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ON WREATH PRODUCT OF PERMUTATION GROUPS

IBRAHIM A. A.
USMANU DANFODIYO UNIVERSITY, SOKOTO - NIGERIA
and
AUDU M. S.
UNIVERSITY OF JOS, JOS - NIGERIA

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Abstract

This report is essentially an upgrade of the results of Audu (see [1] and [2]) on some finite permutation groups. It consists of the basic procedure for computing wreath product of groups. We also discussed the conditions under which the wreath products of permutation groups are faithful, transitive and primitive. Further, the centre of the stabilizer and the centre of wreath products was investigated. and finally, an illustration was supplied to support our findings.

Key words: *Centre, Faithful, Groups, primitive, Stabiliser, Transitive, Wreath product.*

1. INTRODUCTION

Recently, wreath prouct of groups has been used to explore some useful characteristics of finite groups in connection with permutation designes and construction of lattices [3] as well as in the study of interconnection networks [4] for instance. Further, Audu (see [2]) used wreath product to study the structure of some finite permutation groups.

This report provides a theoretical frame work for interpreting various structural properties of finite permutation groups.

1.1. GROUP ACTION

Let G be a group and Ω be a non empty set. We say that G acts on the set Ω (or that G permutes Ω) if to each g in G and each α in Ω, there corresponds a unique point αg in Ω such that, for all α in Ω and g_1, g_2 in G we have that

$(\alpha g_1)g_2 = \alpha g_1 g_2$ and $\alpha 1 = \alpha$.

To be explicit, we say under the condition that G acts on the set Ω on the right.

Let H and K be two groups. We say that H acts on K as a group if to each k in K there corresponds a unique element K^h in K such that for h_1, h_2, h in H and k_1, k_2, k in K

$$(k^{h_1})^{h_2} = k^{h_1 h_2},\ k^1 = k \text{ and } (k_1 k_2)^h = k_1^h k_2^h \tag{1.1}$$

Theorem 1.1

Let C and D be permutation groups on Γ and Δ respectively. Let C^Δ be the set of all maps of Δ into the permutation group C. That is $C^\Delta = \{f : \Delta \to C\}$. For any f_1, f_2 in C^Δ, let $f_1 f_2$ in C^Δ be defined for all δ in Δ by

$$(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta).$$

Thus composition of functions is point-wise and the operator is placed on the right. With respect to this operation of multiplication, C^Δ acquires the structure of a group.

Proof

(i) C^{Δ} is non-empty and is closed with respect to multiplication. If $f_1, f_2 \in C^{\Delta}$, then $f_1(\delta), f_2(\delta) \in C$.

Hence $f_1(\delta).f_2(\delta) \in C$. This implies that $(f_1f_2)(\delta) \in C$ and so $f_1f_2 \in C^{\Delta}$.

(ii) Since multiplication is associative so also is the multiplication in C^{Δ}.

(iii) The identity element in C^{Δ} is the map $e : \Delta \to C$ given by $e(\delta) = 1$ for all $\delta \in \Delta$ and $1 \in C$.

(iv) Every element $f \in C^{\Delta}$ is defined for all $\delta \in \Delta$ by $f^{-1}(\delta) = f(\delta)^{-1}$.

Thus C^{Δ} is a group with respect to the multiplication defined above. (We denote this group by P).

Lemma 1.2

Assume that D acts on P as follows: $f^d(\delta) = f(\delta d^{-1})$ for all $\delta \in \Delta, d \in D$. Then D acts on P as a group.

Proof

Take $f, f_1, f_2 \in P$ and $d, d_1, d_2 \in D$ then

(i)

$$\begin{aligned}(f^{d_1})^{d_2}(\delta) &= f^{d_1}(\delta d_2^{-1}) \\ &= f(\delta d_2^{-1} d_1^{-1}) \\ &= f^{d_1 d_2}(\delta)\end{aligned}$$

(ii)

$$\begin{aligned}f^1(\delta) &= f(\delta 1^{-1}) \\ &= f(\delta)\end{aligned}$$

(iii)

$$\begin{aligned}(f_1f_2)^d(\delta) &= f_1f_2(\delta d^{-1}) \\ &= f_1(\delta d^{-1})f_2(\delta d^{-1}) \\ &= f_1^d(\delta)f_2^d(\delta).\end{aligned}$$

Thus D acts on P as a group (refer to(3.1)).

