphi_i)\}_{i \in I}$ is an atlas of Σ, i.e. a collection of charts such that $\cup_{i \in I} U_i = \Sigma$, then we define the boundary of Σ as

$$\partial\Sigma = \{z \in \Sigma \mid$$

there is $i \in I$ with $z \in U_i$, $\varphi_i\,(U_i) \subset \mathbf{C}^+$ and $\varphi_i\,(z) \in R = \partial C^+ \cup \cup \{z \in \Sigma \mid$ there is $i \in I$ with $z \in U_i$, $\varphi_i\,(U_i) \subset (\mathbf{C}^+)^2$ and $\varphi_i\,(z) \in (\partial\,(C^+))^2$.

If $z \in \Sigma$ and there is $i \in I$ with $z \in U_i$, $\varphi_i\,(U_i) = \mathcal{M}$ and $\varphi_i\,(z) = (0, 0)$ then we say that z is a conic node. We denote the set of conic nodes by $N\,(\Sigma, 1)$.

If $z \in \mathcal{D}$ then we say that z is an inessential node. We denote the set of inessential nodes by $N\,(\Sigma, 2)$.

If $z \in \Sigma$ and there is $i \in I$ with $z \in U_i$, $\varphi_i\,(U_i) = \mathcal{M}^+$ and $\varphi_i\,(z) = (0, 0)$ then we say that z is a boundary node. We denote the set of boundary nodes by $N\,(\Sigma, 3)$.

If z belongs to $N\,(\Sigma) = N\,(\Sigma, 1) \cup N\,(\Sigma, 2) \cup N\,(\Sigma, 3)$ then we say that z is a node.

Finally, we call part of Σ each connected component of $\Sigma \setminus N\,(\Sigma)$.

We say that two charts, (U_i, φ_i), (U_j, φ_j), have analytic (resp. dianalytic) transition if $U_i \cap U_j \neq \emptyset$ and the transition map

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i\,(U_i \cap U_j) \to \varphi_j\,(U_i \cap U_j)$$

is analytic (resp. dianalytic) in the image under φ_i of the complementary of the nodes. A Riemann surface with nodes is a pair $X = (\Sigma, \mathcal{U})$ where $S = (\Sigma, \emptyset)$ is a surface with nodes without boundary and $\mathcal{U}$ is an analytic and maximal atlas of Σ. A stable Riemann surface is a Riemann surface with nodes whose parts have negative Euler characteristic. The reader can observe that $N(\Sigma) = N(\Sigma, 1)$. In this case, we define the genus of Σ by:

$$g(\Sigma) = \frac{1}{2}(2 + \#N(\Sigma) - \chi(\Sigma)).$$

A Klein surface with nodes is a triple $X = (\Sigma, \mathcal{D}, \mathcal{U})$ where $(\Sigma, \mathcal{D})$ is a surface with nodes and $\mathcal{U}$ is a dianalytic and maximal atlas of Σ.

Let X be a Riemann or Klein surface with nodes where $N(\Sigma) = \{z_i\}_{i\in I}$. Let $f_i : U_i \to V_i$ be the charts, with

$$V_i = V_{i,1} \underset{w_i}{\sqcup} V_{i,2} = (V_{i,1} \times \{1\} \cup V_{i,2} \times \{2\}) / \sim$$

where $\sim$ is the identification $(w_i, 1) \sim (w_i, 2)$. We denote by

$$U_i = U_{i,1} \underset{z_i}{\sqcup} U_{i,2}$$

with $U_{i,k} = f_i^{-1}(V_{i,k}). We take \Sigma \backslash N(\Sigma)$ and construct $\widehat{X} = \left(\widehat{\Sigma}, \widehat{\mathcal{U}}\right)$ identifying $U_{i,k} \backslash \{z_i\}$ with $V_{i,k}$ using f_i and we assign charts in the obvious way. We obtain a Riemann or Klein surface, $\widehat{X}$, that, in general, is not connected and we call it the normalization of the Riemann or Klein surface with nodes X. The projection

$$p: \begin{array}{ccc} \widehat{\Sigma} & \to & \Sigma \\ z & \to & z \end{array}$$

is an identification map and $\#p^{-1}(z) = 2$ if and only if $z \in N(\Sigma, 1) \cup N(\Sigma, 3)$.

