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ASYMPTOTICS FOR SECOND ORDER DELAYED DIFFERENTIAL EQUATIONS

SAMUEL CASTILLO *
UNIVERSIDAD DEL BÍO - BÍO, CHILE
and
MANUEL PINTO †
UNIVERSIDAD DE CHILE, CHILE

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Abstract

In this work we present a way to find asymptotic formulas for some solutions of second order linear differential equations with a retarded functional perturbation.

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1. Introduction

The initial motivation of this work is the conjecture of Haddock-Sacker [16] (1980). They consider a differential system

$$y' = \Lambda y + R(t)y(t-r), \tag{1.1}$$

where Λ and R are $N \times N$ matrix valued functions of $t \geq 0$ such that $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_N)$, with $\Re e\lambda_i \neq \Re e\lambda_j$ for $i \neq j$ and the operator norm $\|R(t)\|$ is in L^2. Then, they conjecture that the fundamental matrix of (1.1) $Y = Y(t)$ such that $Y(0) = I$ satisfies

$$Y(t) = (I + o(1)) \exp\left(t\Lambda + e^{-\Lambda r} \int_0^t \operatorname{diag}[R(s)]ds\right),$$

as $t \to +\infty$. This conjecture is a version of the asymptotic theorem of Hartman-Wintner [18] (1955) for an autonomous diagonal differential system with a L^p $(p = 2)$ linear perturbation with delayed argument. This and similar problems were considered by Arino-Győri [2] (1989), Ai [1] (1992) and Cassel-Hou [6] (1993) (here $p \geq 2$ is considered) for a system where the non-perturbed system is diagonal and satisfies the hypotheses of the Hartman-Wintner's asymptotic theorem.

A result which extend the Conjecture of Haddock and Sacker [16] and the result of Cassel and Hou [6] is the following:

Proposition 1. *(See Castillo [7, 2003] Consider the linear differential system*

$$y'(t) = B(t)y(t) + R(t, y_t), \tag{1.2}$$

where $y_t(s) = y(t+s)$ the matrix B is in Jordan form, that is

$$B(t) = \left[\oplus_{i=1}^{k-1} J_{n_i}(\lambda_i(t))\right] \oplus \lambda_k(t) \oplus \left[\oplus_{i=k+1}^{m} J_{n_i}(\lambda_i(t))\right],$$

where $J_{n_i}(\lambda)$ are the $n_i \times n_i$ Jordan matrices

$$J_{n_i}(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ ... & ... & ... & ... & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

and the λ_i's are functions from **R** *into* **C** *satisfying*

$$\Re e(\lambda_i(t) - \lambda_k(t)) < -\eta < 0, \qquad i = 1, \cdots, k-1$$

$$\Re e(\lambda_i(t) - \lambda_k(t)) > \eta > 0, \qquad i = k+1, \cdots, m,$$

where η is a constant, $y_t : [-\tau, 0] \to \mathbf{C}^N$ defined as $y_t(s) = y(t+s)$ for all $t \geq 0$, $\{R(t, \cdot)\}_{t \geq 0}$ is a family of bounded linear functionals from the set of the essentially bounded functions $[-\tau, 0] \to \mathbf{C}^N$ into $\mathbf{C}^N$, $N = \sum_{j=1}^m n_j$ and
$\|R(t, \exp\left(\int_t^{t+\cdot} \lambda_k(\tau) d\tau\right) I)\| \in L^p, 1 \leq p \leq 2$. *Then (2) has a solution $y = y_0(t)$ such that*

$$\begin{aligned} y_0(t) &= \exp\left(\int_0^t [\lambda_k(s) + e_{\tilde{k}}^* \cdot R(s, \exp\left(\int_s^{s+\cdot} \lambda_k(\tau) d\tau\right) e_{\tilde{k}})]\right) \\ &\times \quad (e_{\tilde{k}} + o(1)) \end{aligned} \tag{1.3}$$

as $t \to \infty$, where $\tilde{k} = n_1 + \cdots + n_{k-1} + 1$.

