FIXED POINTS OF A FAMILY OF EXPONENTIAL MAPS

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Abstract

We consider the family of functions $f_\lambda(z) = \exp(i\lambda z)$, $\lambda$ real. With the help of MATLAB computations, we show $f_\lambda$ has a unique attracting fixed point for several values of $\lambda$. We prove there is no attracting periodic orbit of period $n \geq 2$.

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1. Introduction

In this note we show the existence and uniqueness of an attracting fixed point for the map $f_\lambda(z) = \exp(i\lambda z), \ z \in \mathbb{C}$, for certain (real) values of the parameter $\lambda$. The proofs depend on MATLAB calculations, and as such can be viewed as computer-assisted proofs. By contrast, the exponential map $z \mapsto \exp(z)$ admits no attracting fixed point. ([2], see Remark 12)

We began with a MATLAB program in which one inputs a value $\lambda$, a point $z_0$ a positive integer $N$, and a tolerance $\epsilon$. The program outputs $f^n(z_0)$, where $n$ is the least integer $k \leq N$ for which $|f^{k-1}(z_0) - f^k(z_0)| < \epsilon$ if there is such an $n$. Here $f^n(z)$ is the $n$–fold iterate of $f$ at $z$: thus, $f^1(z) = f(z)$, and $f^n(z) = f(f^{n-1}(z))$ for $n > 1$.

We experimented with various values of $\lambda$ and the computations indicated the function $f_\lambda$ had a fixed point inside the circle $|z| \leq 1/|\lambda|$ for $|\lambda| \leq 1.96$ (approximately). However in order to prove the existence of a fixed point, one must use a theorem, which usually involves an invariant domain. Since it was not easy to find such a domain, we took another approach, using the Maximum Modulus Theorem. While the analysis is elementary, the computer-assisted proofs have yielded new results.

Definition 1. A point $z \in \mathbb{C}$ is called an attracting fixed point of a map $f$ if $z$ satisfies $f(z) = z$ and $|f'(z)| < 1$.

Suppose that $z_0$ is an attracting fixed point of the map $f(z)$. Then for $z$ sufficiently close to $z_0$, the iterates $z, f(z), f^2(z), \ldots$ converge to $z_0$.

Let’s begin by stating some facts. Let $f$ be the function $f_\lambda$.

Fact 1. If $z_0$ is an attracting fixed point of $f$, then $|z_0| < 1/|\lambda|$.

Conversely, any fixed point in the disk $|z| < 1/|\lambda|$ is attracting.

Proof. We have $|f'(z_0)| = |i\lambda f(z_0)| = |\lambda||z_0| < 1 \iff |z_0| < 1/|\lambda|$.

Fact 2. For $z \in \mathbb{C}, \ z = x + iy$ ($x, \ y$ real), $|f'(z)| < 1 \iff (\text{sgn} \lambda)y > \frac{1}{|\lambda|}\log(|\lambda|)$.

Proof. If $z = x + iy$,

$$|f'(z)| = |i\lambda \exp(i\lambda z)| = |\lambda|e^{-\lambda y},$$
The conclusion is an easy exercise. □

Fact 3. The region \( R_\lambda \) defined by the inequalities

\[
|z| < \frac{1}{|\lambda|}, \quad \text{sgn}(\lambda)y > \frac{1}{|\lambda|} \log(|\lambda|), \quad z = x + iy
\]

is empty if \( |\lambda| \geq e \).

Proof. \( \frac{1}{|\lambda|} \log(|\lambda|) < \text{sgn}(\lambda)y \leq |z| < \frac{1}{|\lambda|} \), so \( \log(|\lambda|) < 1 \), or \( |\lambda| < e \). □

Our first goal was to prove the existence of a fixed point inside the circle \( |z| = 1/|\lambda| \). We did this using the Maximum Modulus Theorem, or rather a Corollary, called the Minimum Modulus Theorem. Our MATLAB calculations indicated the attracting fixed point should lie in the intersection of \( |z| \leq 1/\lambda \) with the first quadrant (for \( \lambda > 0 \), denoted \( Q_\lambda \) (or simply \( Q \) if \( \lambda \) is fixed), so it was in that region we applied the the Minimum Modulus Principle.

A more conventional approach to the existence of a fixed point using, say, the Brouwer Theorem, requires a having region which is mapped into itself by \( f \). But neither \( Q_\lambda \) nor \( R_\lambda \) (see Fact 3) is invariant. Say \( \lambda > 1 \), \( 0 \in Q_\lambda \) but \( 1 = f_\lambda(0) \notin Q_\lambda \). Also \( \frac{x}{\lambda} \in R_\lambda \), but \( f_\lambda \left( \frac{x}{\lambda} \right) = e^{-1} \notin R_\lambda \) since \( R_\lambda \) is disjoint from the real axis if \( \lambda > 1 \). (But see Remark 9.)

