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ABOUT DECAY OF SOLUTION OF THE WAVE EQUATION WITH DISSIPATION *

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Abstract

In this work, we consider the problem of existence of global solutions for a scalar wave equation with dissipation. We also study the asymptotic behaviour in time of the solutions. The method used here is based in nonlinear techniques.

Key words: wave equation, evolution model, decay of solution, asymptotic behaviour.

*Support: see acknowledgements.

1. Introduction

We will study the following evolution problem

$$
\begin{aligned}
&(1.1) \qquad & u_{tt} - \Delta u + a(x)u_t &= 0 \text{ in } \Omega \times \mathbf{R}^+ \,, \\
&(1.2) \qquad & u &= 0 \text{ on } , \partial\Omega \times \mathbf{R}^+ \,, \\
&(1.3) \qquad & u(0) = u^0 \,, \qquad & u_t(0) = u^1.
\end{aligned}
$$

where Ω is an open bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and a is a suitable, smooth and should not identically zero on $\overline{\Omega}$; besides, a can vanish in some part of $\overline{\Omega}$.

We define by

$$
E(t) = \frac{1}{2}\int_{\Omega} |u_t|^2 + |\nabla u|^2 \, dx \tag{1.4}
$$

the energy associated to the system (1.1)-(1.3). By Lemma 5.1 E is non increasing. Thus, we are interested in finding out what happens to $E(t)$ as t goes to infinity and what is its rate of decay.

In this work, we study the existence of global solution and the asymptotic behaviour of the wave equation with dissipation, where the initial conditions satisfy the mth-order compatibility condition, with allows us to obtain a more regular solution.

We use the semigroup theory [17], [6] to prove the existence and uniqueness of solution to the problem (1.1)-(1.3), as well as its continuous dependency of initial data. Likewise, we study the regularity of this solution.

In section 4 we make a complete study of certain integral inequalities [15]. Also we prove that

$$
\int_t^{\infty} f(\tau)^{1+\tau} d\tau \leq CF(t) \text{ implies } f(t) \leq \frac{C}{(1+t)^{\frac{1}{\sigma}}}
$$

and we use it in the proof of Lemma 4.2. Besides, in this section we introduce Lemma 4.4 which is an analogous version of Lemma 4.3.

Making use of the multiplicative techniques [13], we obtain important estimations like (5.12), (5.16), (5.23). And by adapting the Conrad and Rao methods [1], we obtain the estimation (5.44) which allow us to prove Lemma 5.3 and then the hypothesis of the Lemma 4.4.

Another study can be seen in Nakao [16]. We are strongly motive by the most challenging mathematical results have already been obtained in related topics, see for instance [9, 10, 11, 12], [7, 8] [18], [19], [20], [21] and [14, 22], among others.

2. Main Results

We state the result for existence of solution to the problem (1.1)-(1.3).

Theorem 2.1. *Given* $(u_0, u_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, *there is only one solution* $u(x,t)$ *of (1.1)- (1.3) in* $C^2([0,\infty), L^2(\Omega)) \cap C^1([0,\infty), H_0^1(\Omega)) \cap C([0,\infty), H^2(\Omega) \cap H_0^1(\Omega))$.

Also, we will need the following result for regularity of the solution, for which we cite Kesavan [6] and Ikawa [5]. We introduce the following definition

Definition 2.1. *The initial condition* $(u^0, u^1) \in H^{m+1} \times H^m$ *satisfies the* mth *- order compatibility condition associated to (1.1)-(1.3) if*

$$u^k \in H^{m+1-k} \cap H_0^1 \ \text{ for } k = 0, 1, \ldots, m \ \text{ and } u^{m+1} \in L^2\,, \tag{2.1}$$

where the sequence $(u^k)_k$ *is defined by induction from* (u^0, u^1) *by the formula*

$$u^{k+2} = \Delta u^k - a(x) u^{k+1}. \tag{2.2}$$

Proposition 2.1. *Let* $m \geq 1$ *be an integer. Let us suppose that* $a \in C^{m-1}(\overline{\Omega})$ *and* (u^0, u^1) *satisfies the* mth*- order compatibility condition associated to (1.1)- (1.3). Then, there is only one solution* $u(t)$ *of the problem (1.1)-(1.3) such that*

$$u \in X_m = \bigcap_{k=0}^{m} C^k(\mathbf{R}^+, H^{m+1-k} \cap H_0^1) \cap C^{m+1}(\mathbf{R}^+, L^2)\,, \tag{2.3}$$

and the linear application

$$(u^0, u^1) \in H^{m+1}(\Omega) \times H^m(\Omega) \longrightarrow u \in X_m \tag{2.4}$$

is continuous. That is exists $C > 0$ *such that* $\sum_{k=0}^{m+1} ||D^k u(t)||_{L^2}^2 \leq C ||(u^0, u^1)||_{H^{m+1} \times H^m}^2$, *where* D^k *denotes any kth order partial differentiation with respect to* t *and* x.

Let us suppose that Ω is the open ball in $\mathbf{R}^N$ centered on 0 and the radius R. Let us also assume that $\forall\ |x| \geq \frac{R}{2}$, $a(x) := \tilde{a}(|x|)$, where

$\tilde{a}$:$[\frac{R}{2}, R] \to \mathbf{R}^+$ is a strictly decreasing function which satisfies $\tilde{a}(R) = 0$. (Note that $\frac{R}{2}$ could be replaced by any $R - \epsilon$ with $\epsilon > 0$). Set

$$\forall r \in [0, \frac{R}{2}], \quad b(r) := \tilde{a}(R - r) \text{ and } B(r) := rb(r). \tag{2.5}$$

We observe that B is continuous and strictly increasing on $[0, \frac{R}{2}]$ and that $B(0) = 0$. Also,

$$b(r) \to 0 = \tilde{a}(R) \text{ as } r \to 0$$

That is

$$a(x) \to 0 \text{ as } x \to \partial\Omega\,.$$

We will use the Lemma 4.4 in the proof the following Main Theorem.

Theorem 2.2 (Main result). *Let us suppose that a goes to zero at the boundary quickly enough so that there exist $p > 0$ and $C > 0$ such that*

$$\forall \rho \in (0, \frac{R}{2}), \quad \int_\rho^{\frac{R}{2}} \frac{1}{b(r)^p} dr \leq C \frac{\rho}{b(\rho)^p}. \tag{2.6}$$

Set $m > \frac{N}{2}$. Then, if (u^0, u^1) satisfies the mth-order compatibility condition; there exists $C > 0$ which depends on the norm of the initial condition on $H^{m+1}(\Omega) \times H^m(\Omega)$ such that the solution u of (1.1)-(1.3) verifies

$$E(t) \leq C \left(B^{-1} \left(\frac{1}{t} \right) \right)^{\frac{2m}{N}} \,, \tag{2.7}$$

where B^{-1} denotes the inverse function of B.

2.1. Remarks of Theorem

Remark 2.1. *If exists n in $\mathbf{N}$ such that $n \geq 2$ and exists $p > 0$ such that $b(r) \leq nr^{\frac{1}{p}} b(nr)$, for all $r \in [0, \frac{R}{2n}]$, then (2.6) holds.*

Remark 2.2. *If $b(r) = r^k$ with $k > 0$ and $pk > 1$, then (2.6) holds. Therefore, by the main theorem we obtain*

$$E(t) \leq \frac{C}{t^{\frac{1}{(k+1)\theta}}}\,.$$

Remark 2.3. *If $b(r) = \frac{1}{|Ln\, r|}$ then (2.6) is not true.*

Remark 2.4. *If $b(r) = r^q e^{-\frac{1}{r^k}}$ with $k > 0$ then, we can apply remark 2.1. Therefore, by the main theorem we get*

$$E(t) \leq \frac{C}{[(Ln\, t)^{\frac{1}{k}}]^{\frac{2m}{N}}}\,.$$

3. Existence of solution

From the equation (1.1) writing $v = u_t$ we get:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} u_t \\ u_{tt} \end{pmatrix} = \begin{pmatrix} u_t \\ \Delta u - au_t \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - av \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & -aI \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

we define the Operator $A : D(A) \subset H \longrightarrow H$,

$$A = \begin{pmatrix} 0 & I \\ \Delta & -aI \end{pmatrix}$$

where $H = H_0^1(\Omega) \times L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

Thus (1.1)-(1.3) is equivalent to

$$\text{I.V.P} \begin{cases} U_t(t) &= AU(t) \\ U(0) &= U_0 := \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D(A). \end{cases} \tag{3.1}$$

Theorem 3.1. *The Operator A defined above generates a contraction semigroup $S(t)$ on the Hilbert Space H.*

Proof.- Observe that $D(A)$ is dense in H. We will prove that A is dissipative. Let $U = (u, v)^T \in D(A)$ then

$$\begin{aligned} < AU, U > &= \sum_{i=1}^{N} \int_\Omega \frac{\partial v}{\partial x_i} \overline{\frac{\partial u}{\partial x_i}} dx + \int_\Omega (\Delta u - a(x)v)\overline{v} dx \\ &= \sum_{i=1}^{N} \int_\Omega \frac{\partial v}{\partial x_i} \overline{\frac{\partial u}{\partial x_i}} dx + \int_\Omega \Delta u \overline{v} - a(x)|v|^2 dx \\ &= \sum_{i=1}^{N} \int_\Omega \frac{\partial v}{\partial x_i} \overline{\frac{\partial u}{\partial x_i}} - \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_i} dx - \int_\Omega a(x)|v|^2 dx \end{aligned}$$

$$= - \sum_{i=1}^{N} \int_\Omega 2iIm(\frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_i}) dx - \int_\Omega a(x)|v|^2 dx, \tag{3.2}$$

where $Im(z)$ is the imaginary part of $z \in \mathbf{C}$. Taking the real part of the equality (3.2), we have

$$Re(< AU, U >_{H_0^1(\Omega)\times L^2(\Omega)}) = -\int_\Omega a(x)|v|^2 dx \leq 0$$

Now, we will prove that $0 \in \rho(A)$. In fact, let $F = (f,g)^T \in H_0^1(\Omega) \times L^2(\Omega) = H$, we will prove that there is $U = (u,v)^T \in D(A)$, such that $AU = F$. Let us consider the equations

$$v = f \in H_0^1(\Omega) \tag{3.3}$$

$$\Delta u - a(x)v = g \in L^2(\Omega)\,. \tag{3.4}$$

Replacing (3.3) in (3.4), we have

$$\Delta u = a(x)f + g \in L^2(\Omega)\,. \tag{3.5}$$

By standard results on Elliptic linear equations, we have that (3.5) has only one solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$. From (3.3) we obtain $v = f$. That is, A is an onto map.