A scalar version with the more general perturbation is given by the following result:

Proposition 2. *(Castillo-Pinto [11, 2004]) Consider the linear functional differential equation*

$$y'(t) = \lambda_0(t) y(t) + b(t, y_t), t \geq 0, \tag{1.4}$$

where $\lambda_0 : [0, +\infty[\to \mathbf{C}$ is a locally integrable function and

$$b : [0, +\infty[\times L^\infty([-\tau, 0], \mathbf{C}) \to \mathbf{C}$$

is a continuous function and linear in the second variable function such that

$$\sup_{t \geq T} \int_{t-\tau}^t |b(s, \exp\left(\int_s^{s+\cdot} \lambda_0(\xi) d\xi\right) | ds < \frac{1}{e},$$

for T large enough. Then, every solution of (1.4) has the following asymptotic formula

$$\begin{aligned} y(t) &= \exp\left(\int_\tau^t \left[\lambda_0(s) + b(s, e^{\int_s^{s+\cdot} \lambda_0(\xi) d\xi}) + \sum_{n=1}^{+\infty} \Delta_n(s)\right] ds\right) \\ &\times \quad (c + o(1)), \end{aligned} \tag{1.5}$$

as $t \to +\infty$, where $\Delta_n(t) = b(t, e^{\int_t^{t+\cdot} \lambda_0(\xi) d\xi} [e^{\int_t^{t+\cdot} \mu_n(\xi) d\xi} - e^{\int_t^{t+\cdot} \mu_{n-1}(\xi) d\xi}])$, $\mu_n(t) = b(t, e^{\int_t^{t+\cdot} \lambda_0(\xi) d\xi} e^{\int_t^{t+\cdot} \mu_{n-1}(\xi) d\xi})$, for all $t \geq n\tau$, $\mu_n(t) = 0$ for all $t \in [0, n\tau[$, $\mu_0 = 0$ and $n \in \mathbf{N}$. Conversely, given $c \in \mathbf{C}$ there is a solution $y = y(t)$ of (1.4) satisfying (1.5).

2. Preliminaries

It is assumed that:

Hypothesis 1. *The second order ordinary differential equation*

$$x''(t) = a(t)x'(t) + b(t)x(t),\ t \geq 0, \tag{2.1}$$

has a complex valued solution $x = x_0(t)$ such that $x_0(t) \neq 0$ for all $t \geq 0$, where $a, b : [0, +\infty[\to \mathbf{R}$ are locally integrable functions.

Here, it is considered as a perturbation of the equation (2.1), the second order retarded functional differential equation

$$y''(t) = [a(t) + \hat{a}(t)]y'(t) + b(t)y(t) + c(t, y_t),\ t \geq 0, \tag{2.2}$$

where $a, b : [0, +\infty[\to \mathbf{R}$ are locally integrable functions, $\{c(t, \cdot)\}_{t \geq 0}$ is a family of linear functional from the set of the locally integrable functions $[-\tau, 0] \to \mathbf{C}$, with $\tau > 0$, into $\mathbf{C}$ and for every function $y : [-\tau, +\infty[\to \mathbf{C}$ it is denoted $y_t(s) = y(t + s)$ for all $(t, s) \in [0, +\infty[\times [-\tau, 0]$.

Remark 1. *Notice that the equation (2.2) is the perturbation of equation (2.1). The terms $\hat{a}(t)y'(t)$ and $c(t, y_t)$ are the perturbations which will satisfy smallness conditions as it is mentioned below.*

Hypothesis 2. *Every member of the family $\{c(t, \cdot)\}_{t \geq 0}$ leads the set of the locally integrable functions $[-\tau, 0] \to \mathbf{R}$, with $\tau > 0$, into $\mathbf{R}$.*

It is wanted to get an asymptotic formula for a solution of (2.2) which is a small perturbation of x_0. For that, one aditional definition and a hypothesis are given.