A map $f : \Sigma_1 \to \Sigma_2$ between Riemann or Klein surfaces with nodes is a continuous map such that $f(\partial\Sigma_1) \subset \partial\Sigma_2$ and $f^{-1}(N(\Sigma_2, i)) \subset N(\Sigma_1, 1) \cup N(\Sigma_1, i)$. This map induces a unique continuous map $\widehat{f} : \widehat{\Sigma_1} \to \widehat{\Sigma_2}$ satisfying $f \circ p_1 = p_2 \circ \widehat{f}$ and is called the lifting of f.

A map $f : \Sigma_1 \to \Sigma_2$ between Riemann or Klein surfaces with nodes is complete if $\widehat{f}\left(p_1^{-1}(z)\right) = p_2^{-1}(f(z))$. Finally, a morphism between Riemann (resp. Klein) surfaces with nodes is an analytic (resp. dianalytic) and complete map $f : X_1 \to X_2$ between Riemann (resp. Klein) surfaces with nodes. We have that f is analytic (resp. antianalytic, dianalytic, ...) if and only if $\widehat{f}$ is analytic (resp. antianalytic, dia- nalytic, ...). The homeomorphisms, isomorphisms and automorphisms are defined in the obvious way and we denote them by $Homeo(X_1, X_2)$, $Iso(X_1, X_2)$ and $Aut(X)$.

A symmetric Riemann surface with nodes is a pair (X, σ) where X is a Riemann surface with nodes and $\sigma : X \to X$ is an antianalytic involution.

A map (resp. ho- meomorphism, morphism, antianalytic morphism,...) $f:(X_1,\sigma_1)\rightarrow(X_2,\sigma_2)$ between symmetric Riemann surfaces with nodes is a map (resp. homeomorphism, morphism, antianalytic morphism,...) $f:X_1\rightarrow X_2$ satisfying $f\circ\sigma_1=\sigma_2\circ f$.

We know that if X is a Klein surface, then there exists a triple (X_c,π_c,σ_c) such that (X_c,σ_c) is a symmetric Riemann surface and $\pi_c:X_c\rightarrow X$ is an unbranched double cover satisfying $\pi_c\circ\sigma_c=\pi_c$. If (X_c',π_c',σ_c') is another triple with the same property, then there exists a unique analytic isomorphism $f:(X_c',\sigma_c')\rightarrow(X_c,\sigma_c)$ between symmetric Riemann surfaces such that $\pi_c'=\pi_c\circ f$. This triple, that is unique up to isomorphism, is called the complex double of X.

Let Y be a Klein surface with nodes. We say that a triple (X,π,σ) is a complex double of Y if (X,σ) is a symmetric Riemann surface with nodes and $\pi:X\rightarrow Y$ is an unbranched double cover satisfying $\pi\circ\sigma=\pi$ and $\pi\left(N\left(X\right)\right)=N\left(Y\right)$. We say that two complex doubles, (X_1,π_1,σ_1), (X_2,π_2,σ_2), of Y are isomorphic if there is an isomorphism $f:(X_1,\sigma_1)\rightarrow(X_2,\sigma_2)$ between symmetric Riemann surfaces with nodes such that $\pi_2\circ f=\pi_1$.

If we have a Klein surface with nodes X, then there exist exactly $2^{\#N(X,1)}$ triples (X_c,π_c,σ_c) that are non-isomorphic complex doubles of X (see [G2]). We say that X is a stable Klein surface if any one of its complex doubles is a stable symmetric Riemann surface. We define the algebraic genus of X as the genus of any one of its complex doubles.

2. Topological classification

Let Σ be a compact Riemann surface with nodes and let $\widehat{\Sigma}=\underset{i\in I}{\cup}\widehat{\Sigma}_i$ be its resolution, where $\widehat{\Sigma}_i$ are the connected components of $\widehat{\Sigma}$. We are going to construct a graph with weights in its vertices:

- Let us assign a vertex to each $\widehat{\Sigma}_i$ with weight $g\left(\widehat{\Sigma}_i\right)$=genus of $\widehat{\Sigma}_i$.
- Let us assign an edge to each node $z\in N\left(\Sigma\right)$ connecting the vertices assigned to $\widehat{\Sigma}_{i_1}$, $\widehat{\Sigma}_{i_2}$ if $p^{-1}\left(z\right)\cap\widehat{\Sigma}_{i_j}\neq\emptyset$; it can happen that $i_1=i_2$ (i. e. loops can appear).

This graph is called the graph of Σ and we denote it by $G\left(\Sigma\right)$.