Theorem 1.2

Let D act on P as a group. Then the set of all ordered pairs (f, d) with $f \in P$ and $d \in D$ acquires the structure of a group when we define for all $f_1, f_2 \in P$ and $d_1, d_2 \in D$

$$(f_1, d_1)(f_2, d_2) = (f_1 f_2^{d_1^{-1}}, d_1 d_2).$$

Proof

(i) Closure property follows from the definition of multiplication.
(ii) Take $f_1, f_2, f_3 \in P$ and $d_1, d_2, d_3 \in D$. Then

$$\begin{aligned}
[(f_1, d_1)(f_2, d_2)](f_3, d_3) &= (f_1 f_2^{d_1^{-1}}, d_1 d_2)(f_3, d_3) \\
&= (f_1 f_2^{d_1^{-1}} f_3^{(d_1 d_2)^{-1}}, d_1 d_2 d_3) \\
&= (f_1 f_2^{d_1^{-1}} f_3^{d_2^{-1} d_1^{-1}}, d_1 d_2 d_3).
\end{aligned}$$

Also, we have in the same manner that

$$\begin{aligned}
(f_1, d_1)[(f_2, d_2)(f_3, d_3)] &= (f_1, d_1)(f_2 f_3^{d_2^{-1}}, d_2 d_3) \\
&= (f_1 (f_2 f_3^{d_2^{-1}})^{d_1^{-1}}, d_1 d_2 d_3) \\
&= (f_1 f_2^{d_1^{-1}} f_3^{d_2^{-1} d_1^{-1}}, d_1 d_2 d_3).
\end{aligned}$$

hence multiplication is associative.
(iii) We know that for every $f \in P, f^1 = f$. Now for every $d \in D$, the map $f \to f^d$ is an automorphism of P. Also if e is the identity element in P, then $e^d = e$. Also, $(f^{-1})^d = (f^d)^{-1}$. Now

$$\begin{aligned}
(f, d)(e, 1) &= (f e^{d^{-1}}, d1) = (f e^{d^{-1}}, d) \\
&= (f(e^{-1}), d) = (f, d).
\end{aligned}$$

Also, using the reverse order, we have that

$$(e, 1)(f, d) = (e f^{1^{-1}}, 1d) = (ef, d) = (f, d)$$

Thus identity element exists.
(iv)

$$\begin{aligned}
(f, d)((f^{-1})^d, d^{-1}) &= (f(f^{-1})^d)^{-d}, dd^{-1}) = (f(f^{-1})^{dd^{-1}}, dd^{-1}) \\
&= (f(f^{-1})^1, dd^{-1}) = (e, 1)
\end{aligned}$$

Also,

$$\begin{aligned}((f^{-1})^d, d^{-1})(f,d) &= ((f^{-1})^d f^d, d^{-1}d) = (ff^{-1})^d, d^{-1}d) \\ &= (e^d, 1) = (e, 1)\end{aligned}$$

Thus when D acts on P, the set of all ordered pairs (f,d) with $f \in D, d \in D$, is a group if we define

$(f_1, d_1)(f_2, d_2) = (f_1 f_2^{d_1^{-1}}, d_1 d_2)$.

In what follows, we supply a formal definition of Wreath Product of permutation groups.

2. WREATH PRODUCT

The Wreath product of C by D denoted by $W = C\,wr\,D$ is the semi-direct product of P by D, so that, $W = \{(f,d) \mid f \in P,\ d \in D\}$, with multiplication in W defined as

$$(f_1, d_1)(f_2, d_2) = ((f_1 f_2^{d_1^{-1}}), (d_1 d_2))$$

for all $f_1, f_2 \in P$ and $d_1, d_2 \in D$.

Henceforth, we write fd instead of (f,d) for elements of W.

Theorem 2.1

Let D act on P as $f^d(\delta) = f(\delta d^{-1})$ where $f \in P$, $d \in D$ and $\delta \in \Delta$. Let W be the group of all juxtaposed symbols fd, with $f \in P$, $d \in D$ and multiplication given by

$$(f_1, d_1)(f_2, d_2) = (f_1 f_2^{d_1^{-1}})(d_1 d_2).$$

Then W is a group called the semi-direct product of P by D with the defined action.

Proof

Same as in **Theorem 1.1.**

Based on the forgoing we note the following:

- If C and D are finite groups, then the wreath product W determined by an action of D on a finite set is a finite group of order

$$|W| = |C|^{|\Delta|} . |D| .$$

- P is a normal subgroup of W and D is a subgroup of W.
- The action of W on $\Gamma \times \Delta$ is given by
  $(\alpha, \beta)fd = (\alpha f(\beta), \beta d)$ where $\alpha \in \Gamma$ and $\beta \in \Delta$.

We shall now identify the conditions under which the wreath products will be transitive, faithful and primitive. We shall also determine the centre and the stabilizer of the wreath product W.