We claim that A is one to one. In fact, let $AU = 0$ then

$$v = 0 \tag{3.6}$$

$$\Delta u - a(x)v = 0\,. \tag{3.7}$$

Replacing (3.6) in (3.7) we have $\Delta u = 0$ and using the Green's Identity we have

$$|u|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla u|^2 dx = \int_\Omega \Delta u \overline{u} dx = 0$$

hence $u = 0$ in $H_0^1(\Omega)$. From (3.6) we have that $v = 0$. Therefore $U = 0$. i.e. A is one to one.

Thus, there is $A^{-1} : H \longrightarrow D(A)$ because A is one to one and H is the image of A. Now, we will prove that A^{-1} is bounded. Multiplying the equation (3.5) by $\overline{u}$ and integrating on Ω, we have

$$\int_\Omega \Delta u \overline{u} dx = \int_\Omega (af + g)\overline{u} dx$$

but since $\int_\Omega |\nabla u|^2 dx = \int_\Omega \Delta u \overline{u} dx$, using the Holder and Poincaré inequalities, we obtain

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega (af + g)\overline{u} dx \leq |u|_{L^2}\, |af + g|_{L^2}$$

$$
\begin{aligned}
&\leq \epsilon \int_\Omega |u|^2 dx + C(\epsilon) \int_\Omega |af+g|^2 dx \\
&\leq \epsilon C_p \int_\Omega |\nabla u|^2 dx + C(\epsilon) \int_\Omega |af+g|^2 dx \,.
\end{aligned}
$$

Then, taking $\epsilon > 0$ such that $1 - \epsilon C_p > 0$ we have

$$
(1-\epsilon C_p) \int_\Omega |\nabla u|^2 dx \leq C(\epsilon) \int_\Omega |af+g|^2 dx \,,
$$

that is

$$
\sqrt{1-\epsilon C_p}\, |\nabla u|_{L^2} \leq \sqrt{C(\epsilon)}\, |af+g|_{L^2} \,.
$$

Hence we have

$$
(3.8)\ |\nabla u|_{L^2} \leq \frac{\sqrt{C(\epsilon)}}{\sqrt{1-\epsilon C_p}}\, |af+g|_{L^2} \leq \frac{\sqrt{C(\epsilon)}}{\sqrt{1-\epsilon C_p}}\, \{|a|_\infty |f|_{L^2} + |g|_{L^2}\} \,.
$$

Thus, using (3.8), $v = f$, and the Holder and Poincaré inequalities we get

$$
\begin{aligned}
|U|_H \; = \; |\nabla u|_{L^2} + |v|_{L^2} \; = \; |\nabla u|_{L^2} + |f|_{L^2} \; &\leq \; \hat{C}\{|f|_{L^2} + |g|_{L^2}\} \\
&\leq \; \hat{\hat{C}}\{|\nabla f|_{L^2} + |g|_{L^2}\} \,.
\end{aligned}
$$

Then,

$$
|U|_H \leq \hat{\hat{C}}\, |AU|_H \,,
$$

that is

$$
|A^{-1}F|_H \leq \hat{\hat{C}}\, |F|_H \,,
$$

which allow us to say that A^{-1} is bounded. Now, by the Lummer-Phillips theorem, we have that A is the infinitesimal generator of a C_0 semigroup of contraction on $H : S(t)$.

Remark 3.1. *By Theorem 4.3.2 in [6], if $D(A) \ni U$ then $S(t)U \in C^1([0,\infty), H) \cap C([0,\infty), D(A))$.*

Remark 3.2. *By Remark 4.3.3 in [6], $U(t) := S(t)U_0$ is the solution of IVP (3.1) and it is the unique.*

From these two remarks, we get the following result.

Proposition 3.1. *There exists only one solution of (3.1), $U(t) \in C^1([0,\infty), H_0^1(\Omega) \times L^2(\Omega)) \bigcap C([0,\infty), (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega))$.*

Now, we will finish the proof of Theorem 2.1

Since $U_0 = (u^0, u^1) \in D(A)$, by Proposition 3.1 we obtain that there exists $U(t) \in C^1([0,\infty), H_0^1(\Omega) \times L^2(\Omega)) \bigcap C([0,\infty), (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega))$ solution of (3.1) such that $U(0) = U_0$, $U(t) \in D(A)\,, \forall t \in \mathbf{R}^+$.

Since U satisfies (3.1) we have $u_t = v$ and $v_t = \Delta u - av$. By one hand, we have $u_t \in C^0(I\!R^+, H_0^1(\Omega))$, but since $u \in C^0(\mathbf{R}^+, H_0^1(\Omega))$ then $u \in C^1(\mathbf{R}^+, H_0^1(\Omega))$. Also, $u_{tt} = v_t = \Delta u - au_t \in C(\mathbf{R}^+, L^2(\Omega))$, but u_t and u belong to $C(\mathbf{R}^+, L^2(\Omega))$ then $u \in C^2(\mathbf{R}^+, L^2(\Omega))$. We also obtain that $u \in C(\mathbf{R}^+, H^2(\Omega) \cap H_0^1(\Omega))$.

Remark 3.3. *By the Hille-Yosida Theorem (Theorem 4.4.3 in [6]), since A is the infinitesimal generator of contraction semigroup, A is closed, $D(A)$ is dense in H and $\forall \lambda > 0$, $\exists (\lambda I - A)^{-1}$ bounded, moreover $||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}$.*

Remark 3.4. *Since A is closed and there exists A^{-1} (It was proved in $0 \in \rho(A)$), then A^{-1} is also closed.*

4. Integral Inequalities

Lemma 4.1. *Let $E : \mathbf{R}^+ \to \mathbf{R}^+$ be a no increasing function and $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ be an strictly increasing C^1 function such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \longrightarrow +\infty \text{ as } t \to +\infty. \tag{4.1}$$

Let us suppose that there are $\sigma \geq 0$, and $w > 0$ such that

$$\forall s \geq 0\,, \quad \int_s^{+\infty} E(t)^{1+\sigma} \phi'(t) dt \leq \frac{1}{w} E(0)^\sigma E(s). \tag{4.2}$$

Then, E satisfies the following estimates:

$$\text{If } \sigma = 0, \text{ then } E(t) \leq E(0) e^{1 - w\phi(t)}\,, \quad \text{for all } t \geq 0. \tag{4.3}$$

$$\text{If } \sigma > 0, \text{ then } E(t) \leq E(0) \left(\frac{1+\sigma}{1 + w\sigma\phi(t)} \right)^{\frac{1}{\sigma}}\,, \quad \forall t \geq 0. \tag{4.4}$$

Proof.- Is enough to prove the case $E(0) = 1$, because if $1 \neq E(0) = d > 0$, we define $F(t) := \frac{E(t)}{E(0)}$, then $F(0) = 1$ and applying (4.2) to $E(t)$ we have

$$\begin{aligned}
\int_s^\infty F(t)^{(1+\sigma)} \phi'(t) dt &= \frac{1}{E(0)^{1+\sigma}} \int_s^{+\infty} E(t)^{(1+\sigma)} \phi'(t) dt \\
&\leq \frac{1}{w} E(0)^\sigma E(s) \frac{1}{E(0)^{(1+\sigma)}} \\
&= \frac{1}{w} \frac{E(s)}{E(0)} = \frac{1}{w} F(s)\,,
\end{aligned}$$

i.e. (4.2) holds for $F(s)$. Then

$$\begin{aligned} \text{If } \sigma = 0\,, \text{ then } F(t) &\leq e^{1-w\phi(t)}\,, \quad \forall t \geq 0, \\ \text{i.e. } E(t) &\leq E(0)e^{1-w\phi(t)}\,, \quad \forall t \geq 0\,. \end{aligned}$$

$$\begin{aligned} \text{If } \sigma > 0\,, \text{ then } F(t) &\leq \left(\frac{1+\sigma}{1+w\sigma\phi(t)}\right)^{\frac{1}{\sigma}}\,, \quad \forall t \geq 0, \\ \text{i.e. } E(t) &\leq E(0)\left(\frac{1+\sigma}{1+w\sigma\phi(t)}\right)^{\frac{1}{\sigma}}\,, \quad \forall t \geq 0\,. \end{aligned}$$

Now, we prove for $E(0) = 1$.

We introduce the following function $f : I\!R^+ \longrightarrow I\!R^+$ defined by

$$f(\tau) := E(\phi^{-1}(\tau)),$$

then f is no increasing. Making a change of variable and using (4.2) with $E(0) = 1$, we obtain the following: $\forall\, 0 < S < T < \infty$,

$$\begin{aligned} \int_{\phi(S)}^{\phi(T)} f(\tau)^{(1+\sigma)} d\tau &= \int_{\phi(S)}^{\phi(T)} E(\phi^{-1}(\tau))^{(1+\sigma)} d\tau \\ &= \int_S^T E(t)^{(1+\sigma)} \phi'(t) dt \\ &\leq \frac{1}{w} E(S) \quad = \quad \frac{1}{w} f(\phi(S))\,. \end{aligned}$$

Since $\lim_{T\to\infty} \phi(T) = +\infty$, then f satisfies:

$$\forall S \geq 0\,, \quad \int_S^{+\infty} f(\tau)^{1+\sigma} d\tau \leq \frac{1}{w} f(S)\,. \tag{4.5}$$

Let us denote $h : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$, $h(t) := \int_t^{+\infty} f(\tau)^{(1+\sigma)} d\tau$. So, h is well defined, no increasing, no negative and satisfies the following differential inequality.

$$\forall t \geq 0 \qquad -h' \geq (wh)^{1+\sigma}\,. \tag{4.6}$$

In fact, from $h(t) = -\int_{+\infty}^t f(\tau)^{(1+\sigma)} d\tau > 0$ we have $h'(t) = -f(t)^{(1+\sigma)} < 0$. And so, using (4.5) we have that $-h'(t) = f(t)^{(1+\sigma)} \geq (wh(t))^{(1+\sigma)}$.

Let us define $T_0 := \sup\{t \,:\, h(t) > 0\}$. Then, if $\sigma = 0$, h satisfies:

$$\forall\, 0 \leq t < T_0\,, \quad h(t) \leq h(0) e^{-wt} \leq \frac{1}{w} e^{-wt}\,. \tag{4.7}$$

In fact, from (4.6) with $\sigma = 0$ we have $-h' \geq wh$, that is $h' + wh \leq 0$, then $[e^{-wt}h]' \leq 0$, from where $e^{wt}h(t) \leq h(0)$ holds.

By other hand, from (4.5) and $f(0) = E(0) = 1$, we have that $h(0)w \leq f(0) = 1$, that is $h(0) \leq \frac{1}{w}$. Therefore, $h(t) \leq \frac{1}{w}e^{-wt}$. We observe that the estimate (4.7) holds if $t \geq T_0$. In fact, if $t \geq T_0$ then $h(t) \leq 0 < \frac{1}{w}e^{-wt}$.