Theorem 2.1. Let Σ_1, Σ_2 be two compact Riemann surfaces with nodes and let $G\left(\Sigma_1\right),G\left(\Sigma_2\right)$ be its graphs. Σ_1 is homeomorphic to Σ_2

if and only if $G(\Sigma_1) = G(\Sigma_2)$. Conversely, every pseudomultigraph with natural weights in its vertices defines a unique topological type of Riemann surfaces with nodes.

Now, we shall classify Klein surfaces with nodes. Let us recall some theorems and definitions about Klein surfaces without nodes (see [A-G]). Let X be a compact and connected Klein surface without nodes, $\chi(X)$ its Euler characteristic, $k(X)$ the number of connected components of ∂X and

$$\varepsilon(X) = \begin{cases} 0 & \text{if } X \text{ is orientable,} \\ 1 & \text{if } X \text{ is non orientable.} \end{cases}$$

Then, the triple $(\chi(X), k(X), \varepsilon(X))$ classifies X topologically.

Let X be a connected and compact Klein surface, the topological genus of X is:

$$g_t(X) = \begin{cases} \frac{1}{2}(2 - \chi(X) - k(X)) & \text{if } X \text{ is orientable,} \\ 2 - \chi(X) - k(X) & \text{if } X \text{ is non orientable.} \end{cases}$$

The algebraic genus of X is $g(X) = 1 - \chi(X)$ and the topological type of X is the triple $(g(X), k(X), \varepsilon(X))$. The topological type of X classify topologically X. Conversely, Harnack's Theorem says that there exists a Klein surface with topological type (g, k, ε) if and only if:

$$\begin{array}{ll} 1 \leq k \leq g+1, \quad k \equiv g+1 \ mod.\ 2 & \text{if } \varepsilon = 0 \\ \qquad 0 \leq k \leq g & \text{if } \varepsilon = 1. \end{array}$$

Let X be a compact Klein surface with nodes and let $\widehat{X} = \cup_{i=1}^{n} \widehat{X}_i$ be its resolution, where $\widehat{X}_i$ are the connected components of $\widehat{X}$. We shall construct a pseudomultidigraph with weight in its vertices and in its edges:

- Let us assign a vertex to each $\widehat{X}_i$, with weight (g, r) if $\widehat{X}_i$ is a Riemann surface of genus g with r inessential nodes, or weight (g, k, ε, r) if $\widehat{X}_i$ is a Klein surface with topological type (g, k, ε) and r inessential nodes.

- If $\widehat{X}_i$ is orientable and with boundary, then we fix an orientation on $\widehat{X}_i$ and then we have oriented the connected components of the boundary. If, on the contrary, $\widehat{X}_i$ is non orientable and with boundary, then we orient the connected components of the boundary arbitrarily. We also number its connected components.

- If one connected component of the boundary of $\widehat{X}_i$ contains boundary nodes, then we number these boundary nodes following the orientation.

- Let us assign an edge with weight (n_1, m_1, n_2, m_2) and direction $\overrightarrow{AB}$ to each boundary node if this node connects the node m_1, of the connected component n_1 of A, with the node m_2 of the connected component n_2 of B.
- Let us assign an edge with weight $(0,0)$ and an arbitrary direction to each conic node connecting the vertices corresponding to the connected components that this node conects.

Remark 2.2. Constructing the previous pseudomultidigraph we have made the following choices:

- If $\widehat{X}_i$ is orientable and with boundary, the choice of the orientation of $\widehat{X}_i$.
- If $\widehat{X}_i$ is non orientable and with boundary, the choice of the orientations of the connected components of the boundary.
- The numbering of the connected components of the boundary of $\widehat{X}_i$.
- The first boundary node of each connected component of the boundary of $\widehat{X}_i$.
- The direction of the edge with weight $(0,0)$.
- The edge $\overrightarrow{AB}$ with weight (n_1, m_1, n_2, m_2) is the same that the edge $\overrightarrow{BA}$ with weight (n_2, m_2, n_1, m_1).

Each compact Klein surface with nodes, X, defines different pseudomultidigraphs depending on the previous choices. These choices define an equivalence relation in the set of pseudomultidigraphs with weights in the vertices and in the edges. We call the graph of X to the equivalence class of its pseudomultidigraphs, and we denote it by $G(X)$.