2.1. Transitivity of Won $\Gamma \times \Delta$

Suppose that we take two arbitrary points (α_1, δ_1) and (α_2, δ_2) in $\Gamma \times \Delta$. Then W will be transitive on $\Gamma \times \Delta$ if there exists $fd \in W$, $f \in P, d \in D$ such that $(\alpha_1, \delta_1)^{fd} = (\alpha_2, \delta_2)$. This will hold if $(\alpha_1 f(\delta_1), \delta_1 d) = (\alpha_2, \delta_2)$; that is, if $\alpha_1 f(\delta_1) = \alpha_2, \delta_1 d = \delta_2$. Thus such fd exists if C and D are transitive on Γ and Δ respectively, which is the neccessary condition for W to be transitive on $\Gamma \times \Delta$.

2.2. The Stabilizer $W_{(\alpha,\delta)}$of a point (α, δ) in $\Gamma \times \Delta$

Under the action of W on $\Gamma \times \Delta$, the stabilizer of any point (α, δ) in $\Gamma \times \Delta$ denoted by $W_{(\alpha,\delta)}$ is given by

$$\begin{aligned} W_{(\alpha,\delta)} &= \{fd \mid (\alpha, \delta)^{fd} = (\alpha, \delta)\} \\ &= \{fd \in W \mid (\alpha f(\delta), \delta d) = (\alpha, \delta)\} \\ &= \{fd \in W \mid \alpha f(\delta) = \alpha, \delta d = \delta\} \ = F(\delta)_\alpha D_\delta \end{aligned}$$

where $F(\delta)_\alpha$ is the set of all $f(\delta)$ that stabilize α, and D_δ is the stabilizer of δ under the action of D on Δ.

2.3. Faithfulness of W on $\Gamma \times \Delta$

We recall that W is faithful on $\Gamma \times \Delta$ if the identity element of W is the only element that fixes every point of $\Gamma \times \Delta$. Now the identity element of W is 1 and thus if W is to be faithful on $\Gamma \times \Delta$ then for any (α, δ) in $\Gamma \times \Delta$, the stabilizer of W on $(\alpha, \delta), W_{(\alpha,\delta)}$, must be $F(\delta)_\alpha D_\delta = 1.$ Hence, $F(\delta)_\alpha = 1$ and $D_\delta = 1$ for all $\alpha \in \Gamma,\ \delta \in \Delta$. But $\alpha f(\delta) = \alpha,\ \delta d = \delta$ for all $f(\delta) \in F(\delta)_\alpha$ and for all $d \in D_\delta$.

This must imply that $f(\delta) = 1$ and $d = 1$. Thus we deduce that W would be faithful on $\Gamma \times \Delta$, if the stabilizers of any $\alpha \in \Gamma$and $\delta \in \Delta$ are the identity elements in P and D respectively. Therefore we conclude that W would be faithful on $\Gamma \times \Delta$, if P or C and D are faithful on Γ and Δ respectively.

2.4. Primitivity of Won $\Gamma \times \Delta$

Recall that W would be primitive on $\Gamma \times \Delta$, if and only if given any (α, δ) in $\Gamma \times \Delta, W_{(\alpha,\delta)}$, the stabilizer of (α, δ), is a maximal subgroup of W.

In what follows, we provide this necessary and sufficient condition for the primitivity of W on $\Gamma \times \Delta$.

Now $W_{(\alpha,\delta)} = F(\delta)_\alpha D_\delta$, where $F(\delta)_\alpha$ is the set of those f in P such that $f(\delta)$ fixes α, and D_δ is the stabilizer of δ under the action of D on Δ.

As $F(\delta)_\alpha$ does not include those f in P which do not stabilize α, we have that $F(\delta)_\alpha <\ P$.

We note that in general, $P < W$, PD_δ is a subgroup of W and it is proper unless $D_\delta = D$. We therefore conclude that

$$W_{(\alpha,\delta)} = F(\delta)_\alpha D_\delta < PD_\delta < PD = W$$

Hence $W_{(\alpha,\delta)}$ is not a maximal subgroup of W. Thus W would be imprimitive on $\Gamma \times \Delta$ in a natural way.

However, if $|\Gamma| = 1$; that is, $\Gamma = \{\alpha\}$ say, then $C_\Gamma = C_\alpha = C$. In particular,

$$\alpha f(\delta) = \alpha \text{ for all } f \text{ in } P.$$

Thus $F(\delta)_\alpha = P$ and hence $F(\delta)_\alpha D_\delta = PD_\delta$. And if in addition, D is primitive on Δ then D_δ is maximal in D and hence $PD_\delta = F(\delta)_\alpha D_\delta = W_{(\alpha,\delta)}$ will be maximal in W. That is, W will be primitive on $\Gamma \times \Delta$.