Let $\epsilon > 0$. Since f is no increasing, we have that

$$\forall t \geq \epsilon\,, \quad f(t) \leq \frac{1}{\epsilon}\int_{t-\epsilon}^{t} f(\tau)d\tau \leq \frac{1}{\epsilon}h(t-\epsilon) \leq \frac{1}{w\epsilon}e^{w\epsilon}e^{-wt}\,. \tag{4.8}$$

In fact, in the last inequality of (4.8) it is used (4.7).

On the other hand, since f is no increasing, we have that $f(t).\epsilon \leq \int_{t-\epsilon}^{t} f(\tau)d\tau$. Also $\int_{t-\epsilon}^{t} f(\tau)d\tau \leq \int_{t-\epsilon}^{+\infty} f(\tau)d\tau = h(t-\epsilon)$.

If we take $\epsilon = \frac{1}{w}$ in (4.8), we get

$$\forall t \geq \frac{1}{w}\,, \quad f(t) \leq e^{1-wt}\,. \tag{4.9}$$

Since $E(t) = f(\phi(t))$, by using (4.9), we get (4.3).

If $\sigma > 0$, h satisfies:

$$\text{for all} t \in [0, T_0[\,, \quad (h^{-\sigma})' \geq \sigma w^{(1+\sigma)}\,. \tag{4.10}$$

In fact, $(h^{-\sigma})' = -\sigma h^{-(1+\sigma)}.h' = -\sigma h^{-(1+\sigma)}.(-f^{1+\sigma}) = \sigma h^{-(1+\sigma)} f^{1+\sigma} = \sigma(\frac{f}{h})^{1+\sigma}$. But from (4.5) we have that $w \leq \frac{f}{h}$, from here $w^{1+\sigma} \leq (\frac{f}{h})^{1+\sigma}$ then the result holds.

Integrating the inequality (4.10) from 0 to t, we obtain

$$[h(t)]^{-\sigma} - [h(0]^{-\sigma} \geq \sigma w^{1+\sigma}t \quad \forall\, 0 \leq t < T_0,$$

Thus

$$h(t) \leq (h(0)^{-\sigma} + \sigma w^{1+\sigma}t)^{-\frac{1}{\sigma}}\,, \quad \forall 0 \leq t < T_0. \tag{4.11}$$

Since f is no increasing, we have for all $s \geq 0$

$$(\frac{1}{w} + \sigma s)\, f(\frac{1}{w} + (\sigma+1)s)^{\sigma+1} \leq \int_{s}^{\frac{1}{w}+(\sigma+1)s} f(\tau)^{\sigma+1}d\tau \;\leq\; h(s)\,. \tag{4.12}$$

By the other hand, from (4.5) we have that $h(0) \leq \frac{1}{w}f(0) = \frac{1}{w}E(0) = \frac{1}{w}$, that is $w^{\sigma} \leq \frac{1}{h(0)^{\sigma}}$, then

$$\Big(h(0)^{-\sigma} + \sigma w^{1+\sigma}t\Big)^{-\frac{1}{\sigma}} \leq \Big(w^{\sigma} + \sigma w^{1+\sigma}t\Big)^{-\frac{1}{\sigma}} = \frac{1}{w[1+\sigma ws]^{\frac{1}{\sigma}}}\,. \tag{4.13}$$

Using (4.11), (4.12) and (4.13) we have

$$\left(\frac{1+w\sigma s}{w}\right) f(\frac{1}{w}+(\sigma+1)s)^{\sigma+1} \leq \frac{1}{w[1+\sigma ws]^{\frac{1}{\sigma}}}\,,$$

hence

$$\forall s \geq 0, \quad f(\frac{1}{w}+(\sigma+1)s) \leq \frac{1}{(1+w\sigma s)^{\frac{1}{\sigma}}}\,. \tag{4.14}$$

Putting $t=\frac{1}{w}+(\sigma+1)s$ on (4.14), we get

$$f(t) \leq \left(\frac{1+\sigma}{1+w\sigma t}\right)^{\frac{1}{\sigma}}, \forall t \geq 0. \tag{4.15}$$

Finally, since $E(t)=f(\phi(t))$, using (4.15) we get (4.4).

From (4.4) we deduce the following result:

Corollary 4.1. *Let $f:\mathbf{R}^{+}\to\mathbf{R}^{+}$ be a no increasing and continuous function. Let us assume that there are $\sigma>0$, $\sigma'>0$, and $c>0$ such that*

$$\forall t \geq 0\,, \quad \int_t^{+\infty} f(\tau)^{1+\sigma}d\tau \leq c\frac{f(0)^{\sigma}f(t)}{(1+t)^{\sigma'}}\,. \tag{4.16}$$

Then, there exists $C>0$ such that,

$$\forall t > 0\,, \quad f(t) \leq f(0)\frac{C}{(1+t)^{\frac{(1+\sigma')}{\sigma}}}\,. \tag{4.17}$$

Proof.- Is enough to prove the case $f(0)=1$, because if $1\neq f(0)=d>0$ we define $g(s):=\frac{f(s)}{f(0)}$, then $g(0)=1$ and

$$\begin{aligned}\int_t^{+\infty} g(s)^{1+\sigma}ds &= \frac{1}{f(0)^{(1+\sigma)}}\int_t^{+\infty} f(s)^{(1+\sigma)}ds\\ &\leq \frac{1}{f(0)^{\sigma}f(0)}c\frac{f(0)^{\sigma}f(t)}{(1+t)^{\sigma'}}\\ &= cg(0)^{\sigma}\frac{g(t)}{(1+t)^{\sigma'}}\,,\end{aligned}$$

that is (4.16) holds. Using (4.17) for $g(0)=1$ we have

$$g(t) \leq g(0)\frac{C}{(1+t)^{\frac{(1+\sigma')}{\sigma}}}\,,$$

that is,

$$f(t) \leq \frac{Cf(0)}{(1+t)^{\frac{(1+\sigma')}{\sigma}}}\,.$$

Now, we prove for $f(0)=1$.

If $t \geq 0$, let us define

$$g(t) = \frac{f(t)}{(1+t)^{\sigma'}}$$

then g is no increasing. Since $g(\tau)^{1+\sigma}(1+\tau)^{\sigma'(1+\sigma)} = f(\tau)^{1+\sigma}$ and using (4.16) we get

$$\forall t \geq 0\,, \quad \int_t^{+\infty} g(\tau)^{1+\sigma}(1+\tau)^{\sigma'(1+\sigma)}d\tau \;\leq c\,g(t)\,. \tag{4.18}$$

Define $\phi(t) = (1+t)^{\sigma'(1+\sigma)+1}-1$ then $\phi(0)=0$, $\phi(t)\to+\infty$ as $t\to+\infty$ and $\phi'(t) = (\sigma'(1+\sigma)+1)(1+t)^{\sigma'(1+\sigma)}$. Replacing ϕ' on (4.18) we obtain

$$\int_t^{\infty} g(\tau)^{1+\sigma}\phi'(\tau)d\tau \leq \underbrace{c[\sigma'(1+\sigma)+1]}_{=\frac{1}{w}} g(t)$$

and since $g(0)=f(0)=1$, we can apply Lemma 4.1, to get

$$g(t) \leq g(0)\left(\frac{1+\sigma}{1+w\sigma\phi(t)}\right)^{\frac{1}{\sigma}} = \left(\frac{1+\sigma}{1+w\sigma\{(1+t)^{\frac{1}{wc}}-1\}}\right)^{\frac{1}{\sigma}}\,.$$

Define $r = \min\{1, w\sigma\}$ then $1+w\sigma\{(1+t)^{\frac{1}{wc}}-1\} > r\{1+(1+t)^{\frac{1}{wc}}-1\} = r(1+t)^{\frac{1}{wc}}$, that is $\frac{1}{r(1+t)^{\frac{1}{wc}}} > \frac{1}{1+w\sigma\{(1+t)^{\frac{1}{wc}}-1\}}$, from where we can deduce that g decays like

$$g(t) \leq \frac{C}{(1+t)^{\frac{(\sigma'(1+\sigma)+1)}{\sigma}}} = \frac{C}{(1+t)^{\sigma'}(1+t)^{\frac{1+\sigma'}{\sigma}}}\,.$$

Thus, (4.17) holds.

□

Lemma 4.2. *Let $f : I\!R^+ \to I\!R^+$ be a continuous and no increasing function. Let us assume that there are $\sigma > 0$, $\sigma' > 0$ and $c > 0$ such that*

$$\forall t > 0\,, \quad \int_t^{\infty} f(\tau)^{1+\sigma}d\tau \leq cf(t)^{1+\sigma} + \frac{cf(0)^{\sigma}f(t)}{(1+t)^{\sigma'}}\,. \tag{4.19}$$

Then, there exists $C > 0$ such that

$$\forall t > 0\,, \quad f(t) \leq f(0)\frac{C}{(1+t)^{\frac{(1+\sigma')}{\sigma}}}\,. \tag{4.20}$$

Proof.- Is enough to prove the case $f(0) = 1$, because if $1 \neq f(0) = d > 0$, we define $g(t) = \frac{f(t)}{f(0)}$, then $g(0) = 1$ and

$$\begin{aligned}
\int_t^\infty g(\tau)^{1+\sigma} d\tau &= \frac{1}{f(0)^{1+\sigma}} \int_t^{+\infty} f(\tau)^{1+\sigma} d\tau \\
&\leq \frac{1}{f(0)^{1+\sigma}} \left\{ cf(t)^{1+\sigma} + \frac{cf(0)^\sigma f(t)}{(1+t)^{\sigma'}} \right\} \\
&= cg(t)^{1+\sigma} + c\frac{g(t)}{(1+t)^{\sigma'}}\,.
\end{aligned}$$

Thus, (4.19) holds for g with $g(0) = 1$. Then (4.20) holds for g:

$$g(t) \leq g(0)\frac{C}{(1+t)^{\frac{(1+\sigma')}{\sigma}}}\,,$$

Hence we obtain

$$f(t) \leq f(0)\frac{C}{(1+t)^{\frac{(1+\sigma')}{\sigma}}}\,.$$

Now, we prove for $f(0) = 1$.