Remark 2. It is enough to determine fixed points of \( f_\lambda \) for \( \lambda > 0 \), for

\[
f_{-\lambda}(z) = \exp(-i\lambda z) = \exp((i\lambda z)) = f_\lambda(z),
\]

the last equality resulting from the fact the power series has real coefficients. Also, it is an easy observation that \( \overline{z_0} \) is an attracting fixed point for \( f_{-\lambda} \) iff \( z_0 \) is an attracting fixed point for \( f_\lambda \).

Notation 1. We use both ‘ \( \exp(\cdot) \) ’ and ‘ \( e \cdot \) ’ to denote the exponential function.

2. Existence and Uniqueness of the Fixed Point

Theorem 3. Minimum Modulus Principle ([3], p. 313)

Let \( U \) be a bounded open set in \( \mathbb{C} \), and let \( f \) be analytic on \( U \) and continuous on the closure \( \bar{U} \). Assume that \( f \) never vanishes on \( \bar{U} \). Then the minimum value of \( |f| \) on \( \bar{U} \) occurs on the boundary, \( \partial U \).
Theorem 4.
Let $\lambda \in \mathbb{R}$ be one of the values $0.1, 0.2, 0.3, \ldots, 1.8, 1.9, 1.95,$ or $1.96.$ Let $Q$ be the intersection of the first quadrant $\Re z \geq 0, \Im z \geq 0,$ with the closed disk $|z| \leq 1/\lambda.$ Then there exists a fixed point $z$ in the interior $Q^o$ of $Q$ for the map $f(z) = \exp(i\lambda z)$.

Proof. Our calculations using MATLAB show that

$$\left[ \min \{ z \in Q : |z - f(z)| \} \right] < 10^{-6}.$$ 

The idea of the proof is to show $|z - f(z)|$ is bounded below along the boundary of $Q$ by some constant which is greater than $10^{-6}.$ Theorem would then assert the existence of a fixed point for $f$ in $Q$. Writing $z = x + iy$, for $x, y \in \mathbb{R}$ we have

$$|f(z) - z|^2 = |\exp(i\lambda z) - z|^2 = |e^{-\lambda y}e^{i\lambda x} - (x + iy)|^2 = e^{-2\lambda y} + x^2 + y^2 - 2e^{-\lambda y}(x \cos(\lambda x) + y \sin(\lambda x)).$$

Denote the right hand side of the above by $g(x, y).$ We now have to check the values of $g(x, y)$ along the boundaries of $Q.$ We will start with $y = 0.$ Then for $0 \leq x \leq 1/\lambda$

$$g(x, 0) = x^2 + 1 - 2x \cos(\lambda x).$$

Thus

$$g(x, 0) = (x - 1)^2 + 2x(1 - \cos(\lambda x)).$$

Since $0 < \lambda < e$ we have $g(x, 0) \geq (1 - e^{-1})^2$ on $0 \leq x \leq e^{-1}$ and by $2e^{-1}(1 - \cos(\frac{\lambda}{2})) \geq 2e^{-1}(1 - \cos(1))$ on $e^{-1} \leq x \leq \lambda^{-1}.$

Next we check the boundary $x = 0, 0 \leq y \leq 1/\lambda$:

$$g(0, y) = y^2 + e^{-2\lambda y} \geq e^{-2\lambda y} \geq \frac{1}{e^2}.$$ 

Finally we need to check the boundary on the quarter circle; it is convenient to convert to polar coordinates

$$x = (1/\lambda) \cos \theta, \quad y = (1/\lambda) \sin \theta, \quad \text{for } 0 \leq \theta \leq \pi/2.$$
Then,
\[ g\left(\frac{1}{\lambda}\cos\theta, \frac{1}{\lambda}\sin\theta\right) = \left(\frac{1}{\lambda}\right)^2 + e^{-2\sin\theta} - e^{-\sin\theta}\left(\frac{2}{\lambda}\cos\theta \cos (\cos \theta) - \frac{2}{\lambda}\sin\theta \sin (\cos \theta)\right) \]

and using the identity \(\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta\) we see that
\[
g\left(\frac{1}{\lambda}\cos\theta, \frac{1}{\lambda}\sin\theta\right) = \left(\frac{1}{\lambda}\right)^2 + e^{-2\sin\theta} - e^{-\sin\theta}\left(\frac{2}{\lambda}\right)^2 + \frac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos \theta)).
\]

Since both \((\lambda^{-1} - e^{-\sin\theta})^2\) and \(\frac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos \theta))\) are nonnegative, we must see where both terms are zero: the second expression is zero only when \(\cos \theta = \theta\). Call this \(\theta_0\); so \(\theta_0\) is approximately \(0.739085\). Putting \(\theta = \theta_0\) into the first term and setting it to zero yields \(\lambda = \lambda_0 := e^{\sqrt{1 - \theta_0^2}},\) or approximately \(1.96131\).

