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FIXED POINTS OF A FAMILY OF EXPONENTIAL MAPS

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Abstract

We consider the family of functions $f_\lambda(z) = \exp(i\lambda z)$, λ *real. With the help of MATLAB computations, we show* f_λ *has a unique attracting fixed point for several values of* λ. *We prove there is no attracting periodic orbit of period* $n \geq 2$.

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1. Introduction

In this note we show the existence and uniqueness of an attracting fixed point for the map $f_\lambda(z) = \exp(i\lambda z),\ z \in \mathbf{C}$, for certain (real) values of the parameter λ. The proofs depend on MATLAB calculations, and as such can be viewed as computer-assisted proofs. By contrast, the exponential map $z \mapsto \exp(z)$ admits no attracting fixed point. ([2], see Remark 12)

We began with a MATLAB program in which one inputs a value λ, a point z_0 a positive integer N, and a tolerance ϵ. The program outputs $f^n(z_0)$, where n is the least integer $k \leq N$ for which

$$|f^{k-1}(z_0) - f^k(z_0)| < \epsilon$$

if there is such an n. Here $f^n(z)$ is the $n-$fold iterate of f at z: thus, $f^1(z) = f(z)$, and $f^n(z) = f(f^{n-1}(z))$ for $n > 1$.

We experimented with various values of λ and the computations indicated the function f_λ had a fixed point inside the circle $|z| \leq 1/|\lambda|$ for $|\lambda| \leq 1.96$ (approximately). However in order to prove the existence of a fixed point, one must use a theorem, which usually involves an invariant domain. Since it was not easy to find such a domain, we took another approach, using the Maximum Modulus Theorem. While the analysis is elementary, the computer-assisted proofs have yielded new results.

Definition 1. A point $z \in \mathbf{C}$ is called an **attracting fixed point** of a map f if z satisfies $f(z) = z$ and $|f'(z)| < 1$.

Suppose that z_0 is an attracting fixed point of the map $f(z)$. Then for z sufficiently close to z_0, the iterates $z,\ f(z),\ f^2(z), \ldots$ converge to z_0.

Let's begin by stating some facts. Let f be the function f_λ.

Fact 1. If z_0 is an attracting fixed point of f, then $|z_0| < 1/|\lambda|$.

Conversely, any fixed point in the disk $|z| < 1/|\lambda|$ is attracting.

Proof. We have $|f'(z_0)| = |i\lambda f(z_0)| = |\lambda||z_0| < 1 \iff |z_0| < 1/|\lambda|$. □

Fact 2. For $z \in \mathbf{C},\ z = x + iy$ ($x,\ y$ real), $|f'(z)| < 1 \iff (\operatorname{sgn}\lambda)y > \frac{1}{|\lambda|}\log(|\lambda|)$.

Proof. If $z = x + iy$,

$$|f'(z)| = |i\lambda \exp(i\lambda z)| = |\lambda|e^{-\lambda y},$$

The conclusion is an easy exercise. □

Fact 3. The region $\mathbf{R}_\lambda$ defined by the inequalities

$$|z| < \frac{1}{|\lambda|}, \quad \mathrm{sgn}(\lambda)y > \frac{1}{|\lambda|}\log(|\lambda|), \; z = x + iy$$

is empty if $|\lambda| \geq e$.

Proof. $\frac{1}{|\lambda|}\log(|\lambda|) < \mathrm{sgn}(\lambda)y \leq |z| < \frac{1}{|\lambda|}$, so $\log(|\lambda|) < 1$, or $|\lambda| < e$. □

Our first goal was to prove the existence of a fixed point inside the circle $|z| = 1/|\lambda|$. We did this using the Maximum Modulus Theorem, or rather a Corollary, called the Minimum Modulus Theorem. Our MATLAB calculations indicated the attracting fixed point should lie in the intersection of $|z| \leq 1/\lambda$ with the first quadrant (for $\lambda > 0$), denoted $\mathcal{Q}_\lambda$ (or simply $\mathcal{Q}$ if λ is fixed), so it was in that region we applied the the Minimum Modulus Principle.