Theorem 2.3. Let be X_1, X_2 two compact Klein surfaces with nodes and let $G(X_1)$, $G(X_2)$ be its graphs. X_1 is homeomorphic to X_2 if and only if $G(X_1) = G(X_2)$. Conversely, every pseudomultidigraph defines a unique topological type of Klein surface with nodes provided that the following two conditions are satisfied::

1. The weights of each vertex is either of type (g, r) or (g, k, ε, r). If the weight is (g, k, ε, r) then (g, k, ε) satisfies the conditions of Harnack's Theorem.

2. The weights of each edge is either $(0,0)$ or (n_1, m_1, n_2, m_2). If the weight of the edge $\overrightarrow{A_1A_2}$ is (n_1, m_1, n_2, m_2), then the weight of the vertex A_i is $(g_i, k_i, \varepsilon_i, r_i)$ and $n_i \leq k_i$.

defines a unique topological type of Klein surfaces with nodes. □

3. Compact stable Riemann surfaces of genus 2

Let X be a compact and connected Riemann surface with nodes, let $\widehat{X} = \cup_{i=1}^{s} \widehat{X_i}$ be its resolution, let $G(X)$ be its graph and let $N = \#N(X)$. As X is connected, then $G(X)$ is also connected and then $s \leq N+1$ because s is the number of vertices of $G(X)$ and N is the number of edges of $G(X)$. Let $g_i = g\left(\widehat{X_i}\right)$, since $\chi(X) = \sum_{i=1}^{s} \chi\left(\widehat{X_i}\right) - N$ then

$$g(X) = \frac{1}{2}(2 + N - \chi(X)) = \frac{1}{2}\left(2 + N - \left(\sum_{i=1}^{s} \chi\left(\widehat{X_i}\right) - N\right)\right) =$$

$$= \frac{1}{2}\left(2 + 2N - \sum_{i=1}^{s} 2(1 - g_i)\right) = 1 + N - \sum_{i=1}^{s}(1 - g_i) = 1 + N - s + \sum_{i=1}^{s} g_i.$$

If X_i is a part of X and $n_i = \#\left(\widehat{X_i} \backslash p^{-1}(X_i)\right)$, since $\chi(X_i) = 2 - 2g_i - n_i$, then $\chi(X_i) < 0$ if and only if $g_i + \frac{1}{2}n_i > 1$. Hence $\sum_{i=1}^{s}\left(g_i + \frac{1}{2}n_i\right) > s$ and $\sum_{i=1}^{s} g_i + N - s \geq 1$. Since $g(X) = 1 + N - s + \sum_{i=1}^{s} g_i$, then $g(X) \geq 2$. With those inequalities in mind we can prove:

Proposition 3.1. Let X be a connected and compact stable Riemann surface and let X_i be a part of X. Then,

a) $\chi(X_i) > 0$ if and only if $g(X) = 0$.
b) $\chi(X_i) = 0$ if and only if $g(X) = 1$.
c) $\chi(X_i) < 0$ if and only if $g(X) \geq 2$.

A pant is a topological surface homeomorphic to $\mathbf{S}^2$ with 3 closed discs or points removed.

A pant decomposition of a Riemann surface with nodes Σ is a collection $\{C_i\}_{i\in I}$ with $C_i \subset \Sigma$, C_i is homeomorphic to either $\mathbf{S}^1$ or a point, $C_i \cap C_j = \emptyset$ and $\Sigma \backslash \bigcup_{i\in I} C_i$ is a disjoint union of pants. We shall call C_i the curves of the decomposition.

It is easy to prove that if X is a compact Riemann surface and $\{z_i\}_{i=1}^{n} \subset X$, then there exists a pant decomposition of X where all the z_i are curves of the decomposition. Using this idea and the projection $p : \widehat{X} \to X$, it is easy to prove:

Proposition 3.2. Let X be a compact and connected Riemann surface with nodes. Then X admits a pant decomposition where the nodes are curves of the decomposition. Moreover, if X is stable and $g(X) \geq 2$ then the curves of the decomposition are homeomorphic to either $\mathbf{S}^1$ or nodes and the number of pants is $2g(X) - 2$.

An easy calculation proves that each pant decomposition has $3g(X) - 3$ curves of the decomposition and then $N \leq 3g(X) - 3$. If $N = 3g(X) - 3$ this surface is called terminal.