Let

$$\Phi(t, s) = \exp\left(\int_s^t [a(\xi) - 2\lambda_0(\xi)]d\xi\right),$$

for all $t, s \geq 0$, where $\lambda_0 = \frac{x_0'}{x_0}$.

Assume that:

Hypothesis 3. *There are positive constants α and K such that*

$$\int_s^t \Re e[a(\xi) - 2\lambda_0(\xi)]d\xi \leq \ln K - \alpha|t - s|,$$

for all $t, s \geq 0$, where $\lambda_0 = \frac{x_0'}{x_0}$ and x_0 is a solution of a non perturbed second order linear equation (2.1) which satifies the Hypothesis 1.

Then, under the Hypothesis 3, we have the following inequality:

$$|G(t,s)| \leq Ke^{-\alpha|t-s|}, \tag{2.3}$$

for all $t, s \geq 0$, where

$$G(t,s) = \begin{cases} \Phi(t,s), & \text{if} \quad \Re e \int_s^t [a(\xi) - 2\lambda_0(\xi)]d\xi < 0 \\ 0, & \text{if} \quad \text{else} \end{cases}$$

When (2.1) is autonomous, i.e., $a(t) = a_0$ and $b(t) = b_0$ are real constants we have

$$\lambda_0 = \frac{1}{2}(a_0 \pm \sqrt{a_0^2 + 4b_0}),$$

which are roots of the characteristic equation of (2.1), i.e,

$$\lambda^2 - a_0\lambda - b_0 = 0 \tag{2.4}$$

and the term

$$a(\xi) - 2\lambda_0(\xi) = a_0 - 2\lambda_0 = -\frac{d}{d\lambda}[\lambda^2 - a_0\lambda - b_0]\Big|_{\lambda=\lambda_0} = \mp\sqrt{a_0^2 + 4b_0},$$

is the difference between the two roots of the characteristic equation (2.4).

2.1. Procedure

We make the change of variables, $y(t) = x_0(t)z(t)$ for $t \geq 0$ and we arrive to the differential equation:

$$\begin{aligned} z''(t) &= [a(t) - 2\lambda_0(t)]z'(t) + \hat{a}(t)[\lambda_0 z + z'] \\ &\quad + c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} z_t), \; t \geq \tau. \end{aligned} \tag{2.5}$$

For simplifying notations it is made $\tilde{c}(t,\varphi) = c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi}\varphi)$ for every locally integrable function $\varphi : [-\tau, 0] \to \mathbf{C}$. Let

$$f_1(t) = \tilde{c}(t,1) + \hat{a}(t)\lambda_0(t),$$

and $f(t,\mu) = \hat{a}(t)\mu(t) + \tilde{c}(t, [e^{\int_t^{t+\cdot} \mu(\xi)d\xi} - 1]) - \mu(t)^2$. Then, for every solution of the equation

$$\mu'(t) = [a(t) - 2\lambda_0(t)]\mu(t) + f_1(t) + f(t,\mu), \quad t \geq 2\tau + t_0, \tag{2.6}$$

we have that $z(t) = e^{\int_{2\tau}^t \mu(s)ds}$ is a solution of the equation (10), for $t \geq 2\tau$.

Notice that $f(t,0) = 0$.

The following steps are

a) Consider

$$\mathcal{T} : X \to X,$$

where $X = (X, \|\cdot\|)$ is a suitable Banach space, defined by

$$(\mathcal{T}\mu)(t) = \begin{cases} \int_{t_0+2\tau}^{+\infty} G(t,s)[f_1(s) + f(s,\mu)]ds, & \text{if } \quad t \geq 2\tau + t_0 \\ 0, & \text{if } \quad t \in [t_0 - \tau, t_0 + 2\tau[. \end{cases} \tag{2.7}$$

b) Find an attractor fixed point for $\mathcal{T}$: $\mu_\infty \in X$ $(\mu_\infty = \mathcal{T}\mu_\infty)$, so equation (2.2) has a solution y such that

$$\begin{aligned} y(t) &= x_0(t)e^{\int_{t_0}^{t} \mu_\infty(\xi)d\xi}, \\ y'(t) &= [\lambda_0(t) + \mu_\infty(t)]y(t). \end{aligned} \tag{2.8}$$

c) Since μ_∞ is an attractor we have that μ_∞ can be written as

$$\mu_\infty = \mathcal{T}(0) + \sum_{n=1}^{+\infty} [\mathcal{T}^{n+1}(0) - \mathcal{T}^n(0)], \tag{2.9}$$

in the chosen norm.