Again, if $|\Delta| = 1$; that is, $\Delta = \{\delta\}$ say, then $D_\delta = D$ and

$$W_{(\alpha,\delta)} = F(\delta)_\alpha D_\delta = F(\delta) D$$

And if in addition, C is primitive on Γ, then C_α will be maximal in $C = \{f(\delta) \mid \delta \in \Delta,\ f \in P\}$. Correspondingly, $F(\delta)_\alpha$ will be maximal in P and hence $W_{(\alpha,\delta)}$ will be maximal in W. That is, W will be primitive on $\Gamma \times \Delta$.

In conclusion, we have shown that W is imprimitive on $\Gamma \times \Delta$ in a natural way, unless $|\Gamma| = 1$ and D is primitive on Δ or $|\Delta| = 1$ and C is primitive on Γ.

2.5. The Centre of W

We denote the centre of W by $Z(W)$ which is defined as
$Z(W) = \{fd \mid (fd)(f_1d_1) = (f_1d_1)(fd)\ \forall f_1 \in P,\ d_1 \in D\}$.
Hence $fd \in Z(W)$ if and only if

$$ff_1^{d^{-1}}dd_1 = f_1f^{d_1^{-1}}d_1d \text{ for all } f_1 \in P,\ d_1 \in D. \tag{2.1}$$

We now solve for f and d . Put $d_1 = 1$. Then (2.1) becomes

$$ff_1^{d^{-1}}d = f_1fd \tag{2.2}$$

Put $f_1 = 1$. Then (2.1) also becomes

$$fdd_1 = f^{d_1^{-1}}d_1d \text{ for all } d_1 \in D. \tag{2.3}$$

From (2.1) it follows that for fd to be in $Z(W)$ it is necessary that $d \in Z(D)$.

Claim

(2.4) If $C \neq 1,\ fd \in Z(W)$ and $d \in Z(D)$, then $\delta d = \delta$ for all $\delta \in \Delta$

To show this, let $\delta \in \Delta$ and choose $f_1 \in P$ such that

$$f_1(\delta) = c \neq 1,\ c \in C \text{ and } f_1(\delta') = 1 \text{ for all } \delta' \neq \delta \tag{2.5}$$

Then from (2.2), we have that $ff_1^{d^{-1}} = f_1f$ and so,

$$f_1(\delta)f(\delta) = f(\delta)f_1(\delta d) = f(\delta), \text{ if } \delta d \neq \delta.$$

Hence, $f_1(\delta) = 1$. But this is false by (2.5) and hence we must have $\delta d = \delta$ for all $\delta \in \Delta$. Accordingly, our claim is correct.
Furthermore, (2.2) implies that for all $\delta \in \Delta$,

$$\begin{aligned} f_1(\delta)f(\delta) &= f(\delta)f_1(\delta d) \\ &= f(\delta)f_1(\delta) \\ \text{Hence } f(\delta) &\in Z(C)\forall\delta \in \Delta \end{aligned} \tag{2.6}$$

Also (2.3) implies that

$$f(\delta d_1) = f(\delta) \tag{2.7}$$

for all $\delta \in \Delta,\ d_1 \in D$ (since $d \in Z(D)$). Now (2.7) shows that f is constant over orbits of D in Δ. Thus from (2.4),(2.6) and (2.7) we conclude that provided $C \neq \{1\}$, $fd \in Z(W)$ if and only if
(i) $d \in Z(D) \cap K$, where $K = \{d \in D \mid \delta d = \delta \text{ for all } \delta \in \Delta\}$.
(ii) $f \in \{\Delta_{i \in I} \to Z(C)\}$ where Δ_i are orbits in Δ.

However, if $C = \{1\}$, then clearly $Z(W) = Z(D)$.

With the above notions, we conclude that

$$Z(W) = \begin{cases} Z(D), \text{ if } C = 1 \\ (\Pi Z_i)(Z(D) \cap K), \text{otherwise} \end{cases}$$

3. APPLICATION

Consider the permutation groups $C = \{(1), (135), (153)\}$ and $D = \{(1), (246), (264)\}$ on the sets $\Gamma = \{1, 3, 5\}$ and $\Delta = \{2, 4, 6\}$ respectively.