Let X be a connected and compact stable Riemann surface of genus $g(X) = 2$. We have that:

$$2 = \sum_{i=1}^{s} g_i + 1 + N - s, \qquad 1 \leq s \leq 2, \qquad N \leq 3 \qquad \text{and} \qquad s \leq N + 1.$$

Then we have seven cases:

1. If $N = 0$, $s = 1$, then $g_1 = 2$ and

$$G(X) = \quad .$$

2. If $N = 1$, $s = 1$, then $g_1 = 1$ and

$$G(X) = \quad .$$

3. If $N = 1$, $s = 2$, then $g_1 + g_2 = 2$. If $g_1 = 0$, as $N = 1$, then X is not stable. Hence $g_1 = g_2 = 1$ and

$$G(X) = \quad .$$

4. If $N = 2$, $s = 1$, then $g_1 = 0$ and

$$G(X) = \quad .$$

5. If $N = 2$, $s = 2$, then $g_1 + g_2 = 1$. Hence $g_1 = 0$, $g_2 = 1$ and

$$G(X) = \quad .$$

6. If $N = 3$, $s = 1$, then $g_1 = -1$ and impossible case.
7. If $N = 3$, $s = 2$, then $g_1 = g_2 = 0$ and there are two possibilities:

$$G(X) = \quad \text{or } G(X) = \quad .$$

We have obtained the following theorem:

Theorem 3.3. There are 7 topological types of compact and connected stable Riemann surfaces of genus 2 whose graphs are:

4. Compact stable Klein surfaces of genus 2 and their complex doubles

As in the previous section we shall classify all compact and connected stable Klein surfaces and symmetric stable Riemann surfaces of genus 2.

Theorem 4.1. Let (X, φ) be a compact and connected symmetric stable Riemann surface with graph

$$G\,(X) = \qquad .$$

Then $G\,(X/\varphi)$ is one of the following five graphs:

Moreover, there exists five non homeomorphic stable symmetric Riemann surfaces whose graph is $G\,(X)$.

Theorem 4.2. Let be (X, φ) a compact and connected symmetric stable Riemann surface with graph:

$$G\,(X) = \qquad .$$

Then $G\,(X/\varphi)$ is one of the following four graphs:

Moreover, there exists four non homeomorphic stable symmetric Riemann surfaces whose graph is $G(X)$.

Proof. Let $\widehat{X} = \widehat{X_1} \cup \widehat{X_2}$ be with $g\left(\widehat{X_i}\right) = 1$ and $N(X) = \{A\}$.

If $\widehat{\varphi}\left(\widehat{X_1}\right) = \widehat{X_2}$, then $\widehat{X}/\widehat{\varphi} \simeq \widehat{X_1}$, $\pi(A)$ is an inessential node and we have the first graph.

If $\widehat{\varphi}\left(\widehat{X_i}\right) = \widehat{X_i}$, then $\varphi(A) = A$ and $\pi(A)$ is a boundary node. Then $\widehat{\varphi}\,|_{\widehat{x_i}}$ has fixed points and its topological type is $(1,2,0)$ or $(1,1,1)$. Therefore we have the other two graphs.

As these Klein surfaces with nodes do not contain conic nodes, then they have a unique complex double that is (X, φ). $\square$

Theorem 4.3. Let (X, φ) be a compact and connected symmetric stable Riemann surface with graph:

$$G(X) = \qquad .$$

Then $G(X/\varphi)$ is one of the following six graphs:

Moreover, there exists six non homeomorphic stable symmetric Riemann surfaces whose graph is $G(X)$.

Proof. In this situation $g\left(\widehat{X}\right) = 1$ and we denote $N(X) = \{A\}$. Then, $\widehat{\varphi}\left(\widehat{X}\right) = \widehat{X}$, $\varphi(A) = A$ and $\pi(A)$ is an inessential or boundary node. We distinguish two cases:

- If $\pi(A)$ is an inessential node, then the topological type of $\widehat{X}/\widehat{\varphi}$ is $(1,2,0)$, $(1,0,1)$ or $(1,1,1)$ and then $G(X/\varphi)$ is one of the first three graphs.

- If $\pi(A)$ is a boundary node, then the topological type of $\widehat{X}/\widehat{\varphi}$ is $(1,2,0)$ or $(1,1,1)$ and then $G(X/\varphi)$ is one of the last three graphs.