We will prove (4.20) by induction. Next, we denote por C every constant. First, let us bound the right hand of (4.19),

$$\begin{aligned}
cf(t)^{1+\sigma} + \frac{cf(0)^\sigma f(t)}{(1+t)^{\sigma'}} &\leq cf(t) \left\{ f(t)^\sigma + \frac{f(0)^\sigma}{(1+t)^{\sigma'}} \right\} \\
&\leq cf(t) \left\{ f(0)^\sigma + \frac{f(0)^\sigma}{(1+t)^{\sigma'}} \right\} \\
&= cf(0)f(0)^\sigma \left\{ 1 + \underbrace{\frac{1}{(1+t)^{\sigma'}}}_{<1} \right\} \\
&< 2cf(0)^{1+\sigma}\,,
\end{aligned}$$

and so we have

$$\int_t^\infty f(\tau)^{1+\sigma} d\tau \leq Cf(t)\,. \tag{4.21}$$

Now, we prove that $\int_t^\infty f(\tau)^{1+\sigma}d\tau \leq Cf(t)$ imply $f(t) \leq \frac{C}{(1+t)^{\frac{1}{\sigma}}}$.

In fact, considering $f(t)$ instead of $E(t)$, $\phi(t) = t$ (i.e. $\phi'(t) = 1$, $\phi(0) = 0$, $\phi(t) \longrightarrow +\infty$ as $t \to +\infty$), $\sigma > 0$, and using (4.21) we deduce from Lemma 4.1 that

$$f(t) \leq \left(\frac{1+\sigma}{1+\frac{\sigma}{C}t}\right)^{\frac{1}{\sigma}} .$$

Taking $r := \min\{1, \frac{\sigma}{C}\}$ then $r(1+t) \leq 1+\frac{\sigma}{C}t$, that is $\frac{1}{1+\frac{\sigma}{C}t} \leq \frac{1}{r(1+t)}$, from where we get

$$f(t) \leq \frac{C}{(1+t)^{\frac{1}{\sigma}}} .$$

That is $f(t)^\sigma \leq \frac{C^\sigma}{1+t}$. Then, using this estimate in (4.19) we obtain

$$\int_t^{+\infty} f(\tau)^{1+\sigma}d\tau \leq C\frac{f(t)}{1+t} + C\frac{f(t)}{(1+t)^{\sigma'}} .$$

Taking $\sigma_1 := \inf\{1, \sigma'\}$, we have

$$\int_t^{+\infty} f(\tau)^{1+\sigma}d\tau \leq C\frac{f(t)}{(1+t)^{\sigma_1}} ,$$

and using Corollary 4.1 we arrive at

$$f(t) \leq \frac{C}{(1+t)^{\frac{(1+\sigma_1)}{\sigma}}} .$$

If $\sigma' \leq 1$ then $\sigma_1 = \sigma'$, from where we get the inequality (4.20). If $\sigma' > 1$ then

$$f(t) \leq \frac{C}{(1+t)^{\frac{2}{\sigma}}} ,$$

that is $f(t)^\sigma \leq \frac{C^\sigma}{(1+t)^2}$. Then, using this estimate on (4.19) we have

$$\int_t^{+\infty} f(\tau)^{1+\sigma}d\tau \leq C\frac{f(t)}{(1+t)^2} + C\frac{f(t)}{(1+t)^{\sigma'}} .$$

Taking $\sigma_2 := \inf\{2, \sigma'\}$, we have

$$\int_t^{+\infty} f(\tau)^{1+\sigma}d\tau \leq C\frac{f(t)}{(1+t)^{\sigma_2}} ,$$

and using Corollary 4.1 we arrive at

$$f(t) \le \frac{C}{(1+t)^{\frac{(1+\sigma_2)}{\sigma}}}\,.$$

If $\sigma' \le 2$ then $\sigma_2 = \sigma'$, from where we get the inequality (4.20). If $\sigma' > 2$ then

$$f(t) \le \frac{C}{(1+t)^{\frac{3}{\sigma}}}\,,$$

that is $f(t)^\sigma \le \frac{C^\sigma}{(1+t)^3}$, and so on. Then, the conclusion holds by induction.

Lemma 4.3. *Let $E : I\!R^+ \to I\!R^+$ be a no increasing function and $\phi : I\!R^+ \to I\!R^+$ an strictly increasing C^1 function such that,*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \longrightarrow +\infty \text{ as } t \to +\infty. \tag{4.22}$$

Let us assume that there are $\sigma > 0$, $\sigma' > 0$ and $c > 0$ such that

$$\forall s \ge 0\,, \quad \int_s^{+\infty} E(t)^{1+\sigma}\phi'(t)dt \le cE(s)^{1+\sigma} + c\frac{E(s)}{(1+\phi(s))^{\sigma'}}\,. \tag{4.23}$$

Then, there exists $C > 0$ depending continuously on $E(0)$, satisfying

$$\forall t > 0\,, \quad E(t) \le \frac{C}{(1+\phi(t))^{\frac{(1+\sigma')}{\sigma}}}\,. \tag{4.24}$$

Proof.- Is enough to define $f(\tau) = E(\phi^{-1}(\tau))$ and use Lemma 4.2.

In analogy to this Lemma, we have the next version.

Lemma 4.4. *Let $E : I\!R^+ \to I\!R^+$ be a no increasing function and $\phi : I\!R^+ \to I\!R^+$ an strictly increasing C^1 function such that*

$$\phi(t) \longrightarrow +\infty \text{ as } t \to +\infty. \tag{4.25}$$

Let us assume that there are $\sigma > 0$, $\sigma' > 0$ and $c > 0$ such that

$$\forall s \ge 1\,, \quad \int_s^{+\infty} E(t)^{1+\sigma}\phi'(t)dt \le cE(s)^{1+\sigma} + c\frac{E(s)}{\phi(s)^{\sigma'}}\,. \tag{4.26}$$

Then there exists $C > 0$ depending continuously on $E(1)$ satisfying

$$\forall t \ge 1\,, \quad E(t) \le \frac{C}{\phi(t)^{\frac{(1+\sigma')}{\sigma}}}\,. \tag{4.27}$$

5. Using the multiplier method

Let $(u^0, u^1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfying the mth - order compatibility condition. Then, the regularity given by (2.3) justifies the calculus we are going to do.

We know that problem (1.1)-(1.3) is dissipative.

Lemma 5.1.

$$E'(t) = -\int_\Omega a(x)|u_t|^2 dx\,, \quad \forall t > 0. \tag{5.1}$$

Proof.- Multiplying the equation (1.1) by u_t and integrating on Ω, and using the Green Identity, we have

$$\begin{aligned}
0 &= \int_\Omega (u_{tt} - \Delta u + a(x)u_t)u_t\, dx \\
&= \frac{\partial}{\partial t}\{\frac{1}{2}\int_\Omega (u_t)^2\, dx\} - \int_\Omega (\Delta u)\, u_t\, dx + \int_\Omega a(x)(u_t)^2\, dx \\
&= \frac{\partial}{\partial t}\{\frac{1}{2}\int_\Omega (u_t)^2\, dx\} + \int_\Omega (\nabla u)\, \nabla u_t\, dx + \int_\Omega a(x)(u_t)^2\, dx \\
&= \frac{\partial}{\partial t}\{\frac{1}{2}\int_\Omega (u_t)^2\, dx\} + \frac{\partial}{\partial t}\{\frac{1}{2}\int_\Omega |\nabla u|^2\, dx\} + \int_\Omega a(x)(u_t)^2\, , dx
\end{aligned}$$

then the result holds.

Let $\sigma \geq 0$, and $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ be a concave and increasing C^2 function. Let w be a neighborhood of the boundary $\partial\Omega$.

Lemma 5.2. *Let $h : \overline{\Omega} \longrightarrow I\!R^N$ be a C^1 vector field, $\sigma \geq 0$ and $0 \leq S \leq T < +\infty$. Then we have,*

$$\begin{aligned}
&\int_S^T E^\sigma \phi' \int_{\partial\Omega} 2\partial_\nu u h.\nabla u + (h\cdot\nu)(|u_t|^2 - |\nabla u|^2) \\
&= [E^\sigma \phi' \int_\Omega q u_t\, h\cdot\nabla u]_S^T - \int_S^T (\sigma E' E^{\sigma-1}\phi' + E^\sigma \phi'') \int_\Omega 2u_t\, h\cdot\nabla u \\
&+ \int_S^T E^\sigma \phi' \int_\Omega (div\, h)(|u_t|^2 - |\nabla u|^2) + 2\sum_{i,j} \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} + 2au_t\, h\cdot\nabla u\,.
\end{aligned} \tag{5.2}$$

Proof.- Multiplying the equation (1.1) by $E^\sigma \phi'\, 2h\cdot\nabla u$ and integrating on $[S,T]\times\Omega$ we have

$$0 = \int_S^T E^\sigma \phi' \int_\Omega 2h\cdot\nabla u(u_{tt} - \Delta u + a(x)u_t)\, dx\, dt$$

$$= \underbrace{\int_S^T E^\sigma \phi' \int_\Omega 2h \cdot (\nabla u) u_{tt}\, dx\, dt}_{I_1:=} \underbrace{- \int_S^T E^\sigma \phi' \int_\Omega 2h \cdot \nabla u(\Delta u)\, dx\, dt}_{I_2:=} +$$

$$\int_S^T E^\sigma \phi' \int_\Omega 2h \cdot \nabla u(a(x)u_t)\, dx\, .dt \tag{5.3}$$

Since $\frac{\partial}{\partial t}\left(\int_\Omega 2h(\nabla u)u_t\, dx\right) = \int_\Omega 2h(\nabla u_t)u_t\, dx + \int_\Omega 2h(\nabla u)u_{tt}\, dx$ and integrating by parts we obtain

$$\begin{aligned} I_1 &= \int_S^T E^\sigma \phi' \frac{\partial}{\partial t}\left(\int_\Omega 2h(\nabla u)u_t\, dx\right) dt - \int_S^T E^\sigma \phi' \int_\Omega 2h(\nabla u_t)u_t\, dx\, dt \\ &= -\int_S^T E^\sigma \phi' \int_\Omega 2h(\nabla u)u_t\, dxdt + [E^\sigma \phi' \int_\Omega 2h(\nabla u)u_t\, dx]_S^T \\ &\quad -\int_S^T E^\sigma \phi' \int_\Omega 2h(\nabla u_t)u_t\, dxdt\, . \end{aligned}$$

Using the Green Identity we have

$$\begin{aligned} I_2 &= -\int_S^T E^{\sigma\phi'} \int_\Omega 2h \cdot \nabla u \Delta u dx\, dt \\ &= \int_S^T E^\sigma \phi' \int_\Omega \nabla u \cdot \nabla(2h \cdot \nabla u) dx\, dt - \int_S^T E^\sigma \phi' \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(2h \cdot \nabla u) dt\, . \end{aligned}$$