From this we can conclude: suppose \(\lambda\) is in the interval \(0 < \lambda \leq 1.96\). Then for \(0 \leq \theta \leq 0.741\),
\[
\lambda^{-1} - e^{-\sin\theta} \geq 1.96^{-1} - e^{-\sin.741} > 10^{-3}.
\]

And for \(0.741 \leq \theta \leq \frac{\pi}{2}\),
\[
\frac{2}{\lambda}e^{-\sin\theta}(1 - \cos(\theta - \cos \theta)) \geq \frac{2}{1.96}e^{-\sin.741}(1 - \cos(0.741 - \cos.741)) > (2.67)10^{-6}
\]

We conclude that \(|g(x, y)|^{\frac{1}{2}} > 10^{-3}\) on the boundary of \(Q\). Since
\[
\min_{z \in \mathbb{Q}} \{|z - f(z)|\} = \min_{(x, y) \in \mathbb{Q}} \{g(x, y)^{\frac{1}{2}}\} < 10^{-6}
\]
for any \(\lambda\) in the Table, it follows from the Minimum Modulus Principle that the function \(f(z) = \exp(\lambda z)\) has a fixed point. \(\Box\)

**Corollary 5.** From Fact 1, any fixed point for \(f\) inside the circle \(|z| < \frac{1}{\lambda}\) is attracting, so that Theorem 4 establishes the existence of an attracting fixed point in \(Q^\circ\). By Fact 2, the fixed point lies in the intersection \(Q^\circ \cap \mathbb{R}\).

Below are the approximate fixed points for different values of \(\lambda\). For each \(\lambda\) the program terminated when \(|f(z) - z| < 10^{-8}\).

Table of Fixed Points
One might infer from our results that \( f_{\lambda} \) has an attracting fixed point for all values of \( \lambda \), \( 0 < \lambda < \lambda_0 \). But our method of proof can only be applied to finitely many \( \lambda \).

**Remark 6.** For \( \theta_0, \lambda_0 \) as in the proof of the theorem, the proof shows that the function \( f_0(z) = \exp(i\lambda_0 z) \) has a fixed point at \( z = z_0 := \lambda_0^{-1} e^{i\theta_0} \). Since \( z_0 \) lies on the circle \( |z| = 1/\lambda_0 \), it follows \( |f_0'(z_0)| = 1 \). Such a point is called a nonhyperbolic, or neutral fixed point.

**Remark 7.** It is possible to express the fixed point \( z \) of \( f_{\lambda} \) as an analytic function of \( \lambda \). Solving \( z = f_{\lambda}(z) \) for \( \lambda = -i \log(z)/z \). The inverse function is given by \( z = g(\lambda) := iW(-i\lambda)/\lambda \) where \( W \) is the Lambert \( W \)-function, or the principal branch of the inverse of \( w \to we^w \). Since the values of \( W \) are not easy to calculate, this does not simplify the question of deciding when the fixed point \( z = g(\lambda) \) is attracting, i.e., when it satisfies \( |g'(\lambda)| < 1 \). (Cf [1].)

Our MATLAB computation indicates the fixed point is unique. That is indeed the case, as we now prove.

**Theorem 8.** Let \( \lambda \) be one of the values in the Table. Then the map \( f(z) = \exp(i\lambda z) \) has a unique attracting fixed point.

**Proof.**

Let \( R \) be the region in \( C \) determined by the two inequalities \( x^2 + y^2 < 1/\lambda^2 \) and \( y > 1/\lambda \log(\lambda) \). The region \( R \) is convex, and by Facts 1, 2, and 3 it is nonempty, and contains all attracting fixed points of the map \( f \).
Suppose now that \( z_1, z_2 \) are two distinct fixed points of \( f \). Then \( z_1, z_2 \) lie in \( \mathbb{R} \), and if \( \mathcal{C} \) is a contour joining \( z_1 \) and \( z_2 \),

\[
|z_1 - z_2| = |f(z_1) - f(z_2)| = \left| \int_{\mathcal{C}} f'(z) \, dx \right| \leq \int_{\mathcal{C}} |f'(z)| \, |dz| \leq \max_{z \in \mathcal{C}} \{|f'(z)|\} \text{length}(\mathcal{C}).
\]

If \( \mathcal{C} \) is the straight line contour joining \( z_1 \) and \( z_2 \), then \( \mathcal{C} \subset \mathbb{R} \) so that \( |f'(z)| < 1 \) for \( z \in \mathcal{C} \), and since \( \mathcal{C} \) is compact, \( \max_{z \in \mathcal{C}} \{|f'(z)|\} < 1 \). Since \( \text{length}(\mathcal{C}) = |z_1 - z_2| \), the calculation above implies \( |z_1 - z_2| < |z_1 - z_2| \), which is absurd. Thus the fixed point is unique.