As these Klein surfaces with nodes do not contain conic nodes, then they have a unique complex double(X,φ). $\square$

Theorem 4.4. Let (X,φ) be a compact and connected symmetric stable Riemann surface with graph:

$$G(X) = \qquad\qquad .$$

Then $G(X/\varphi)$ is one of the following four graphs:

Moreover, there exists four non homeomorphic stable symmetric Riemann surfaces whose graph is $G(X)$.

Proof. Let $\widehat{X} = \widehat{X_1} \cup \widehat{X_2}$ where $g\left(\widehat{X_1}\right) = 1$ and $g\left(\widehat{X_2}\right) = 0$We denote $N(X) = \{A, B\}$,A being the node corresponding to the loop and B the node connecting the two parts.

Since $g\left(\widehat{X_1}\right) \neq g\left(\widehat{X_2}\right)$, then $\widehat{\varphi}\left(\widehat{X_i}\right) = \widehat{X_i}$. As B connects the two parts of X, we have that $\varphi(B) = B$ and $\pi(B)$ is a boundary node. Then, $\widehat{\varphi}\mid_{\widehat{X_i}}$ haz fixed points and the topological type of $\widehat{X_1}/\widehat{\varphi}$ is $(1,2,0)$ or $(1,1,1)$ and the topological type of $\widehat{X_2}/\widehat{\varphi}$ is $(0,1,0)$. The node $\pi(A)$ could be a boundary or inessential node; thus, combining these possibilities, we obtain the 4 possible graphs of X/φ.

As these Klein surfaces with nodes do not contain conic nodes, then they have a unique complex double that is (X,φ). $\square$

Example 4.5. Let $\widehat{X} = \widehat{\mathbf{C}}$ and

$$\begin{array}{rccl} \widehat{\varphi}: & \widehat{X} & \rightarrow & \widehat{X} \\ & z & \rightarrow & \widehat{\varphi}(z) = -\frac{1}{\overline{z}}. \end{array}$$

Then, $\left(\widehat{X}, \widehat{\varphi}\right)$ is a symmetric Riemann surface.

We define $X_1 = \widehat{X}/\sim_1$ where $\sim_1$ is defined using the identifications

$$\frac{1}{3}i \sim_1 2i, \quad -3i \sim_1 -\frac{1}{2}i,$$

and $X_2 = \widehat{X}/\sim_2$ where $\sim_2$ is defined using the identifications

$$\frac{1}{3}i \sim_2 -\frac{1}{2}i, \quad -3i \sim_2 2i.$$

As $\widehat{\varphi}$ is compatible with $\sim_j$, then there exist $\varphi_j : X_j \rightarrow X_j$ with lifting $\widehat{\varphi}$. Therefore, (X_j, φ_j) is a symmetric Riemann surface with nodes whose resolution is $\left(\widehat{X}, \widehat{\varphi}\right)$.

We take $Y = X_1/\varphi_1 = X_2/\varphi_2 = \widehat{\mathbf{C}}^+/\sim_3$
where $\sim_3$ is defined using the identifications:

$$\frac{1}{3}i \sim_3 2i \text{ y } r \sim_3 -\frac{1}{r} \quad \forall r \in \widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}.$$

Hence, Y is a Klein surface with nodes,

$$G(Y) =$$

and (X_1, π_1, φ_1), (X_1, π_2, φ_1) are the two double covers of Y. We define

$$\begin{array}{rccl} \widehat{f}: & \widehat{X} & \rightarrow & \widehat{X} \\ & z & \rightarrow & \widehat{f}(z) = \frac{iz+1}{z+i}. \end{array}$$

Hence, $\widehat{f} \circ \widehat{\varphi}(z) = \widehat{\varphi} \circ \widehat{f}(z) = \frac{\overline{z}-i}{-\overline{z}-1}$ and $\widehat{f} : \left(\widehat{X}, \widehat{\varphi}\right) \rightarrow \left(\widehat{X}, \widehat{\varphi}\right)$ is an isomorphism between symmetric Riemann surfaces. Moreover,

$$\widehat{f}\left(\frac{1}{3}i\right) = -\frac{1}{2}i, \quad \widehat{f}(2i) = \frac{1}{3}i, \quad \widehat{f}(-3i) = 2i, \quad \widehat{f}\left(-\frac{1}{2}i\right) = -3i.$$

Then there exists an isomorphism $f : (X_1, \varphi_1) \to (X_2, \varphi_2)$ with $\widehat{f}$ as its lifting. Thus the two double covers of Y are isomorphic as symmetric Riemann surfaces with nodes.