Replacing I_1 and I_2 on the equality (5.3) we obtain

$$\begin{aligned} 0 &= -\int_S^T E^\sigma \phi' \int_\Omega 2h(\nabla u)u_t\, dxdt + [E^\sigma \phi' \int_\Omega 2h(\nabla u)u_t\, dx]_S^T \\ &\quad \underbrace{-\int_S^T E^\sigma \phi' \int_\Omega 2h(\nabla u_t)u_t\, dxdt}_{I_3:=} + \underbrace{\int_S^T E^\sigma \phi' \int_\Omega \nabla u \cdot \nabla(2h \cdot \nabla u) dx\, dt}_{I_4:=} \\ &\quad -\int_S^T E^\sigma \phi' \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(2h \cdot \nabla u) dt + \int_S^T E^\sigma \phi' \int_\Omega 2h \cdot \nabla u(a(x)u_t)\, dx\, .dt \end{aligned} \tag{5.4}$$

Using the fact that $h\nabla(u_t^2) = div(hu_t^2) - (div\, h)u_t^2$ and the Divergence Theorem $\int_{\partial\Omega} hu_t^2 \cdot \nu = \int_\Omega div(hu_t^2)dx$, we have

$$\begin{aligned} I_3 &= -\int_S^T E^\sigma \phi' \int_\Omega h \cdot \nabla(u_t^2) dxdt \\ &= -\int_S^T E^\sigma \phi' \int_\Omega div\,(hu_t^2) dxdt + \int_S^T E^\sigma \phi' \int_\Omega (div\, h)u_t^2 dxdt \\ &= -\int_S^T E^\sigma \phi' \int_{\partial\Omega} (hu_t^2) \cdot \nu dxdt + \int_S^T E^\sigma \phi' \int_\Omega (div\, h)u_t^2 dxdt\, . \end{aligned}$$

By other hand, we have

$$\nabla u\cdot\nabla(2h\cdot\nabla u)=2\partial_i u\partial_i h_k\partial_k u+h\cdot\nabla(|\nabla u|^2)\,. \tag{5.5}$$

And since $div(h|\nabla u|^2)=(divh)|\nabla u|^2+h\nabla(|\nabla u|^2)$ and using the Divergence Theorem, we have

$$\begin{aligned}\int_\Omega(div\,h)|\nabla u|^2\,dx+\int_\Omega h\nabla(|\nabla u|^2)\,dx &= \int_\Omega div(h|\nabla u|^2)\,dx\\ &= \int_{\partial\Omega} h|\nabla u|^2{\cdot}\nu\,dx\end{aligned}\tag{5.6}$$

Using (5.5) and (5.6) we obtain

$$\begin{aligned}I_4 &= 2\int_S^T E^\sigma\phi'\int_\Omega \partial_i u\partial_i h_k\partial_k u\,dxdt+\int_S^T E^\sigma\phi'\int_\Omega h\cdot\nabla(|\nabla u|^2)dxdt\\ &= 2\int_S^T E^\sigma\phi'\int_\Omega \partial_i u\partial_i h_k\partial_k u\,dxdt+\int_S^T E^\sigma\phi'\int_{\partial\Omega} h|\nabla u|^2\cdot\nu dxdt\\ &\quad-\int_S^T E^\sigma\phi'\int_\Omega(div\,h)|\nabla u|^2dxdt\,.\end{aligned}$$

Replacing I_3 and I_4 in the equality (5.4), we will have the result.

Lemma 5.3. *There exists a constant $C>0$ such that $\forall\, 0\le S<T<\infty$*

$$\int_S^T E(t)^{1+\sigma}\phi'(t)dt\le CE(S)^{1+\sigma}+C\int_S^T E(t)^\sigma\phi'(t)\left(\int_w |u_t|^2dx\right)dt\,. \tag{5.7}$$

holds.

Proof.- Let K_1 be a compact of Ω such that $\Omega-K_1$ be a compact set on w.

Define $h(x):=\beta(x)m(x)$, where $m(x)=x-x_0$ and β is a C^∞ function whoose support is compactly in Ω and equal to 1 on K_1. Since ϕ' is no increasing and positive then ϕ' is bounded on $I\!R^+$ (i.e. $|\phi'(t)|\le M$).

Now, we apply (5.2) to this h and get

$$0 \geq [E^\sigma \phi' \int_\Omega 2u_t\, h \cdot \nabla u dx]_S^T - \int_S^T (\sigma E' E^{\sigma-1}\phi' + E^\sigma \phi'')(\int_\Omega 2u_t h \cdot \nabla u dx)dt$$

$$+\int_S^T E^\sigma \phi' \int_\Omega div\, h\, (u_t^2 - |\nabla u|^2) + 2\sum_{i,j} \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} + 2a(x)u_t h \cdot \nabla u\,.$$

(5.8)

By other hand, using that $\int_\Omega 2h \cdot \nabla u\, u_t dx \leq cE(t)$ and $E(t)^{\sigma+1} < E(S)^{\sigma+1}$ para $S < T$, we have

$$\begin{aligned}
&|\int_S^T (\sigma E' E^{\sigma-1}\phi' + E^\sigma \phi'')(\int_\Omega 2h \cdot \nabla u\, u_t dx)dt| \\
&\leq \int_S^T |(\sigma E' E^{\sigma-1}\phi' + E^\sigma \phi'')(\int_\Omega 2h \cdot \nabla u\, u_t dx)|dt \\
&\leq \int_S^T |\sigma E' E^{\sigma-1}\phi' + E^\sigma \phi''| cE dt \\
&= \int_S^T \{-\sigma E' E^{\sigma-1}\phi' - E^\sigma \phi''\} cE dt \\
&\leq cM \int_S^T -\sigma E' E^\sigma dt + cE(S)^{\sigma+1} \int_S^T -\phi'' dt \\
&= cM[\frac{\sigma}{\sigma+1} E(t)^{\sigma+1}]_T^S + cE(S)^{\sigma+1} \underbrace{[\phi']_T^S}_{\leq 2M} \\
&\leq cM \frac{\sigma}{\sigma+1} E(S)^{\sigma+1} + c2ME(S)^{\sigma+1} \\
&\leq c' E(S)^{\sigma+1}\,. \qquad (5.9)
\end{aligned}$$

And, since $E(T) < E(S)$ and by Holder $\int_\Omega u_t\, h \cdot \nabla u dx \leq ||u_t||\,||\nabla u|| \leq E(t)$, we obtain

$$\begin{aligned}
-[E^\sigma \phi' \int_\Omega 2u_t\, h \cdot \nabla u dx]_S^T \quad &\leq \quad -E^\sigma(T)\phi'(T) \int_\Omega 2u_t(T)\, h(T) \cdot \nabla u(T) dx \\
&\quad\quad +E^\sigma(S)\phi'(S) \int_\Omega 2u_t(S)\, h(S) \cdot \nabla u(S) dx \\
&\leq \quad E^\sigma(S) C\{E(S) + E(T)\} \leq CE^{\sigma+1}(S)\,. \qquad (5.10)
\end{aligned}$$

Here we need to make the following estimate

$$-\int_S^T E^\sigma \phi' \int_\Omega 2\sum_{i,j} \frac{\partial h_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} dxdt \quad \leq \quad \int_S^T E^\sigma \phi' c ||\nabla u||\, ||\nabla u||_{L^2(\Omega - K_1)} dt$$

$$\leq \int_S^T E^\sigma \phi' \{\epsilon E(t) + C(\epsilon)||\nabla u||^2_{L^2(\Omega-K_1)}\} dt\,, \tag{5.11}$$

where ϵ will be considered little enough.

Using the inequalities (5.9), (5.10) and (5.11) on (5.8) we have that there exists $C > 0$ such that

$$\int_S^T E^\sigma \phi' \int_\Omega N u_t^2 + (2-N)|\nabla u|^2 dx\, dt$$
$$\leq C \int_S^T E^\sigma \phi' \int_{\Omega - K_1} \{u_t^2 + |\nabla u|^2\} dxdt + CE(S)^{1+\sigma} + \epsilon \int_S^T E^{1+\sigma}\phi' dt\,, \tag{5.12}$$

with ϵ little enough.

Integrating by parts the expression:

$$0 = (N-1)\int_S^T E^\sigma \phi' \int_\Omega u(u_{tt} - \Delta u + a(x)u_t) dxdt\,,$$

we have

$$\begin{aligned}(N-1)\int_S^T E^\sigma \phi' \int_\Omega |\nabla u|^2 dxdt - (N-1)\int_S^T E^\sigma \phi' \int_\Omega |u_t|^2 dxdt \\ = -(N-1)[E^\sigma \phi' \int_\Omega u u_t dx]_S^T - (N-1)\int_S^T E^\sigma \phi' \int_\Omega u a u_t dxdt \\ +(N-1)\int_S^T (\sigma E^{\sigma-1}E'\phi' + E^\sigma \phi'') \int_\Omega u u_t dxdt\,.\end{aligned} \tag{5.13}$$

By the Poincaré inequality we have $\int_\Omega uu_t dx \le ||u||\,||u_t|| \le C_p||\nabla u||\,||u_t|| \le CE(t)$. Using this in (5.9) we obtain

$$|(N-1)\int_S^T (\sigma E^{\sigma-1}E'\phi' + E^\sigma\phi'')\int_\Omega uu_t dxdt| \le CE(S)^{\sigma+1}\,. \tag{5.14}$$

With a similar proof to (5.10), we also obtain

$$-(N-1)[E^\sigma\phi'\int_\Omega uu_t dx]_S^T \le CE(S)^{\sigma+1}\,. \tag{5.15}$$

Adding (5.12) and (5.13), taking $\epsilon < 1$ and using (5.14) and (5.15) we have

$$\begin{aligned}\int_S^T E^{1+\sigma}\phi' dt &\le \int_S^T E^\sigma\phi'\int_\Omega u_t^2 + |\nabla u|^2 dxdt\\ &\le CE(S)^{1+\sigma} + C\int_S^T E^\sigma\phi'\int_{\Omega-K_1}\{u_t^2 + |\nabla u|^2\}dxdt\,.\end{aligned} \tag{5.16}$$

We want to eliminate the last term of (5.16). To do this, we construct a function $\xi \in C^\infty(\overline{\Omega})$ such that $\xi = 1$ in $\Omega - K_1$ and $\xi = 0$ outside w.