3. Cases

3.1. The perturbations $\hat{a}(t)$ and $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -)$ are bounded

Proposition 3. *Let M, k_1 and k_2 be such that $M \in]0, \frac{\alpha}{2K}[$, $k_2 \in]0, 1 - \frac{2MK}{\alpha}[$ and $k_1 \in]0, \frac{M^2K}{2\alpha}[$. Suppose that in equation (2.1), hypotheses 1, 2 and 3 are satisfied. Assume that in equation (2.2), $\hat{a}(t)$ and $|c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -)|$ are so small that satisfy*

$$\left| \int_{2\tau}^{+\infty} G(t,s) f_1(s) ds \right| \leq k_1 \tag{3.1}$$

and

$$\int_{2\tau}^{+\infty} e^{-\alpha|t-s|} [|\hat{a}(s)| + e^{\frac{\alpha\tau}{2K}} \tau |c(s, e^{\int_s^{s+\cdot} \lambda_0(\xi)d\xi} -)|] ds \leq k_2. \tag{3.2}$$

Then, equation (2.2) has a solution with the following formula

$$
\begin{aligned}
y(t) &= x_0(t)\exp\left(\int_{2\tau}^{t}\left[\mu_1(s)+\sum_{n=1}^{+\infty}\Delta_n(s)\right]ds\right),\\
y'(t) &= \left[\lambda_0(t)+\mu_1(t)+\sum_{n=1}^{+\infty}\Delta_n(t)\right]y(t),
\end{aligned}
\tag{3.3}
$$

where

$$
\Delta_n(t)=\int_{2\tau}^{+\infty}G(t,s)[f(s,\mu_n)-f(s,\mu_{n-1})]ds,
\tag{3.4}
$$

and the functions $\{\mu_n\}_{n=0}^{+\infty}$ *are given by* $\mu_0=0$
and

$$
\mu_n(t)=\begin{cases}\int_{2\tau}^{+\infty}G(t,s)[f_1(s)+f(s,\mu_{n-1})]ds, & \textit{for} \quad t\geq(n+1)\tau\\ 0, & \textit{for} \quad t\in[0,(n+1)\tau[\end{cases}
\tag{3.5}
$$

for $n\in\mathbf{N}$.

Proof of Proposition 3 :
It can be proved that M, k_1 and k_2 satisfy

$$
\begin{aligned}
k_1+Mk_2+\tfrac{M^2K}{\alpha} &\leq \tfrac{M^2K}{2\alpha}+M\left(1-\tfrac{2MK}{\alpha}\right)+\tfrac{M^2K}{\alpha}\\
&= M\left(1-\tfrac{MK}{2\alpha}\right)\\
&< M
\end{aligned}
$$

and $k_2+\frac{2MK}{\alpha}<1$.

Then, if we consider $B[0,M]$ as the closed ball centered in 0 with radius M in $(\mathcal{B}([0,+\infty[,\mathbf{R}),\|\cdot\|_\infty)$, then by the relations (3.1) and (3.2), $\mathcal{T}(B[0,M])\subseteq B[0,M]$ and the restriction

$$\mathcal{T}:B[0,M]\to B[0,M]$$

is a contraction operator. So, there exists a unique

$$\mu_\infty\in B[0,M]:\ \mathcal{T}\mu_\infty=\mu_\infty.$$

Then,

$$
\mu_\infty(t)=\int_{2\tau}^{+\infty}G(t,s)[f_1(s)+f(s,\mu_\infty)]ds,
\tag{3.6}
$$

for $t \geq 2\tau$. From (2.8), the formula (3.3) is obtained.