A more conventional approach to the existence of a fixed point using, say, the Brouwer Theorem, requires a having region which is mapped into itself by f. But neither $\mathcal{Q}_{\lambda\lambda}$ nor $\mathcal{R}_\lambda$ (see Fact 3) is invariant. Say $\lambda > 1$, $0 \in \mathcal{Q}_\lambda$ but $1 = f_\lambda(0) \notin \mathcal{Q}_{\lambda\lambda}$. Also $\frac{i}{\lambda} \in \mathcal{R}_\lambda$, but $f_\lambda(\frac{i}{\lambda}) = e^{-1} \notin \mathcal{R}_\lambda$ since $\mathcal{R}_\lambda$ is disjoint from the real axis if $\lambda > 1$. (But see Remark 9.)

Remark 2. It is enough to determine fixed points of f_λ for $\lambda > 0$, for

$$f_{-\lambda}(\overline{z}) = \exp(-i\lambda\overline{z}) = \exp(\overline{(i\lambda z)}) = \overline{f_\lambda(z)},$$

the last equality resulting from the fact the power series has real coefficients. Also, it is an easy observation that $\overline{z_0}$ is an attracting fixed point for $f_{-\lambda}$ iff z_0 is an attracting fixed point for f_λ.

Notation 1. We use both ' $\exp(\cdot)$ ' and '$e^{\cdot}$ ' to denote the exponential function.

2. Existence and Uniqueness of the Fixed Point

Theorem 3. Minimum Modulus Principle ([3], p. 313)

Let U be a bounded open set in $\mathbf{C}$, and let f be analytic on U and continuous on the closure $\bar{U}$. Assume that f never vanishes on $\bar{U}$. Then the minimum value of $|f|$ on $\bar{U}$ occurs on the boundary, ∂U.

Theorem 4.

Let $\lambda \in \mathbf{R}$ be one of the values $0.1, 0.2, 0.3, \ldots, 1.8, 1.9, 1.95,$ or 1.96. Let $\mathcal{Q}$ be the intersection of the first quadrant $\Re z \geq 0,\ \Im z \geq 0$, with the closed disk $|z| \leq 1/\lambda$. Then there exists a fixed point z in the interior $\mathcal{Q}^o$ of $\mathcal{Q}$ for the map $f(z) = \exp(i\lambda z)$.

Proof. Our calculations using MATLAB show that

$$[\min\{z \in \mathcal{Q} : |z - f(z)|\} < 10^{-6}.$$

The idea of the proof is to show $|z - f(z)|$ is bounded below along the boundary of $\mathcal{Q}$ by some constant which is greater than 10^{-6}. Theorem would then assert the existence of a fixed point for f in $\mathcal{Q}$. Writing $z = x + iy$, for $x, y \in \mathbf{R}$ we have

$$\begin{aligned} |f(z) - z|^2 &= |exp(i\lambda z) - z|^2 \\ &= |e^{-\lambda y} e^{i\lambda x} - (x + iy)|^2 \\ &= |e^{-\lambda y}(\cos(\lambda x) + i\sin(\lambda x)) - (x + iy)|^2 \\ &= e^{-2\lambda y} + x^2 + y^2 - 2e^{-\lambda y}(x\cos(\lambda x) + y\sin(\lambda x)). \end{aligned}$$

Denote the right hand side of the above by $g(x, y)$. We now have to check the values of $g(x, y)$ along the boundaries of $\mathcal{Q}$. We will start with $y = 0$. Then for $0 \leq x \leq 1/\lambda$

$$g(x, 0) = x^2 + 1 - 2x\cos(\lambda x).$$

Thus

$$g(x, 0) = (x - 1)^2 + 2x(1 - \cos(\lambda x)).$$

Since $0 < \lambda < e$ we have $g(x, 0) \geq (1 - e^{-1})^2$ on $0 \leq x \leq e^{-1}$ and by $2e^{-1}(1 - \cos(\frac{\lambda}{e})) \geq 2e^{-1}(1 - \cos(1))$ on $e^{-1} \leq x \leq \lambda^{-1}$.