Theorem 4.6. Let (X, φ) be a compact and connected symmetric stable Riemann surface with graph:

$$G(X) = \qquad .$$

Then $G(X/\varphi)$ is one of the following seven graphs:

Moreover, there exists eight non homeomorphic stable symmetric Riemann surfaces whose graph is $G(X)$.

Proof. In this situation $g\left(\widehat{X}\right) = 0$ and $N(X) = \{A, B\}$. Then, $\widehat{\varphi}\left(\widehat{X}\right) = \widehat{X}$ and $\widehat{X}/\widehat{\varphi}$ has topological type $(0,1,0)$ or $(0,0,1)$. We have four possibilities:

- $\varphi(A) = B$. Then, $\pi(A) = \pi(B)$ is a conic node and we obtain the first two graphs.
- $\pi(A)$ and $\pi(B)$ are inessential nodes. Then we obtain the third and fourth graphs.
- $\pi(A)$ and $\pi(B)$ are boundary nodes. Then, $\widehat{\varphi}$ has fixed points and the topological type of $\widehat{X}/\widehat{\varphi}$ is $(0,1,0)$, and we obtain the fifth and sixth graphs.
- $\pi(A)$ is a boundary node and $\pi(B)$ is an inessential node. In this case, $\widehat{\varphi}$ has fixed points and the topological type of $\widehat{X}/\widehat{\varphi}$ is $(0,1,0)$. Then we obtain the last graph.

Moreover, the two double covers corresponding to the first graph are not homeomorphic as symmetric Riemann surfaces with nodes (see [G2]) and the two double covers corresponding to the second graph are homeomorphic as symmetric Riemann surfaces with nodes (previous example). The other graphs do not contain conic nodes and then they only have one double cover, that is (X, φ). Then, there exists 8 symmetric Riemann surfaces with nodes non homeomorphic with this graph. $\square$

Theorem 4.7. Let (X, φ) be a compact and connected symmetric stable Riemann surface with graph:

$$G(X) = \qquad .$$

Then $G(X/\varphi)$ is one of the following four graphs:

Moreover, there exists four non homeomorphic stable symmetric Riemann surfaces whose graph is $G(X)$.

Proof. Let $\widehat{X} = \widehat{X_1} \cup \widehat{X_2}$ where $g\left(\widehat{X_1}\right) = g\left(\widehat{X_2}\right) = 0$, and let $N(X) = \{A_1, A_2, B\}$, where A_i is the node corresponding to the loop of $\widehat{X_i}$ and B is the node connecting the two connected components of $\widehat{X}$. We have two possibilities:

- If $\widehat{\varphi}\left(\widehat{X_1}\right) = \widehat{X_2}$, then $\widehat{X}/\widehat{\varphi} \simeq \widehat{X_1}$, $\varphi(A_1) = A_2$ and $\varphi(B) = B$. Thus, $\pi(A_1) = \pi(A_2)$ is a conic node and $\pi(B)$ is an inessential node, and we obtain the first graph.

- If $\widehat{\varphi}\left(\widehat{X_i}\right) = \widehat{X_i}$, then $\varphi(B) = B$ and $\pi(B)$ is a boundary node. Thus, $\widehat{\varphi}\mid_{\widehat{X_i}}$ has fixed points and the topological type of $\widehat{X_i}/\widehat{\varphi}$ is $(0,1,0)$. Moreover, $\varphi(A_i) = A_i$ and $\pi(A_i)$ could be an inessential or boundary node. Thus we obtain the last three graphs.

The Klein surface with nodes corresponding to the first graph has two double covers, but only one of them satisfies that its graph is $G(X)$ (see [G2]). The other graphs do not contain conic nodes and then they have only one double cover, that is (X, φ). Then, there exists 4 symmetric Riemann surfaces with nodes non homeomorphic with this graph. □