To make the formula (3.3) easier to be understood, the following remarks are given.

Remark 2. *Let $\lambda_n(t) = \lambda_0(t)$, for $t \in [0, \tau[$ and*

$$\lambda_n(t) = \lambda_0(t) + \mu_1(t) + \sum_{j=1}^{n} \Delta_j(t), \text{ for } t \geq \tau$$

and $n \in \mathbf{N}$. Then, given $n_0 \in \mathbf{N}$, $\lambda_n(t) = \lambda_{n_0}(t)$, for $t \in [0, (n_0+1)\tau[$ and $n \geq n_0$. So, (3.3) can be written as:

$$y(t) = \exp\left(\int_0^t \left[\lim_{n\to+\infty} \lambda_n(s)\right] ds\right)(c + o(1)),$$

as $t \to +\infty$.

Remark 3. *If $\Delta_k \in L^1$ for some $k \in \mathbf{N}$, then $\sum_{m=k}^{+\infty} \Delta_m \in L^1$. In this case (3.3) can be written as*

$$y(t) = \exp\left(\int_{2\tau}^t \left[\lambda_0(s) + \mu_1(s) + \sum_{j=1}^{n-1} \Delta_j(s)\right] ds\right)(\hat{c} + o(1)),$$

as $n \to +\infty$, for some $\hat{c} \in \mathbf{C}$. Here $\sum_{j=1}^{n-1} \Delta_j(s) = 0$ if $n = 1$.

3.2. The perturbations $\hat{a}(t)$ and $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -)$ are in L^p

Similar relations to (3.1) and (3.2) can be given for the cases where $f_1(t)$ and $f(t, \mu)$ and μ are in L^p. So, we have the following result.

Proposition 4. *Suppose that in equation (2.1), hypotheses 1, 2 and 3 are satisfied. Assume that in equation (2.2), $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -) \in L^p$ for some $p \in [2^{n-1}, 2^n[$ and $n \in \mathbf{N}$. Then, (2.2) has a solution with the following asymptotic formula*

$$y(t) = x_0(t) \exp\left(\int_{2\tau}^t \left[\mu_1(s) + \sum_{j=1}^{n-1} \Delta_j(s)\right] ds\right)(c + o(1)), \tag{3.7}$$

as $t \to +\infty$, where the functions $\{\Delta_j\}_{j=1}^{+\infty}$, $\{\mu_j\}_{j=0}^{+\infty}$ are given by the relation (3.4) and (3.5).

Example 1: Consider the second order delayed differential equation

$$y''(t) = (a_0 + \hat{a}(t))y'(t) + (b_0 + \hat{b}(t))y(t) + \hat{c}(t)y(t-\tau),\ t \geq 0, \tag{3.8}$$

where $a_0, b_0 \in \mathbf{R}$ such that $a_0^2 + 4b_0 > 0$ and $\hat{a}, \hat{b}, \hat{c} \in L^2$. If we consider

$$x''(t) = a_0 x'(t) + b_0 x(t),\ t \geq 0,$$

as the equation (2.1) which has the following characteristic roots $\lambda_{0\pm} = \frac{1}{2}(a_0 \pm \sqrt{a_0^2 + 4b_0})$, then

$$a_0 - 2\lambda_{0\pm} = \mp\sqrt{a_0^2 + 4b_0} = \lambda_{0\mp} - \lambda_{0\pm}.$$

Since $a_0^2 + 4b_0 > 0$, (2.3) is satisfied. In (2.2) we take,

$$c(t, \varphi) = \hat{b}(t)\varphi(0) + \hat{c}(t)\varphi(-\tau).$$

Then, when we consider $\lambda_0 = \lambda_{0-}$. So,

$$\mu_\infty(t) = \int_t^{+\infty} e^{(\lambda_{0+} - \lambda_{0-})(t-s)}[\hat{a}(s)\lambda_{0-} + \hat{b}(s) + e^{-\lambda_{0-}\tau}\hat{c}(s)]ds + \Theta(t),$$

for $t \geq 2\tau$, where

$$\begin{array}{rcl} \Theta(t) & = & \int_t^{+\infty} e^{(\lambda_{0+} - \lambda_{0-})(t-s)} \\ & \times & [\hat{a}(s)\mu_\infty(s) \\ & + & \hat{c}(s)e^{-\lambda_{0-}\tau}\left[e^{\int_s^{s+\cdot}\mu_\infty(\xi)d\xi} - 1\right] \\ & - & \mu_\infty(s)^2]ds. \end{array}$$

Clearly, $\Theta \in L^1$.