Let $P = C^\Delta = \{f \mid \Delta \to C\}$. Then $|P| = |C|^{|\Delta|} = 3^3 = 27$. The mappings are as follows

$f_1 : 2 \to (1),\ 4 \to (1),\ 6 \to (1)$
$f_2 : 2 \to (135),\ 4 \to (135),\ 6 \to (135)$
$f_3 : 2 \to (153),\ 4 \to (153),\ 6 \to (153)$
$f_4 : 2 \to (1),\ 4 \to (135),\ 6 \to (153)$
$f_5 : 2 \to (1),\ 4 \to (153),\ 6 \to (135)$

$f_6 \ : \ 2 \to (135), \ 4 \to (1), \ 6 \to (153)$
$f_7 \ : \ 2 \to (135), \ 4 \to (153), \ 6 \to (1)$
$f_8 \ : \ 2 \to (153), \ 4 \to (135), \ 6 \to (1)$
$f_9 \ : \ 2 \to (153), \ 4 \to (1), \ 6 \to (135)$
$f_{10} \ : \ 2 \to (1), \ 4 \to (1), \ 6 \to (135)$
$f_{11} \ : \ 2 \to (1), \ 4 \to (1), \ 6 \to (153)$
$f_{12} \ : \ 2 \to (135), \ 4 \to (135), \ 6 \to (1)$
$f_{13} \ : \ 2 \to (135), \ 4 \to (135), \ 6 \to (153)$
$f_{14} \ : \ 2 \to (153), \ 4 \to (153), \ 6 \to (1)$
$f_{15} \ : \ 2 \to (153), \ 4 \to (153), \ 6 \to (135)$
$f_{16} \ : \ 2 \to (1), \ 4 \to (135), \ 6 \to (135)$
$f_{17} \ : \ 2 \to (153), \ 4 \to (135), \ 6 \to (135)$
$f_{18} \ : \ 2 \to (1), \ 4 \to (153), \ 6 \to (153)$
$f_{19} \ : \ 2 \to (135), \ 4 \to (153), \ 6 \to (1153)$
$f_{20} \ : \ 2 \to (1), \ 4 \to (135), \ 6 \to (1)$
$f_{21} \ : \ 2 \to (1), \ 4 \to (153), \ 6 \to (1)$
$f_{22} \ : \ 2 \to (135), \ 4 \to (1), \ 6 \to (135)$
$f_{23} \ : \ 2 \to (135), \ 4 \to (153), \ 6 \to (135)$
$f_{24} \ : \ 2 \to (153), \ 4 \to (1), \ 6 \to (153)$
$f_{25} \ : \ 2 \to (153), \ 4 \to (135), \ 6 \to (153)$
$f_{26} \ : \ 2 \to (135), \ 4 \to (1), \ 6 \to (1)$
$f_{27} \ : \ 2 \to (153), \ 4 \to (1), \ 6 \to (1)$

We can easily verify that P is a group with respect to the operations $(f_1f_2)(\delta) = f_1(\delta)f_2(\delta)$ where $\delta \in \Delta$. We recall the definition of the action of D on P as $f^d(\delta) = f(\delta d^{-1})$, where $d \in D, \delta \in \Delta$, then D acts on P as groups. We recall the definition of $W = CwrD$, the semi-direct product of P by D in that order; that is, $W = \{fd : f \in P, d \in D\}$. Now, W is a group with respect to the operation

$$(f_1d_1)(f_2d_2) = (f_1f_2^{d_1^{-1}})(d_1d_2).$$

Accordingly, $d_1 = (1), \ d_2 = (246)$, and $d_3 = (264)$. Then the elements of W are

$(f_1d_1), (f_2d_1), (f_3d_1), (f_4d_1), (f_5d_1), (f_6d_1), (f_7d_1), (f_8d_1), (f_9d_1), (f_{10}d_1),$
$(f_{11}d_1), (f_{12}d_1), (f_{13}d_1), (f_{14}d_1), (f_{15}d_1), (f_{16}d_1), (f_{17}d_1), (f_{18}d_1), (f_{19}d_1),$
$(f_{20}d_1), (f_{21}d_1), (f_{22}d_1), (f_{23}d_1), (f_{24}d_1), (f_{25}d_1), (f_{26}d_1), (f_{27}d_1), (f_1d_2),$
$(f_2d_2), (f_3d_2), (f_4d_2), (f_5d_2), (f_6d_2), (f_7d_2), (f_8d_2), (f_9d_2), (f_{10}d_2), (f_{11}d_2),$
$(f_{12}d_2), (f_{13}d_2), (f_{14}d_2), (f_{15}d_2), (f_{16}d_2), (f_{17}d_2), (f_{18}d_2), (f_{19}d_2), (f_{20}d_2),$
$(f_{21}d_2), (f_{22}d_2), (f_{23}d_2), (f_{24}d_2), (f_{25}d_2), (f_{26}d_2), (f_{27}d_2), (f_1d_3), (f_2d_3),$