Example 4.8. Let $\widehat{X} = \widehat{\mathbf{C}}^{+} \times \{1\} \cup \widehat{\mathbf{C}}^{+} \times \{2\}$. We define on $\widehat{X}$ the following identifications

$$(0,1) \sim (0,2)\,,\ (i,1) \sim (i,2)\,,$$

and we define $X = \widehat{X}/\sim$. Hence X is a Klein surface with nodes, $\widehat{X}$ is its resolution and

$$G(X) = \qquad\qquad .$$

Let $\widehat{X_c} = \widehat{\mathbf{C}} \times \{1\} \cup \widehat{\mathbf{C}} \times \{2\}$ and we define

$$\begin{array}{cccc} \widehat{\varphi_c}: & \widehat{X_c} & \to & \widehat{X_c} \\ & (z,n) & \to & (\overline{z},n)\,, \end{array}$$

$$\begin{array}{cccc} \widehat{\pi_c}: & \widehat{X_c} & \to & \widehat{X} \\ & (z,n) & \to & (\phi(z)\,,n) \end{array}$$

where $\phi(z) = \operatorname{Re} z + |\operatorname{Im} z|\sqrt{-1}$. Then $\left(\widehat{X_c}, \widehat{\pi_c}, \widehat{\varphi_c}\right)$ is the double cover of $\widehat{X}$.

We define $X_1 = \widehat{X_c}/\sim_1$ where $\sim_1$ is defined using the identifications

$$(0,1) \sim_1 (0,2)\,,\ (i,1) \sim_1 (i,2)\,,\ (-i,1) \sim_1 (-i,2)$$

and $X_2 = \widehat{X_c}/\sim_2$ where $\sim_2$ is defined using the identifications

$$(0,1) \sim_2 (0,2)\,,\ (i,1) \sim_2 (-i,2)\,,\ (-i,1) \sim_2 (i,2)\,.$$

We take φ_i, π_i in the obvious way. Then (X_1, π_1, φ_1) and (X_2, π_2, φ_2) are the two double covers of X and we define

$$\begin{array}{rccc} f: & (X_1, \varphi_1) & \to & (X_2, \varphi_2) \\ & (z,1) & \to & (z,1) \\ & (z,2) & \to & (-z,2)\,. \end{array}$$

Then, $f \circ \varphi_1 = \varphi_2 \circ f$ and f is an isomorphism between symmetric Riemann surfaces with nodes.

Theorem 4.9. Let (X, φ) be a compact and connected symmetric stable Riemann surface with graph:

$$G(X) = \qquad\qquad .$$

Then $G(X/\varphi)$ is one of the following four graphs:

Moreover, there exists four non homeomorphic stable symmetric Riemann surfaces whose graph is $G(X)$.

Proof. Let $\widehat{X} = \widehat{X_1} \cup \widehat{X_2}$ where $g\left(\widehat{X_1}\right) = g\left(\widehat{X_2}\right) = 0$ and let $N(X) = \{A_1, A_2, A_3\}$. Since $\varphi(N(X)) = N(X)$, then φ can fix 1 or 3 nodes. We have two cases:

- If $\widehat{\varphi}\left(\widehat{X_1}\right) = \widehat{X_2}$, then $\widehat{X}/\widehat{\varphi} \simeq \widehat{X_1}$. If the three nodes are fixed, then $\pi(A_i)$ is an inessential node and we obtain the first graph; if, on the contrary, $\varphi(A_1) = A_1$ and $\varphi(A_2) = A_3$, then $\pi(A_1)$ is an inessential node and $\pi(A_2) = \pi(A_3)$ is a conic node, and we obtain the third graph.

- If $\widehat{\varphi}\left(\widehat{X_i}\right) = \widehat{X_i}$, then $\varphi(A_1) = A_1$ and $\widehat{\varphi}\mid_{\widehat{X_i}}$ has fixed points, then the topological type of $\widehat{X_i}/\widehat{\varphi}$ is $(0,1,0)$. If A_2 and A_3 are fixed nodes, then $\pi(A_i)$ is a boundary node and we obtain the second graph; if, on the contrary, $\varphi(A_2) = A_3$, then $\pi(A_1)$ is a boundary node and $\pi(A_2) = \pi(A_3)$ is a conic node. Then we obtain the fourth graph.

The Klein surface with nodes corresponding to the third graph has two double coverings, but only one of them satisfies that its graph is $G(X)$ (see [G2]). The Klein surface with nodes corresponding to the fourth graph admits two double coverings that are homeomorphic as symmetric Riemann surfaces with nodes (previous example). Finally, the Klein surfaces corresponding to the other graphs do not contain conic nodes and therefore have a unique double cover that is (X, φ). Then, there exists 4 symmetric Riemann surfaces with nodes non homeomorphic with this graph. □

As a consequence we obtain:

Corollary 4.10. There are 33 non homeomorphic stable Klein surfaces of genus 2.

Corollary 4.11. There are 35 non homeomorphic stable symmetric Riemann surfaces of genus 2.

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