Then, by integration by parts

$$\int_0^t \mu_\infty(s)ds = \frac{1}{\lambda_{0-} - \lambda_{0+}} \int_0^t [\hat{a}(s)\lambda_{0-} + \hat{b}(s) + e^{-\lambda_{0+}\tau}\hat{c}(s)]ds + \gamma(t),$$

where $\lim_{t\to+\infty}\gamma(t) = \text{const}$. So, equation (3.8) has a solution $y(t)$ such that

$$y(t) = x_0(t)\exp\left(\frac{1}{\lambda_{0+} - \lambda_{0-}}\int_0^t [\hat{a}(s)\lambda_{0-} + \hat{b}(s) + e^{-\lambda_{0-}\tau}\hat{c}(s)]ds\right)(c+o(1)),$$

as $t \to +\infty$.

Now, when we consider $\frac{x_0'}{x_0} = \lambda_0 = \lambda_{0+}$ Then, (2.2) has a solution y such that

$$y(t) = x_0(t)\exp\left(\frac{1}{\lambda_{0-} - \lambda_{0+}}\int_0^t [\hat{a}(s)\lambda_{0+} + \hat{b}(s) + e^{-\lambda_{0+}\tau}\hat{c}(s)]ds\right)(c+o(1)),$$

as $t \to +\infty$.

3.3. Equation (2.1) satisfies Hyp. 1 and a weaker hypothesis than Hyp. 3 but perturbations $\hat{a}(t)$ and $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -)$ are in L^1

Consider the following hypothesis

Hypothesis 4. *$\int_s^t \Re e[a(\xi) - 2\lambda_0(\xi)]d\xi$ is bounded for all $t, s : s \geq t \geq 0$ or*

$$\lim_{t \to +\infty} \int_s^t \Re e[a(\xi) - 2\lambda_0(\xi)]d\xi = -\infty.$$

Under Hypothesis 4, the relations (3.1) and (3.2) can be extended for the cases where $f_1(t)$ and $f(t, \mu)$ and μ are in L^1 for a parameter t large enough. So, we have the following result.

Proposition 5. *Suppose that in equation (2.1), hypotheses 1, 2 and 4 are satisfied. Assume that in equation (2.2), $c(t, e^{\int_t^{t+\cdot} \lambda_0(\xi)d\xi} -) \in L^1$. Then, (2.2) has a solution with the following asymptotic formula*

$$y(t) = x_0(t)(c + o(1)), \tag{3.9}$$

as $t \to +\infty$, where the functions $\{\Delta_n\}_{n=1}^{+\infty}$, $\{\mu_n\}_{n=0}^{+\infty}$ are given by the relation (3.4) and (3.5).

Example 1: Consider the equation

$$y''(t) + a^2 y(t) = \frac{1}{t^2} y(t - \tau),\ t \geq 0, \tag{3.10}$$

has solutions y_0, y_1 with the following asymptotic formulas

$y_0(t) = \cos(at)(1 + o(1))$ and $y_1(t) = \sin(at)(1 + o(1)),$ as $t \to +\infty.$

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Samuel Castillo
Departamento de Matemática
Facultad de Ciencias
Universidad del Bío-Bío
Casilla 5 - C
Concepción - Chile
e-mail : scastill@ubiobio.cl

and

Manuel Pinto
Departamento de Matemática
Facultad de Ciencias
Universidad de Chile
Casilla 653
Santiago - Chile
e-mail : pintoj@uchile.cl