$(f_3d_3),(f_4d_3),(f_5d_3),(f_6d_3),(f_7d_3),(f_8d_3),(f_9d_3),(f_{10}d_3),(f_{11}d_3),(f_{12}d_3),$
$(f_{13}d_3),(f_{14}d_3),(f_{15}d_3),(f_{16}d_3),(f_{17}d_3),(f_{18}d_3),(f_{19}d_3),(f_{20}d_3),(f_{21}d_3),$
$(f_{22}d_3),(f_{23}d_3),(f_{24}d_3),(f_{25}d_3),(f_{26}d_3),(f_{27}d_3).$

Now define the action of W on $\Gamma \times \Delta$ as

$$(\alpha,\delta)^{fd} = (\alpha f(\delta), \delta d).$$

Further, $\Gamma\times\Delta = \{(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$
We obtain the following permutations by the action of W on $\Gamma \times \Delta$

$$\begin{aligned}
(1,2)^{f_1d_1} &= (1f_1(2),2d_1) = (1(1),2(1)) = (1,2)\\
(1,4)^{f_1d_1} &= (1f_1(4),4d_1) = (1(1),4(1)) = (1,4)\\
(1,6)^{f_1d_1} &= (1f_1(2),2d_1) = (1(1),6(1)) = (1,6)\\
(3,2)^{f_1d_1} &= (3f_1(2),2d_1) = (3(1),2(1)) = (3,2)\\
(3,4)^{f_1d_1} &= (3f_1(4),4d_1) = (3(1),4(1)) = (3,4)\\
(3,6)^{f_1d_1} &= (3f_1(6),6d_1) = (3(1),6(1)) = (3,6)\\
(5,2)^{f_1d_1} &= (5f_1(2),2d_1) = (5(1),2(1)) = (5,2)\\
(5,4)^{f_1d_1} &= (5f_1(4),4d_1) = (5(1),4(1)) = (5,4)\\
(5,6)^{f_1d_1} &= (5f_1(6),6d_1) = (5(1),6(1)) = (5,6)
\end{aligned}$$