Next we check the boundary $x = 0,\ 0 \leq y \leq 1/\lambda$:

$$g(0, y) = y^2 + e^{-2\lambda y} \geq e^{-2\lambda y} \geq \frac{1}{e^2}.$$

Finally we need to check the boundary on the quarter circle; it is convenient to convert to polar coordinates

$$x = (1/\lambda)\cos\theta, \quad y = (1/\lambda)\sin\theta, \quad \text{for } 0 \leq \theta \leq \pi/2.$$

Then,

$$g\left(\tfrac{1}{\lambda}\cos\theta, \tfrac{1}{\lambda}\sin\theta\right) =$$
$$(\tfrac{1}{\lambda})^2 + e^{-2\sin\theta} - e^{-\sin\theta}\left(\tfrac{2}{\lambda}\cos\theta\,\cos(\cos\theta) - \tfrac{2}{\lambda}\sin\theta\,\sin(\cos\theta)\right)$$

and using the identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ we see that

$$\begin{aligned} g(\tfrac{1}{\lambda}\cos\theta, \tfrac{1}{\lambda}\sin\theta) &= (\tfrac{1}{\lambda})^2 + e^{-2\sin\theta} - e^{-\sin\theta}(\tfrac{2}{\lambda}\cos(\theta - \cos\theta)) \\ &= (\lambda^{-1} - e^{-\sin\theta})^2 + \tfrac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos\theta)). \end{aligned}$$

Since both $(\lambda^{-1} - e^{-\sin\theta})^2$ and $\frac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos\theta))$ are nonnegative, we must see where both terms are zero: the second expression is zero only when $\cos\theta = \theta$. Call this θ_0; so θ_0 is approximately .739085. Putting $\theta = \theta_0$ into the first term and setting it to zero yields $\lambda = \lambda_0 := e^{\sqrt{1-\theta_0^2}}$, or approximately 1.96131.

From this we can conclude: suppose λ is in the interval $0 < \lambda \leq 1.96$. Then for $0 \leq \theta \leq .741$,

$$\lambda^{-1} - e^{-\sin\theta} \geq 1.96^{-1} - e^{-\sin .741} > 10^{-3}.$$

And for $.741 \leq \theta \leq \frac{\pi}{2}$,

$$\begin{aligned} &\tfrac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos\theta)) \\ &\geq \tfrac{2}{1.96}e^{-\sin .741}(1 - \cos(.741 - \cos .741)) > (2.67)10^{-6} \end{aligned}$$

We conclude that $|g(x,y)|^{\frac{1}{2}} > 10^{-3}$ on the boundary of $\mathcal{Q}$. Since

$$\min_{z\in\mathcal{Q}}\{|z - f(z)|\} = \min_{(x,y)\in\mathcal{Q}}\{g(x,y)^{\frac{1}{2}}\} < 10^{-6}$$

for any λ in the Table, it follows from the Minimun Modulus Principle that the function $f(z) = \exp(i\lambda z)$ has a fixed point. $\square$

Corollary 5. From Fact 1, any fixed point for f inside the circle $|z| < \frac{1}{\lambda}$ is attracting, so that Theorem 4 establishes the existence of an attracting fixed point in $\mathcal{Q}^\circ$. By Fact 2, the fixed point lies in the intersection $\mathcal{Q}^\circ \cap \mathbf{R}$.

Below are the approximate fixed points for different values of λ. For each λ the program terminated when $|f(z) - z| < 10^{-8}$.