We multiply the equation (1.1) by ξu and integrate it on Ω; then we multiply this expression by $E^\sigma\phi'$, and integrate on $[S,T]$, and integrating by parts we get

$$\begin{aligned}&\int_S^T E^\sigma\phi'\int_\Omega -\xi u a(x) u_t dxdt\\ =\;&\int_S^T E^\sigma\phi'\int_\Omega \xi u(u_{tt} - \Delta u)dxdt\\ =\;&\int_S^T E^\sigma\phi'\int_\Omega \xi u u_{tt} dxdt - \int_S^T E^\sigma\phi'\int_\Omega \xi u\Delta u dxdt\\ =\;&-\int_S^T (\sigma E^{\sigma-1}E'\phi' + E^\sigma\phi'')\int_\Omega \xi u u_t dxdt\\ &-\int_S^T E^\sigma\phi'\int_\Omega \xi u_t^2 dxdt + [E^\sigma\phi'\int_\Omega uu_t dx]_S^T\\ &\underbrace{-\int_S^T E^\sigma\phi'\int_\Omega \xi u\Delta u dxdt}_{I:=}\,.\end{aligned} \tag{5.17}$$

Using the Green Identity and $\nabla(u^2) = (\nabla u)\, u + u\nabla u$ we have

$$
\begin{aligned}
I &= \int_S^T E^\sigma \phi' \int_\Omega \nabla(\xi u)\cdot\nabla u dx dt \\
&= \int_S^T E^\sigma \phi' \int_\Omega \{\nabla(\xi) u\cdot\nabla u + \xi\nabla u\cdot\nabla u\} dx\, dt \\
&= \int_S^T E^\sigma \phi' \int_\Omega \{\frac{1}{2}\nabla(\xi)\cdot\nabla(u^2) + \xi|\nabla u|^2\} dx\, dt \\
&= \int_S^T E^\sigma \phi' \int_\Omega \{-\frac{1}{2}(\Delta\xi)u^2 + \xi|\nabla u|^2\} dx\, .dt \qquad (5.18)
\end{aligned}
$$

Replacing (5.18) on (5.17) we obtain

$$
\begin{aligned}
\int_S^T E^\sigma \phi' \int_\Omega -\xi u a(x) u_t dx dt &= -\int_S^T (\sigma E^{\sigma-1}E'\phi' + E^\sigma\phi'') \int_\Omega \xi u u_t dx dt \\
&\quad + \int_S^T E^\sigma \phi' \int_\Omega \xi|\nabla u|^2 dx dt + [E^\sigma \phi' \int_\Omega u u_t dx]_S^T \\
&\quad - \int_S^T E^\sigma \phi' \int_\Omega \{\frac{1}{2}(\Delta\xi)u^2 + \xi u_t^2\} dx\, dt\,, \qquad (5.19)
\end{aligned}
$$

from where

$$
\begin{aligned}
\int_S^T E^\sigma \phi' \int_{\Omega-K_1} 1\cdot|\nabla u|^2 dx dt &\le \int_S^T E^\sigma \phi' \int_\Omega \xi|\nabla u|^2 dx dt \\
&= \int_S^T (\sigma E^{\sigma-1}E'\phi' + E^\sigma\phi'') \int_\Omega \xi u u_t dx dt \\
&\quad -[E^\sigma \phi' \int_\Omega u u_t dx]_S^T \\
&\quad + \int_S^T E^\sigma \phi' \int_\Omega -\xi u a(x) u_t dx dt \\
&\quad + \int_S^T E^\sigma \phi' \int_\Omega \{\frac{1}{2}(\Delta\xi)u^2 + \xi u_t^2\} dx\, dt\,. \qquad (5.20)
\end{aligned}
$$

Since ξ is bounded, in a similar way to (5.9), we obtain

$$
|\int_S^T (\sigma E^{\sigma-1}E'\phi' + E^\sigma\phi'') \int_\Omega \xi u u_t dx dt| \le C E^{1+\sigma}\,. \qquad (5.21)
$$

Also, using the fact of ξ is bounded, similarly to (5.10) we get

$$
-\,[E^\sigma \phi' \int_\Omega u u_t dx]_S^T \le C E^{1+\sigma}. \qquad (5.22)
$$

And, using (5.21) and (5.22) in (5.20) we obtain

$$\int_S^T E^\sigma \phi' \int_{\Omega - K_1} |\nabla u|^2 dx dt \le C E(S)^{1+\sigma} + C \int_S^T E^\sigma \phi' \int_w (u_t^2 + u^2) dx dt \,. \tag{5.23}$$

Now, to eliminate the last term of (5), we will adapt the Conrad and Rao [1] method.

We start with an arbitrary function $\beta \in C^\infty(\mathbf{R}^N)$ be such that $0 \le \beta \le 1$, $\beta = 1$ on w and $\beta = 0$ outside a neighborhood of w (see [4], Theorem 1.2.2, or [2] p.p 3489).

Now, fix t and consider the solution z to the elliptic problem:

$$\Delta z = \beta(x) u \text{ in } \Omega \tag{5.24}$$

$$z|_{\partial\Omega} = 0 \,. \tag{5.25}$$

Multiplying the equation (5.24) by z, integrating on Ω and using the Green Identity, we have:

$$\int_\Omega \beta(x) u\, z dx = \int_\Omega (\Delta z) z dx = -\int_\Omega |\nabla z|^2 dx \,,$$

hence, using the Holder and Poincaré inequalities, we obtain

$$|z|_{L^2}^2 \le C \int_\Omega |\nabla z|^2 dx = -C \int_\Omega \beta(x) u\, z dx \le c |u|_{L^2(\Omega)} |z|_{L^2(\Omega)} \,; \tag{5.26}$$

then,

$$|z|_{L^2} \le C |u|_{L^2(\Omega)} \,. \tag{5.27}$$

Similarly to (5.26) we have

$$\begin{aligned} |z|_{L^2}^2 \le C \int_\Omega |\nabla z|^2 dx = -C \int_\Omega \beta(x) u\, z dx &\le c |\beta\, u|_{L^2(\Omega)} |z|_{L^2(\Omega)} \\ &\le c |u|_{L^2(w)} |z|_{L^2(\Omega)} \,, \end{aligned} \tag{5.28}$$

then,

$$|z|_{L^2} \le C |u|_{L^2(w)} \,. \tag{5.29}$$

Differentiating with respect to t to the equation (5.24) we have the problem

$$\Delta z_t = \beta(x)u_t \text{ in } \Omega \tag{5.30}$$
$$z_t|_{\partial\Omega} = 0\,. \tag{5.31}$$

Multiplying (5.30) by z_t, integrating on Ω and using the Green Identity we obtain

$$\int_\Omega \beta(x)u_t\, z_t dx = \int_\Omega (\Delta z_t)z_t dx = -\int_\Omega |\nabla z_t|^2 dx\,,$$

hence, using the Holder and Poincaré inequalities, we obtain

$$|z_t|_{L^2}^2 \leq C\int_\Omega |\nabla z_t|^2 dx = -C\int_\Omega \beta(x)u_t\, z_t dx \leq c|u_t|_{L^2(\Omega)}|z_t|_{L^2(\Omega)} \tag{5.32}$$

then

$$|z_t|_{L^2} \leq C|u_t|_{L^2(\Omega)}\,. \tag{5.33}$$

Also

$$\begin{aligned} |z_t|_{L^2}^2 \leq C\int_\Omega |\nabla z_t|^2 dx = -C\int_\Omega \beta(x)u_t\, z_t dx &\leq c|\beta\, u_t|_{L^2(\Omega)}|z_t|_{L^2(\Omega)} \\ &\leq c|u_t|_{L^2(w)}|z_t|_{L^2(\Omega)} \end{aligned} \tag{5.34}$$

then

$$|z_t|_{L^2} \leq C|u_t|_{L^2(w)}\,. \tag{5.35}$$

By other hand,

$$\begin{aligned} 0 &= \int_S^T E^\sigma\phi' \int_\Omega z(u_{tt} - \Delta u + au_t)dx\, dt \\ &= [E^\sigma\phi' \int_\Omega zu_t dx]_S^T - \int_S^T (\sigma E' E^{\sigma-1}\phi' + E^\sigma\phi'')\int_\Omega zu_t dx\, dt \\ &\quad + \int_S^T E^\sigma\phi' \int_\Omega (-z\Delta u + azu_t - z_t u_t)dx\, dt\,. \end{aligned} \tag{5.36}$$

Using the Green Identity and the fact that z is solution of (5.24)-(5.25) we obtain

$$\begin{aligned} \int_S^T E^\sigma\phi' \int_\Omega z\Delta u\, dx\, dt &= \int_S^T E^\sigma\phi' \int_\Omega (\Delta z)u\, dx\, dt \\ &= \int_S^T E^\sigma\phi' \int_\Omega (\beta(x)u)u\, dx\, dt \\ &= \int_S^T E^\sigma\phi' \int_w u^2 dx\, dt\,. \end{aligned} \tag{5.37}$$

Using (5.37) on (5.36) we have

$$\begin{aligned}\int_S^T E^\sigma \phi' \int_w u^2 dx\,dt &= \int_S^T E^\sigma \phi' \int_\Omega z\Delta u\,dx\,dt \\ &= [E^\sigma \phi' \int_\Omega z u_t dx]_S^T - \int_S^T (\sigma E' E^{\sigma-1}\phi' + E^\sigma \phi'') \int_\Omega z u_t dx\,dt \\ &\quad + \int_S^T E^\sigma \phi' \int_\Omega (az u_t - z_t u_t) dx\,dt\,. \end{aligned} \tag{5.38}$$

We can observe that the following inequality holds

$$|[E^\sigma \phi' \int_\Omega z u_t dx]_S^T - \int_S^T (\sigma E' E^{\sigma-1}\phi' + E^\sigma \phi'') \int_\Omega z u_t dx\,dt| \le C E(S)^{1+\sigma}. \tag{5.39}$$

By other hand, let $\eta > 0$, using (5.35) we obtain

$$\begin{aligned}|\int_S^T E^\sigma \phi' \int_\Omega z_t u_t dx\,dt| &\le \int_S^T E^\sigma \phi' C |z_t|_{L^2} |u_t|_{L^2}\,dt \\ &\le \int_S^T E^\sigma \phi' C |u_t|_{L^2(w)} |u_t|_{L^2}\,dt \\ &\le \frac{C}{2\eta} \int_S^T E^\sigma \phi' \int_w u_t^2 dx\,dt + \frac{\eta}{2} \int_S^T E^\sigma \phi' \int_\Omega u_t^2 dx\,dt \\ &\le \frac{C}{2\eta} \int_S^T E^\sigma \phi' \int_w u_t^2 dx\,dt + \eta \int_S^T E^{\sigma+1}\phi'\,dt\,, \end{aligned} \tag{5.40}$$

where η will be taken little enough.

Also, we have

$$\begin{aligned}|\int_S^T E^\sigma \phi' \int_\Omega z a u_t dx\,dt| &\le \int_S^T E^\sigma \phi' |\int_\Omega z a u_t dx|\,dt \\ &\le \int_S^T E^\sigma \phi' |z|_{L^2(\Omega)} |a u_t|_{L^2}\,dt \\ &\le \int_S^T E^\sigma \phi' C |u|_{L^2(w)} |\sqrt{a} u_t|_{L^2}\,dt \\ &\le \gamma \int_S^T E^\sigma \phi' \int_w u^2 dx\,dt + C(\gamma) \int_S^T E^\sigma \phi' \int_\Omega a u_t^2 dx\,dt \\ &= \gamma \int_S^T E^\sigma \phi' \int_w u^2 dx\,dt + C(\gamma) \int_S^T E^\sigma \phi' (-E')\,dt \\ &\le \gamma \int_S^T E^\sigma \phi' \int_w u^2 dx\,dt - C(\gamma) \int_S^T E^\sigma (E')\,dt\,, \end{aligned} \tag{5.41}$$

where γ will be taken very little.