**Remark 9.** An alternative, more conventional approach to the existence of a fixed point may be possible using a standard fixed point theorem, such as the Brouwer Theorem. Assume that \( z_0 \) satisfies \( |f(z_0) - z_0| < 10^{-6} \), and \( |f'(z_0)| < 1 \). Let \( \epsilon = 1 - |f'(z_0)| \). There is a \( \delta > 0 \) such that \( |f'(z)| < 1 - \epsilon/2 \) for \( |z - z_0| < \delta \). So for \( |z - z_0| \leq \delta \),

\[
|f(z) - z_0| \leq |f(z) - f(z_0)| + |f(z_0) - z_0| < \max \{|f'(z)| : z \in \mathcal{C}\} \delta + 10^{-6} < (1 - \epsilon/2)\delta + 10^{-6}
\]

(where \( \mathcal{C} \) is the line segment joining \( z \) and \( z_0 \)) is valid as long as \( 10^{-6} < \frac{\epsilon}{2} \delta \). A tolerance finer than \( 10^{-6} \) may be required. We have not carried out these calculations. However, we do not see how the critical value \( \lambda_0 \) could be obtained through this approach.

### 3. Attracting Orbits

If \( z_0, z_1, \ldots, z_{n-1} \) is a set of points satisfying

**Remark 1.**

\[
z_1 = f(z_0), \ z_2 = f(z_1), \ldots, z_0 = f(z_{n-1})
\]

then \( z_0, z_1, \ldots, z_{n-1} \) is called an orbit of period \( n \).

**Definition 10.** Let \( z_0, z_1, \ldots, z_{n-1} \) be an orbit of period \( n \). It is said to be an attracting orbit if \( |f'(z_k)| < 1, \ 0 \leq k \leq n - 1 \).

Let \( f(z) = \exp(i\lambda z), \ \lambda > 0 \), and \( z_0, z_1, \ldots, z_{n-1} \) a period \( n \) orbit, and assume the orbit is attracting. Observe
\[ |f'(z_0)| = |i\lambda \exp(i\lambda z_0)| = \lambda |z_1| < 1. \]

Similarly, \( z_2, \ldots, z_{n-1}, \ z_0 \) lie in the circle \( |z| < 1/\lambda \).

Furthermore, it follows from Fact 2 that \( z_k \) satisfy \( y_k > \frac{1}{4} \log(\lambda) \), where \( z_k = x_k + iy_k \). Thus, \( z_0, z_1, \ldots, z_{n-1} \) lie in the region \( \mathbf{R} \) in the complex plane determined by the two inequalities \( x^2 + y^2 < 1/\lambda^2 \) and \( y > \frac{1}{4} \log(\lambda) \).

**Theorem 11.** Let \( 0 \neq \lambda \in \mathbb{R} \). Then the map \( f(z) = \exp(i\lambda z) \) does not have any attracting periodic orbit of period \( n \), for \( n \geq 2 \).

**Proof.** As noted above (cf Remark 2) it is enough to prove the assertion for \( \lambda > 0 \). The proof is in the spirit of the uniqueness proof (Theorem 8).

\( \square \)

Note our definition of attracting orbit is stronger than the standard definition ([2]): \( |(f^n)'(z_0)| < 1 \), or equivalently that \( |f'(z_0) f'(z_1) \cdots f'(z_{n-1})| < 1 \).

**Remark 12.** Recall that the Julia set of a map \( f \) is the closure of the repelling periodic points. It’s interesting to note the difference between the maps \( f_\lambda \) and the exponential map, \( z \to e^z \). For the exponential map, it is shown in [2] that the Julia set is all of \( \mathbb{C} \). But for the maps \( f_\lambda \) (at least for the values of \( \lambda \) in the table), the Julia set is a proper subset: indeed, there is an open neighborhood \( U \) of the attracting fixed point, which is invariant under \( f_\lambda \), not containing any other periodic points. Thus, the Julia set of \( f_\lambda \) is a proper subset of \( \mathbb{C} \).

Of course each \( f_\lambda \) has a Julia set which is unbounded, and hence \( f_\lambda \) has infinitely many repelling periodic points.

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