And in summary,

$$(\Gamma\times\Delta)^{f_1d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6) \end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_2d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,2),(3,4),(3,6),(5,2),(5,4),(5,6),(1,2),(1,4),(1,6) \end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_3d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (5,2),(5,4),(5,6),(1,2),(1,4),(1,6),(3,2),(3,4),(3,6) \end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_4d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,2),(3,4),(5,6),(3,2),(5,4),(1,6),(5,2),(1,4),(3,6) \end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_5d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,2),(5,4),(3,6),(3,2),(1,4),(5,6),(5,2),(3,4),(1,6) \end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_6d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,2),(1,4),(5,6),(5,2),(3,4),(1,6),(1,2),(5,4),(3,6) \end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_7d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,2),(5,4),(1,6),(5,2),(1,4),(3,6),(1,2),(3,4),(5,6) \end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_8d_1} = \begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (5,2),(3,4),(1,6),(1,2),(5,4),(3,6),(3,2),(1,4),(5,6) \end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_9d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,2),(1,4),(3,6),(1,2),(3,4),(5,6),(3,2),(5,4),(1,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{10}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,2),(1,4),(3,6),(3,2),(3,4),(5,6),(5,2),(5,4),(1,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{11}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,2),(1,4),(5,6),(3,2),(3,4),(1,6),(5,2),(5,4),(3,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{12}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,2),(3,4),(1,6),(5,2),(5,4),(3,6),(1,2),(1,4),(5,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{13}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,2),(3,4),(5,6),(5,2),(5,4),(1,6),(1,2),(1,4),(3,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{14}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,2),(5,4),(1,6),(1,2),(1,4),(3,6),(3,2),(3,4),(5,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{15}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,2),(5,4),(3,6),(1,2),(1,4),(5,6),(3,2),(3,4),(1,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{16}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,2),(3,4),(3,6),(3,2),(5,4),(5,6),(5,2),(1,4),(1,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{17}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,2),(3,4),(3,6),(1,2),(5,4),(5,6),(3,2),(1,4),(1,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{18}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,2),(5,4),(5,6),(3,2),(1,4),(1,6),(5,2),(3,4),(3,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{19}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,2),(5,4),(5,6),(5,2),(1,4),(1,6),(1,2),(3,4),(3,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{20}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,2),(3,4),(1,6),(3,2),(5,4),(3,6),(5,2),(1,4),(5,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{21}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,2),(5,4),(1,6),(3,2),(1,4),(3,6),(5,2),(3,4),(5,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{21}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,2),(5,4),(1,6),(3,2),(1,4),(3,6),(5,2),(3,4),(5,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{22}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,2),(1,4),(3,6),(5,2),(3,4),(5,6),(1,2),(5,4),(1,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{23}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,2),(5,4),(3,6),(5,2),(1,4),(5,6),(1,2),(3,4),(1,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{24}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,2),(1,4),(5,6),(1,2),(3,4),(1,6),(3,2),(5,4),(3,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{25}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,2),(3,4),(5,6),(1,2),(5,4),(1,6),(3,2),(1,4),(3,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{26}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,2),(1,4),(1,6),(5,2),(3,4),(3,6),(1,2),(5,4),(5,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{27}d_1}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,2),(1,4),(1,6),(1,2),(3,4),(3,6),(3,2),(5,4),(5,6)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{1}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(1,6),(1,2),(3,4),(3,6),(3,2),(5,4),(5,6),(5,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{2}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(3,6),(3,2),(5,4),(5,6),(5,2),(1,4),(1,6),(1,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{3}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(5,6),(5,2),(1,4),(1,6),(1,2),(3,4),(3,6),(3,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{4}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(3,6),(5,2),(3,4),(5,6),(1,2),(5,4),(1,6),(3,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{5}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(5,6),(3,2),(3,4),(1,6),(5,2),(5,4),(3,6),(1,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{6}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(1,6),(5,2),(5,4),(3,6),(1,2),(1,4),(5,6),(3,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{7}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(5,6),(1,2),(5,4),(1,6),(3,2),(1,4),(3,6),(5,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{8}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(3,6),(1,2),(1,4),(5,6),(3,2),(3,4),(1,6),(5,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{9}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(1,6),(3,2),(1,4),(3,6),(5,2),(3,4),(5,6),(1,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{10}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6),(1,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{11}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(1,6),(5,2),(3,4),(3,6),(1,2),(5,4),(5,6),(3,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{12}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(3,6),(1,2),(5,4),(5,6),(3,2),(1,4),(1,6),(5,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{13}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(3,6),(5,2),(5,4),(5,6),(1,2),(1,4),(1,6),(3,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{14}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(5,6),(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{15}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(5,6),(3,2),(1,4),(1,6),(5,2),(3,4),(3,6),(1,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{16}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(3,6),(3,2),(3,4),(5,6),(5,2),(5,4),(1,6),(1,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{17}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(3,6),(3,2),(1,4),(5,6),(5,2),(3,4),(1,6),(1,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{18}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(5,6),(5,2),(3,4),(1,6),(1,2),(5,4),(3,6),(3,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{19}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(5,6),(5,2),(5,4),(1,6),(1,2),(1,4),(3,6),(3,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{20}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(3,6),(1,2),(3,4),(5,6),(3,2),(5,4),(1,6),(5,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{21}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,4),(5,6),(1,2),(3,4),(1,6),(3,2),(5,4),(3,6),(5,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{22}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(1,6),(3,2),(5,4),(3,6),(5,2),(1,4),(5,6),(1,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{23}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(5,6),(3,2),(5,4),(1,6),(5,2),(1,4),(3,6),(1,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{24}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(1,6),(5,2),(1,4),(3,6),(1,2),(3,4),(5,6),(3,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{25}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(3,6),(5,2),(1,4),(5,6),(1,2),(3,4),(1,6),(3,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{26}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,4),(1,6),(1,2),(5,4),(3,6),(3,2),(1,4),(5,6),(5,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{27}d_2}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,4),(1,6),(1,2),(1,4),(3,6),(3,2),(3,4),(5,6),(5,2)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{1}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,6),(1,2),(1,4),(3,6),(3,2),(3,4),(5,6),(5,2),(5,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{2}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,6),(3,2),(3,4),(5,6),(5,2),(5,4),(1,6),(1,2),(1,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{3}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,6),(5,2),(5,4),(1,6),(1,2),(1,4),(3,6),(3,2),(3,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{4}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(1,6),(3,2),(5,4),(3,6),(5,2),(1,4),(5,6),(1,2),(3,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_5 d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,6),(5,2),(3,4),(3,6),(1,2),(5,4),(5,6),(3,2),(1,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_6 d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,6),(1,2),(5,4),(5,6),(3,2),(1,4),(1,6),(5,2),(3,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_7 d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,6),(5,2),(1,4),(5,6),(1,2),(3,4),(1,6),(3,2),(5,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_8 d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (5,6),(3,2),(1,4),(1,6),(5,2),(3,4),(3,6),(1,2),(5,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_9 d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (5,6),(1,2),(3,4),(1,6),(3,2),(5,4),(3,6),(5,2),(1,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{10} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,6),(1,2),(3,4),(3,6),(3,2),(5,4),(5,6),(5,2),(1,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{11} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,6),(1,2),(5,4),(3,6),(3,2),(1,4),(5,6),(5,2),(3,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{12} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,6),(3,2),(1,4),(5,6),(5,2),(3,4),(1,6),(1,2),(5,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{13} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,6),(3,2),(5,4),(5,6),(5,2),(1,4),(1,6),(1,2),(3,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{14} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (5,6),(5,2),(1,4),(1,6),(1,2),(3,4),(3,6),(3,2),(5,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{15} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (5,6),(5,2),(3,4),(1,6),(1,2),(5,4),(3,6),(3,2),(1,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{16} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6),(1,2),(1,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{17} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (5,6),(3,2),(3,4),(1,6),(5,2),(5,4),(3,6),(1,2),(1,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{18} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,6),(5,2),(5,4),(3,6),(1,2),(1,4),(5,6),(3,2),(3,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{19} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (3,6),(5,2),(5,4),(5,6),(1,2),(1,4),(1,6),(3,2),(3,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{20} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,6),(3,2),(1,4),(3,6),(5,2),(3,4),(5,6),(1,2),(5,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{21} d_3}=\begin{pmatrix} (1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\ (1,6),(5,2),(1,4),(3,6),(1,2),(3,4),(5,6),(3,2),(5,4)\end{pmatrix}$$