Since

$$-\int_S^T E^\sigma E' dt = -\frac{1}{\sigma+1}[E(t)^{\sigma+1}]_S^T = \frac{1}{\sigma+1}\{E(S)^{\sigma+1} - E(T)^{\sigma+1}\} \leq \frac{1}{\sigma+1}\{E(S)^{\sigma+1}\}$$

then (5.41) becomes

$$|\int_S^T E^\sigma \phi' \int_\Omega z a u_t dx\, dt| \leq \gamma \int_S^T E^\sigma \phi' \int_w u^2 dx\, dt + C(\gamma) E(S)^{\sigma+1}. \tag{5.42}$$

Using (5.39), (5.40) and (5.42) in (5.38), we have

$$(1-\gamma)\int_S^T E^\sigma \phi' \int_w u^2 dx\, dt \leq \frac{C}{\eta}\int_S^T E^\sigma \phi' \int_w u_t^2 dx\, dt + CE(S)^{1+\sigma} + \eta \int_S^T E^{1+\sigma}\phi' dt. \tag{5.43}$$

Taking $\gamma < 1$, from (5.43) we have that there exists $C > 0$ such that $\forall \eta > 0$

$$\int_S^T E^\sigma \phi' \int_w u^2 dx\, dt \leq \frac{C}{\eta}\int_S^T E^\sigma \phi' \int_w u_t^2 dx\, dt + CE(S)^{1+\sigma} + \eta \int_S^T E^{1+\sigma}\phi' dt \tag{5.44}$$

holds, where η is little enough.

Replacing (5.44) in (5.23) we get

$$\int_S^T E^\sigma \phi' \int_{\Omega - K_1} |\nabla u|^2 dxdt \leq CE(S)^{1+\sigma} + (C + \frac{C}{\eta})\int_S^T E^\sigma \phi' \int_w u_t^2 dxdt + \eta \int_S^T E^{1+\sigma}\phi' dt. \tag{5.45}$$

Replacing (5.45) in (5.16) we have

$$(1-\eta)\int_S^T E^{1+\sigma}\phi' dt \leq CE(S)^{1+\sigma} + (C + \frac{C}{\eta})\int_S^T E^\sigma \phi' \int_w u_t^2 dxdt$$

and taking $\eta < 1$ we obtain

$$\int_S^T E^{1+\sigma}\phi' dt \leq CE(S)^{1+\sigma} + C\int_S^T E^\sigma \phi' \int_w u_t^2 dxdt\,.$$

6. Finishing the proof of the Theorem

Let $\rho : t \longrightarrow \rho(t)$ a decreasing function which goes to zero as t goes to the infinite. Later on, we choose ρ.

Let us define the function $\tilde{\alpha}$ by

$$\begin{aligned} \tilde{\alpha}(r,t) &:= \tilde{\alpha}(r) && \text{if} \quad \frac{R}{2} \leq r \leq R-\rho(t) \\ \tilde{\alpha}(r,t) &:= \tilde{\alpha}(R-\rho(t)) && \text{if} \quad r \geq R-\rho(t) \end{aligned} \tag{6.1}$$

and the function α on $w \times \mathbf{R}^+$ by

$$\alpha(x,t) := \tilde{\alpha}(|x|,t), \quad \forall |x| \geq \frac{R}{2}. \tag{6.2}$$

Lemma 6.1 (Gagliardo-Niremberg). *If $m > \frac{N}{2}$, there exists $c > 0$ such that for every $v \in H^m(\Omega)$*

$$||v||_{L^\infty(\Omega)} \leq c||v||^\theta_{H^m(\Omega)} ||v||^{1-\theta}_{L^2(\Omega)} \quad \textit{with} \quad \theta = \frac{N}{2m}\,. \tag{6.3}$$

Now, using (2.3) and (2.4) it is possible to apply (6.3) and deduce

$$\begin{aligned} ||u_t||^2_{L^\infty(\Omega)} &\leq c||u_t||^2_{H^m(\Omega)} \underbrace{||u_t||^{2(1-\theta)}_{L^2(\Omega)}}_{\leq E(t)^{1-\theta}} \\ &\leq C_m||(u^0,u^1)||^{2\theta}_{H^{m+1}\times H^m} E(t)^{1-\theta} \end{aligned} \tag{6.4}$$

Let $p > 0$ such that (2.6) holds. Then, using the Jensen Inequality and (6.4) we will estimate the last term of (5.7).

$$\begin{aligned} &\int_S^T E(t)^\sigma \phi'(t) (\int_w u_t^2 dx)\, dt \\ = &\int_S^T E(t)^\sigma \phi'(t) (\int_w \frac{1}{\alpha(x,t)} u_t^2\, \alpha(x,t) dx)\, dt \\ \leq &\int_S^T E(t)^\sigma \phi'(t)\, ||u_t^2 \alpha^{\frac{p}{p+1}}||_{L^{\frac{p+1}{p}}} ||\alpha^{-\frac{p}{p+1}}||_{L^{p+1}} dt \end{aligned}$$

$$
\begin{aligned}
&= \int_S^T E(t)^\sigma \phi'(t)\, (\int_w \frac{1}{\alpha(x,t)^p} dx)^{\frac{1}{p+1}} (\int_w u_t^{\frac{2(p+1)}{p}} \alpha(x,t) dx)^{\frac{p}{p+1}} dt \\
&= \int_S^T E(t)^\sigma \phi'(t)^{\frac{1}{(p+1)}}\, (\phi'(t)^p \int_w \frac{1}{\alpha(x,t)^p} dx)^{\frac{1}{p+1}} (\int_w u_t^{2+\frac{2}{p}} \alpha(x,t) dx)^{\frac{p}{p+1}} dt \\
&\leq \int_S^T E(t)^\sigma \phi'(t)^{\frac{1}{(p+1)}}\, (\phi'(t)^p \int_w \frac{1}{\alpha(x,t)^p} dx)^{\frac{1}{p+1}} \\
&\qquad \cdot (\int_w u_t^2 \alpha(x,t) dx)^{\frac{p}{p+1}} ||u_t(t)||_{L^\infty(\Omega)}^{\frac{2}{p+1}} dt \\
&\leq C_m \int_S^T E(t)^{\sigma+\frac{(1-\sigma)}{(p+1)}} \phi'(t)^{\frac{1}{(p+1)}}\, (\phi'(t)^p \int_w \frac{1}{\alpha(x,t)^p} dx)^{\frac{1}{p+1}} \\
&\qquad \cdot (\int_w u_t^2 \alpha(x,t) dx)^{\frac{p}{p+1}} dt\,. \qquad (6.5)
\end{aligned}
$$

To simplify notations, we introduce

$$
\varepsilon(t) = \phi'(t) (\int_w \frac{1}{\alpha(x,t)^p} dx)^{\frac{1}{p}}\,. \qquad (6.6)
$$

Let $\epsilon > 0$. Applying the Young inequality we get the following estimation

$$
\begin{aligned}
& \int_S^T E(t)^\sigma \phi'(t) (\int_w \frac{1}{\alpha(x,t)^p} dx) dt \\
\leq\ & C_m \int_S^T E(t)^{\sigma+\frac{(1-\theta)}{(p+1)}} \phi'(t)^{\frac{1}{(p+1)}}\, \varepsilon(t)^{\frac{p}{p+1}} (\int_w u_t^2 \alpha(x,t) dx)^{\frac{p}{p+1}} dt \\
\leq\ & C_m \int_S^T \underbrace{E(t)^{\sigma+\frac{(1-\theta)}{(p+1)}} \phi'(t)^{\frac{1}{(p+1)}}}_{\in L^{p+1}}\ \underbrace{(\varepsilon(t) \int_w u_t^2 \alpha(x,t) dx)^{\frac{p}{p+1}}}_{\in L^{\frac{p+1}{p}}} dt \\
\leq\ & C_m (\int_S^T E(t)^{\sigma(p+1)+(1-\theta)} \phi'(t) dt)^{\frac{1}{(p+1)}} \cdot (\int_S^T \varepsilon(t) \int_w u_t^2 \alpha(x,t) dx\, dt)^{\frac{p}{p+1}} \\
\leq\ & C_m \frac{\epsilon}{p+1} \int_S^T E(t)^{\sigma(p+1)+(1-\theta)} \phi'(t) dt + C_m \frac{p}{(p+1)\epsilon} \int_S^T \varepsilon(t) \int_w u_t^2 \alpha(x,t) dx\, dt\,. \\
& (6.7)
\end{aligned}
$$

σ is defined such that

$$
\sigma(p+1) + (1-\theta) = \sigma + 1\,, \qquad \text{that is} \qquad \sigma = \frac{\theta}{p} = \frac{N}{2mp}\,. \qquad (6.8)
$$

From (5.7) and (6.7) we can deduce: if ϵ is little enough, there exists a positive constant C such that

$$
\int_S^T E(t)^{1+\sigma} \phi'(t) dt \ \leq\ C E(S)^{\sigma+1} + C \int_S^T \varepsilon(t) \int_w u_t^2 \alpha(x,t) dx dt\,. \qquad (6.9)
$$

Now, choosing ρ and ϕ carefully, we will estimate the last term of (6.9).

The Choice of the function ρ.