Table of Fixed Points

λ	Fixed Point	λ	Fixed point
.1	$0.9854986 + 0.0974364i$	1.1	$0.5463106 + 0.3745068i$
.2	$0.9470891 + 0.1815722i$	1.2	$0.5191342 + 0.3729401i$
.3	$0.8954147 + 0.2464877i$	1.3	$0.4945130 + 0.3703950i$
.4	$0.8396697 + 0.2931240i$	1.4	$0.4721254 + 0.3671575i$
.5	$0.7852571 + 0.3251993i$	1.5	$0.4516947 + 0.3634355i$
.6	$0.7346292 + 0.3465471i$	1.6	$0.4329840 + 0.3593815i$
.7	$0.6885773 + 0.3602367i$	1.7	$0.4157904 + 0.3551080i$
.8	$0.6470987 + 0.3685124i$	1.8	$0.3999401 + 0.3506985i$
.9	$0.6098583 + 0.3729619i$	1.9	$0.3852840 + 0.3462151i$
1.0	$0.5764127 + 0.3746990i$	1.95	$0.3783627 + 0.3439606i$
		1.96	$0.3770090 + 0.3435093i$

One might infer from our results that f_λ has a attracting fixed point for all values of $\lambda,\ 0 < \lambda < \lambda_0$. But our method of proof can only be applied to finitely many λ.

Remark 6. For $\theta_0,\ \lambda_0$ as in the proof of the theorem, the proof shows that the function $f_0(z) = \exp(i\lambda_0 z)$ has a fixed point at $z = z_0 := \lambda_0^{-1} e^{i\theta_0}$. Since z_0 lies on the circle $|z| = 1/\lambda_0$, it follows $|f_0'(z_0)| = 1$. Such a point is called a nonhyperbolic, or neutral fixed point.

Remark 7. It is possible to express the fixed point z of f_λ as an analytic function of λ. Solving $z = f_\lambda(z)$ for λ yields $\lambda = -i\,\log(z)/z$. The inverse function is given by $z = g(\lambda) := iW(-i\lambda)/\lambda$ where W is the Lambert W-function, or the principal branch of the inverse of $w \to we^w$. Since the values of W are not easy to calculate, this does not simplify the question of deciding when the fixed point $z = g(\lambda)$ is attracting, i.e., when it satisfies $|g(\lambda)| < \frac{1}{|\lambda|}$. (Cf [1].)

Our MATLAB computation indicates the fixed point is unique. That is indeed the case, as we now prove.

Theorem 8. Let λ be one of the values in the Table. Then the map $f(z) = \exp(i\lambda z)$ has a unique attracting fixed point.

Proof.

Let $\mathbf{R}$ be the region in $\mathcal{C}$ determined by the two inequalities $x^2 + y^2 < 1/\lambda^2$ and $y > \frac{1}{\lambda}\log(\lambda)$. The region $\mathbf{R}$ is convex, and by Facts 1, 2, and 3 it is nonempty, and contains all attracting fixed points of the map f.

Suppose now that $z_1,\ z_2$ are two distinct fixed points of f. Then z_1, z_2 lie in $\mathbf{R}$, and if $\mathcal{C}$ is a contour joining $z_1$and z_2,

$$\begin{aligned}|z_1 - z_2| = |f(z_1) - f(z_2)| = |\textstyle\int_{\mathcal{C}} f\prime(z)\,dx| f\prime(z)\,dx| \\ \leq \textstyle\int_{\mathcal{C}} |f\prime(z)|\,d|z| \leq \max_{z\ \in\mathcal{C}}\{|f\prime(z)|\} length(\mathcal{C}).\end{aligned}$$

If $\mathcal{C}$ is the straight line contour joining $z_1$and z_2, then $\mathcal{C} \subset \mathbf{R}$ so that $|f\prime(z)| < 1$ for $z \in \mathcal{C}$, and since $\mathcal{C}$ is compact, $\max_{z\in\mathcal{C}}\{|f\prime(z)|\} < 1$.Since length($\mathcal{C}$) $= |z_1 - z_2|$ the calculation above implies $|z_1 - z_2| < |z_1 - z_2|$, which is absurd. Thus the fixed point is unique. □