$$(\Gamma\times\Delta)^{f_{22}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,6),(1,2),(3,4),(5,6),(3,2),(5,4),(1,6),(5,2),(1,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{23}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,6),(5,2),(3,4),(5,6),(1,2),(5,4),(1,6),(3,2),(1,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{24}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,6),(1,2),(5,4),(1,6),(3,2),(1,4),(3,6),(5,2),(3,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{25}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,6),(3,2),(5,4),(1,6),(5,2),(1,4),(3,6),(1,2),(3,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{26}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(3,6),(1,2),(1,4),(5,6),(3,2),(3,4),(1,6),(5,2),(5,4)\end{pmatrix}$$
$$(\Gamma\times\Delta)^{f_{27}d_3}=\begin{pmatrix}(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\\(5,6),(1,2),(1,4),(1,6),(3,2),(3,4),(3,6),(5,2),(5,4)\end{pmatrix}$$

Rename the symbols as

$$\begin{array}{lllllllll}(1,2)&\rightarrow&1;&(3,4)&\rightarrow&5;&(3,2)&\rightarrow&4;\\(1,4)&\rightarrow&2;&(3,6)&\rightarrow&6;&(5,4)&\rightarrow&8;\\(1,6)&\rightarrow&3;&(5,2)&\rightarrow&7;&(5,6)&\rightarrow&9.\end{array}$$

Then the permutations in cyclic form are

$(1),(147)(258)(369),(174)(285)(396),(258)(396),(285)(369),(147)(396),$
$(147)(285),(174)(258),(174)(369),(369),(396),(147)(258),$
$(147)(258)(396),(174)(285),(174)(285)(369),(258)(369),(174)(258)(369),$
$(285)(396),(147)(285)(396),(258),(285),(147)(369),(147)(285)(369),$
$(174)(396),(174)(258)(396),(147),(174),(123)(456)(789),$
$(159)(267)(348),$
$(189)(294)(375),(126)(378)(459),(129)(345)(678),(156)(237),(489),$
$(153)(297)(486),(183)(264)(597),(189)(234)(567),(123456789),$
$(123789456),$
$(159726483),(159483726),(186429753),(186753429),$
$(126783459)(183426759),(129453786),(153729486),(126459783),$
$(129786453),(156723489),(153486729),(189423756),(183759426),$
$(156489723),(189756423),(132)(465)(798),(168)(249)(357),(195)(276)$
$(384),$
$(138)(246)(579),(135)(279)(468),(162)(387)(495),(165)(273)(498),$
$(198)(243)(576),(192)(354)(687),(135468792),(138795462),$
$(165732498),(162495738),(1984532765),(192768435),(13572468),$
$(192435768),(138462795),(162738495),(132465798),(13279465),$
$(168735492),(168492735),(195438762),(195762438),(165498732),$

(198765432).

It is observed that by acting W on $\Gamma \times \Delta$
(a) A permutation group is obtained which is found to be transitive since C and D are transitive on Γ and Δ respectively.
(b) W is faithful on $\Gamma \times \Delta$, since C and D are faithful on Γ and Δ respectively.
(c) W is imprimitive on $\Gamma \times \Delta$. The subsets of imprimitivity are $\{1, 4, 7\}$, $\{2, 5, 8\}$ and $\{3, 6, 9\}$.

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Ibrahim A. A.
Mathematics Department
Usmanu Danfodiyo University
PMB 2346 Sokoto
Nigeria
e-mail : aminualhaji@yahoo.co.uk

and

Audu M. S.
Mathematics Department
University of Jos
PMB 2084 JOS
PLATEAU STATE
NIGERIA 930003
Nigeria
e-mail : aminualhaji@yahoo.co.uk