Let us assume that ϕ is a concave and strictly increasing C^2 function such that

$$\phi(t) \longrightarrow +\infty \quad \text{and} \quad \phi'(t) \longrightarrow 0 \quad \text{as } t \to +\infty\,. \tag{6.10}$$

Lemma 6.2. *If b satisfies (2.6), then there exists $C > 0$ such that*

$$\int_{\frac{R}{2}}^{R} \frac{1}{\tilde{\alpha}(r,t)^p} dr \leq C \frac{\rho(t)}{b(\rho(t))^p}\,. \tag{6.11}$$

Proof.- If b satisfies (2.6), then

$$\begin{aligned}
\int_{\frac{R}{2}}^{R} \frac{1}{\tilde{\alpha}(r,t)^p} dr &= \int_{\frac{R}{2}}^{R-\rho(t)} \frac{1}{\tilde{\alpha}(r,t)^p} dr + \int_{R-\rho(t)}^{R} \frac{1}{\tilde{\alpha}(r,t)^p} dr \\
&= \int_{\frac{R}{2}}^{R-\rho(t)} \frac{1}{\tilde{a}(r)^p} dr + \int_{R-\rho(t)}^{R} \frac{1}{\tilde{a}(R-\rho(t))^p} dr \\
&= \int_{\rho(t)}^{\frac{R}{2}} \frac{1}{\tilde{a}(R-r)^p} dr + \int_{R-\rho(t)}^{R} \frac{1}{\tilde{a}(R-\rho(t))^p} dr \\
&= \int_{\rho(t)}^{\frac{R}{2}} \frac{1}{b(r)^p} dr + \int_{R-\rho(t)}^{R} \frac{1}{b(\rho(t))^p} dr \\
&\leq C \frac{\rho(t)}{b(\rho(t))^p} + \frac{\rho(t)}{b(\rho(t))^p}\,.
\end{aligned}$$

Using (6.11) we obtain the following estimation for ε.

$$\varepsilon(t) \leq \phi'(t) \frac{\rho(t)^{\frac{1}{p}}}{b(\rho(t))}\,. \tag{6.12}$$

Since b is strictly increasing near to 0, we define ρ:

$$\rho(t) := b^{-1}(\phi'(t))\,. \tag{6.13}$$

We observe that ρ is decreasing, since b is increasing and ϕ' is decreasing. From definition of ρ and (6.12) we have

$$\varepsilon(t) \leq C\rho(t)^{\frac{1}{p}}\,. \tag{6.14}$$

Also, we obtain

$$\begin{aligned}
\int_w \alpha(x,t)u_t^2 dx &= \int_{\frac{R}{2}\le|x|\le R-\rho(t)} \alpha(x,t)u_t^2 dx + \int_{|x|>R-\rho(t)} \alpha(x,t)u_t^2 dx \\
&\le \int_\Omega a(x)u_t^2 dx + \int_{|x|>R-\rho(t)} \tilde{a}(R-\rho(t))u_t^2 dx \\
&\le -E'(t) + b(\rho(t))E(t) \\
&= -E'(t) + \phi'(t)E(t)\,. \qquad (6.15)
\end{aligned}$$

From (6.9), using (6.15) , (6.14), $\rho(t)^{\frac{1}{p}} \le \rho(S)^{\frac{1}{p}}$ and the fact that E decreases, we have

$$\begin{aligned}
&\int_S^T E(t)^{\sigma+1}\phi'(t)dt \\
\le\ & CE(S)^{1+\sigma} + C\int_S^T \varepsilon(t)(-E'(t)+\phi'(t)E(t))dt \\
\le\ & CE(S)^{1+\sigma} + C\int_S^T \rho(S)^{\frac{1}{p}}(-E'(t))dt + C\int_S^T \rho(t)^{\frac{1}{p}}\phi'(t)E(S)dt \\
\le\ & CE(S)^{1+\sigma} + C\rho(S)^{\frac{1}{p}}\{E(S)-E(T)\} + CE(S)\int_S^T \rho(t)^{\frac{1}{p}}\phi'(t)dt \\
\le\ & CE(S)^{1+\sigma} + C\rho(S)^{\frac{1}{p}}E(S) + CE(S)\int_S^T \rho(t)^{\frac{1}{p}}\phi'(t)dt\,. \qquad (6.16)
\end{aligned}$$

The choice of the function ϕ.

Here, we will show how to define ϕ such that $\int_1^{+\infty}\rho(t)^{\frac{1}{p}}\phi'(t)dt$ is finite. Let $p' > 1+p$ and define

$$\psi(t) := 1 + \int_1^t \frac{1}{b(\frac{1}{r^{p'}})}dr\,, \quad \forall t > 1\,. \tag{6.17}$$

Then, ψ is an strictly increasing, convex and C^2 function, which satisfies

$$\psi(t) \longrightarrow +\infty \quad \text{and} \quad \psi'(t) = \frac{1}{b(\frac{1}{t^{p'}})} \longrightarrow +\infty \quad \text{as } t\to+\infty\,.$$

Define

$$\phi(t) := \psi^{-1}(t)\,, \quad \forall t \ge 1\,. \tag{6.18}$$

Then ϕ is a strictly increasing, concave and C^2 function, satisfying

$$\phi(t) \longrightarrow +\infty \quad \text{and} \quad \phi'(\psi(t)) = \frac{1}{\psi'(t)} \to 0 \quad \text{as} \quad t\to+\infty. \tag{6.19}$$

Thus ϕ has all the properties that we use to get (5.7) and (6.16). Using $\rho(t) = b^{-1}(\phi'(t))$ and making a change of variable $t = \psi(\tau)$, using (6.19) and making another change of variable $\tau = \phi(t)$, and using $b^{-1}(\frac{1}{\psi'(\tau)}) = b^{-1}(b(\frac{1}{\tau^{p'}})) = b^{-1} \circ b(\frac{1}{\tau^{p'}}) = \tau^{-p'}$ we obtain

$$\begin{aligned}
\int_1^{+\infty} \rho(t)^{\frac{1}{p}} \phi'(t)dt &= \int_1^{+\infty} [b^{-1}(\phi'(t))]^{\frac{1}{p}} \phi'(t)dt \\
&= \int_1^{+\infty} [b^{-1}(\phi'(\psi(\tau)))]^{\frac{1}{p}} d\tau \\
&= \int_1^{+\infty} [b^{-1}(\frac{1}{\psi'(\tau)})]^{\frac{1}{p}} d\tau \\
&= \int_1^{+\infty} \frac{1}{\tau^{\frac{p'}{p}}} d\tau \\
&= \lim_{M \to +\infty} \frac{1}{-\frac{p'}{p}+1} \{M^{-\frac{p'}{p}+1} - 1\} \text{ and since } -\frac{p'}{p} + 1 < 0 \\
&= \frac{1}{\frac{p'}{p} - 1} > 0\,.
\end{aligned}$$

Estimation depending on ϕ.

Since $\psi \circ \phi = I$ then $\psi'(\phi(t))\phi'(t) = 1$, and then $\phi'(t) = \frac{1}{\psi'(\phi(t))} = b(\frac{1}{\phi(t)^{p'}})$, from which we deduce

$$\rho(t) = b^{-1}(\phi'(t)) = b^{-1} \circ b(\frac{1}{\phi(t)^{p'}}) = \frac{1}{\phi(t)^{p'}}\,. \tag{6.20}$$

By other hand, using (6.20), we obtain

$$\begin{aligned}
\int_S^T \rho(t)^{\frac{1}{p}} \phi'(t)dt &= \int_S^T \frac{1}{\phi(t)^{\frac{p'}{p}}} dt \\
&= \frac{1}{1-\frac{p'}{p}} [\phi(t)^{1-\frac{p'}{p}}]_S^T \\
&= \frac{1}{1-\frac{p'}{p}} \{\phi(T)^{1-\frac{p'}{p}} - \phi(S)^{1-\frac{p'}{p}}\} \\
&= \frac{1}{\frac{p'}{p}-1} \{\phi(S)^{1-\frac{p'}{p}} - \phi(T)^{1-\frac{p'}{p}}\} \\
&\leq \frac{1}{\frac{p'}{p}-1} \{\phi(S)^{1-\frac{p'}{p}}\} \quad \text{since } \phi = \psi^{-1} > 0\,.
\end{aligned} \tag{6.21}$$

Using (6.20) and (6.21) in (6.16) we have

$$\begin{aligned}\int_S^T E(t)^{\sigma+1}\phi'(t)dt &\leq CE(S)^{1+\sigma} + E(S)\frac{C}{\phi(S)^{\frac{p'}{p}}} + E(S)\frac{C}{\phi(S)^{\frac{p'}{p}-1}} \\ &\leq CE(S)^{1+\sigma} + E(S)\frac{C}{\phi(S)^{\frac{p'}{p}-1}}\,. \qquad (6.22)\end{aligned}$$

Then apply Lemma 4.4 since (4.26) holds with $\sigma' = \frac{p'}{p} - 1 > 0$, and deduce that exists a constant C depending continuously on $E(1)$, such that

$$E(t) \leq \frac{C}{\phi(t)^{\frac{p'}{p\sigma}}} = \frac{C}{\phi(t)^{\frac{p'}{\theta}}}\,, \quad \forall t \geq 1\,. \qquad (6.23)$$

Growth of ϕ.

Estimate the growth of ϕ is equivalent to bound the function $\phi^{-1} = \psi$.

Let T_0 such that $b(\frac{1}{\tau^{p'}}) \leq 1\,, \quad \forall \tau \geq T_0$.

If $s < \tau$ and since b is increasing we have $b(\frac{1}{\tau^{p'}}) \leq b(\frac{1}{s^{p'}})$, i.e. $\frac{1}{b(\frac{1}{s^{p'}})} \leq \frac{1}{b(\frac{1}{\tau^{p'}})}$.

On other hand, we have that: if $p' \geq 1$, $1 + (\tau - 1)z \leq \tau^{p'}z$ holds for $z \geq 1$ and $\tau \geq 1$. In fact, we only have to prove that $1 \leq (1 - \tau + \tau^{p'})z$. If $\tau \geq 1$ then $\tau^{p'} - \tau \geq 0$ and so $1 + \tau^{p'} - \tau \geq 1$; then multiplying by $z \geq 1$ we have $(1 - \tau + \tau^{p'})z \geq 1$. Using these remarks we obtain

$$\begin{aligned}\psi(\tau) &= 1 + \int_1^\tau \frac{1}{b(\frac{1}{s^{p'}})}ds \leq 1 + \frac{1}{b(\frac{1}{\tau^{p'}})}\int_1^\tau ds \leq 1 + (\tau - 1)\frac{1}{b(\frac{1}{\tau^{p'}})} \\ &\leq \tau^{p'} \cdot \frac{1}{b(\frac{1}{\tau^{p'}})} = \frac{1}{B(\frac{1}{\tau^{p'}})}\,. \qquad (6.24)\end{aligned}$$

Then, letting $t = \frac{1}{B(\frac{1}{\tau^{p'}})}$ (that is $\frac{1}{\tau^{p'}} = B^{-1}(\frac{1}{t})$) and using (6.24) we have $\psi(\tau) \leq t$, from which $\tau \leq \psi^{-1}(t) = \phi(t)$, that is

$$\frac{1}{\phi(t)} \leq \frac{1}{\tau}\,. \qquad (6.25)$$

Thus, using (6.25) in (6.23) we obtain

$$E(t) \leq \frac{C}{\phi(t)^{\frac{p'}{\theta}}} \leq \frac{1}{\tau^{\frac{p'}{\theta}}} = \left(B^{-1}(\frac{1}{t})\right)^{\frac{1}{\theta}}\,, \qquad (6.26)$$

where $\theta = \frac{N}{2m}$.

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