Remark 9. An alternative, more conventional approach to the existence of a fixed point may be possible using a standard fixed point theorem, such as the Brouwer Theorem. Assume that $z_0$satisfies $|f(z_0) - z_0| < 10^{-6}$, and $|f'(z_0)| < 1$.Let $\epsilon = 1 - |f'(z_0)|$.There is a $\delta > 0$such that $|f'(z)| < 1 - \epsilon/2$ for $|z - z_0| < \delta$. So for $|z - z_0| \leq \delta$,

$$\begin{aligned}|f(z) - z_0| &\leq |f(z) - f(z_0)| + |f(z_0) - z_0| \\ &< \max\{|f'(z)| : z \in \mathcal{C}\}\,\delta + 10^{-6} \\ &< (1 - \epsilon/2)\delta + 10^{-6} \\ &< \delta\end{aligned}$$

(where $\mathcal{C}$ is the line segment joining z and z_0) is valid as long as $10^{-6} < \frac{\epsilon}{2}\delta$. A tolerance finer than 10^{-6} may be required. We have not carried out these calculations. However, we do not see how the critical value λ_0 could be obtained through this approach.

3. Attracting Orbits

If $z_0,\ z_1, \ldots, z_{n-1}$ is a set of points satisfying

Remark 1.

$$z_1 = f(z_0),\ z_2 = f(z_1), \ldots, z_0 = f(z_{n-1})$$

then $z_0,\ z_1, \ldots, z_{n-1}$ is called an *orbit of period* n.

Definition 10. Let $z_0,\ z_1, \ldots, z_{n-1}$ be an orbit of period n. It is said to be an *attracting* orbit if $|f'(z_k)| < 1,\ 0 \leq k \leq n-1$.

Let $f(z) = \exp(i\lambda z),\ \lambda > 0$, and $z_0,\ z_1, \ldots, z_{n-1}$ a period n orbit, and assume the orbit is attracting. Observe

$$|f'(z_0)| = |i\lambda \exp(i\lambda z_0)| = \lambda |z_1| < 1.$$

Similarly, $z_2, \ldots, z_{n-1},\ z_0$ lie in the circle $|z| < 1/\lambda$.

Furthermore, it follows from Fact 2 that z_k satisfy $y_k > \frac{1}{\lambda}\log(\lambda)$, where $z_k = x_k + iy_k$. Thus, $z_0,\ z_1, \ldots, z_{n-1}$ lie in the region **R** in the complex plane determined by the two inequalities $x^2 + y^2 < 1/\lambda^2$ and $y > \frac{1}{\lambda}\log(\lambda)$.

Theorem 11. Let $0 \neq \lambda \in \mathbf{R}$. Then the map $f(z) = \exp(i\lambda z)$ does not have any attracting periodic orbit of period n, for $n \geq 2$.

Proof. As noted above (cf Remark 2) it is enough to prove the assertion for $\lambda > 0$. The proof is in the spirit of the uniqueness proof (Theorem 8). $\square$

Note our definition of attracting orbit is stronger than the standard definition ([2]): $|(f^n)'(z_0)| < 1$, or equivalently that $|f'(z_0)\, f'(z_1) \cdots f'(z_{n-1})| < 1$.

Remark 12. Recall that the Julia set of a map f is the closure of the repelling periodic points. It's interesting to note the difference between the maps f_λ and the exponential map, $z \to e^z$. For the exponential map, it is shown in [2] that the Julia set is all of **C**. But for the maps f_λ (at least for the values of λ in the table), the Julia set is a proper subset: indeed, there is an open neighborhood U of the attracting fixed point, which is invariant under f_λ, not containing any other periodic points. Thus, the Julia set of f_λ is a proper subset of **C**.

Of course each f_λ has a Julia set which is unbounded, and hence f_λ has infinitely many repelling periodic points.

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