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ON SOME INFINITESIMAL AUTOMORPHISMS OF RIEMANNIAN FOLIATION

MOHAMED ALI CHAOUCH
UNIVERSITÉ DU 7 NOVEMBRE À CARTHAGE, TUNISIE

NABILA TORKI - HAMZA
UNIVERSITÉ DU 7 NOVEMBRE À CARTHAGE, TUNISIE

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Abstract

In Riemannian foliation, a transverse affine vector field preserves the curvature and its covariant derivatives. In this paper we solve the converse problem. Actually, we show that an infinitesimal automorphism of a Riemannian foliation which preserves the curvature and its covariant derivatives induces a transverse almost homothetic vector field. If in addition the manifold is closed and the foliation is irreducible harmonic , then a such infinitesimal automorphism induces a transverse killing vector field.

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1. Introduction

It is well-known that an affine vector field on Riemannian manifold preserves the curvature tensor and its covariant differentials. Using the definition of a transverse affine vector field on a Riemannian manifold endowed with a Riemannian foliation, we can easily show in the same way that an affine infinitesimal automorphism of the foliation preserves the curvature and its covariant derivatives.

In [10] the authors discussed the inverse problem and they used the decomposition of de Rham to prove that a vector field which preserves the curvature and its covariant derivatives is homothetic.

In this paper we use the basic connection [11] and the Blumenthal decomposition of a Riemannian foliation [3] to extend the Nomizu-Yano theorem to the Riemannian foliation case building on their proof idea. We show that an infinitesimal automorphism of Riemannian analytic foliation which preserves the connection and its covariant derivatives is a transverse almost homothetic.

The following theorems are the main results of this work.

Theorem 1. *Let $\mathcal{F}$ be an irreducible analytic harmonic g_M-Riemannian foliation of codimension ≥ 2 on a closed analytic Riemannian manifold (M, g_M), and let X be an infinitesimal automorphism of $\mathcal{F}$ such that*

$$\Theta(X)\nabla^m R = 0 \quad \text{for all } m \in \mathbf{N}.$$

Then $\pi(X)$ is transverse Killing.

Theorem 2. *Let $\mathcal{F}$ be an analytic g_M-Riemannian foliation without Euclidean part on an analytic Riemannian manifold (M, g_M), and let X be an infinitesimal automorphism of $\mathcal{F}$ such that*

$$\Theta(X)\nabla^m R = 0 \quad \text{for all } m \in \mathbf{N}.$$

Then $\pi(X)$ is transverse almost homothetic.

The paper is organized as follows. In section 2 we recall some definitions and we give some examples. In section 3 we introduce a transverse tensor field of type (1,2) measuring the deviation of transverse vector fields to be transverse affine and we provide some preliminary results. Section 4 is devoted to some integral formulas. Moreover, we prove that on a closed Riemannian manifold endowed with harmonic Riemannian foliation any transverse affine vector field is a transverse Killing. In section 5 we prove some preliminaries theorems. The proofs of the main theorems are given in section 6.

2. Preliminaries

2.1. Adapted connection

Let (M, g_M) be a Riemannian connected manifold of dimension n with a foliation $\mathcal{F}$ of codimension q. The foliation $\mathcal{F}$ is given by an integrable subbundle E of the tangent bundle TM over M. Let $E^{\perp}$ denote the orthogonal complement bundle of E, and let Q indicate the normal bundle TM/E. The bundle TM splits orthogonally as $TM = E \oplus E^{\perp}$ with the map $\sigma : Q \longrightarrow E^{\perp} \subset TM$ splitting the following exact sequence:

$$0 \longrightarrow E \longrightarrow TM \xrightarrow{\pi} Q \longrightarrow 0.$$

Then the metric g_M on TM is the direct sum $g_M = g_E \oplus g_{E^{\perp}}$ with $g_Q = \sigma^{\star} g_{E^{\perp}}$. The splitting map $\sigma : (Q, g_Q) \longrightarrow (E^{\perp}, g_{E^{\perp}})$ is a metric isomorphism.

Let ∇^Mbe the Riemannian connection on (M, g_M). We can define an *adapted connection* ∇ in Q by putting

$$\nabla_X s = \begin{cases} \pi([X, Z_s]) & \text{for } X \in \Gamma(E) \\ \pi(\nabla^M_X Z_s) & \text{for } X \in \Gamma(E^{\perp}) \end{cases}$$

for $s \in \Gamma(Q)$ and $Z_s = \sigma(s) \in \Gamma(E^{\perp})$, where $\Gamma(L)$ denotes the space of all sections of the bundle L.

The *torsion* T_{∇} of ∇ is given by

$$T_{\nabla}(X, Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi([X, Y])$$

for all $X, Y \in \Gamma(TM)$, and the *curvature* R_{∇} of ∇ is defined by

$$R_{\nabla}(X, Y).\pi(Z) = \nabla_X \nabla_Y (\pi(Z)) - \nabla_Y \nabla_X (\pi(Z)) - \nabla_{[X,Y]} \pi(Z)$$

for all $X, Y, Z \in \Gamma(TM)$.

We know that $T_{\nabla} = 0$ and $R_{\nabla}(X, Y) = 0$ for $X, Y \in \Gamma(E)$ [9].

2.2. Q-tensor field, Lie-differentiation and covariant differential

We recall [7] that a *Q-tensor field* $\mathcal{K}$ of type $(0, r)$ (resp., $(1, r)$) on M is an r-linear mapping of $\Gamma(Q) \times ... \times \Gamma(Q)$ into $A(M)$ (resp., into $\Gamma(Q)$) such that

$$\mathcal{K}(f_1 s_1, ..., f_r s_r) = f_1 ... f_r . \mathcal{K}(s_1, ..., s_r)$$

for $f_i \in A(M)$ and $s_i \in \Gamma(Q)$, where $A(M)$ is the algebra of real-valued functions of class C^∞ on M.

A section $X \in \Gamma(TM)$ is an *infinitesimal automorphism* of $\mathcal{F}$ if $[X,Y] \in \Gamma(E)$ for all $Y \in \Gamma(E)$. We denote by $\mathcal{V}(\mathcal{F})$ the space of all infinitesimal automorphisms of $\mathcal{F}$. For $X \in \mathcal{V}(\mathcal{F})$, the *Lie-differentiation* $\Theta(X)$ with respect to X is defined by

$$\Theta(X)s = \pi[X, \sigma(s)] \quad \text{for all } s \in \Gamma\left(Q\right),$$

and

$$\begin{aligned}(\Theta(X).\mathcal{K})(s_1, ..., s_r) &= \Theta(X).(\mathcal{K}(s_1, ..., s_r)) \\ &- \sum_{i=1}^{r} \mathcal{K}(s_1, .., \pi([X, \sigma(s_i)]), .., s_r)\end{aligned}$$

for all $s \in \Gamma(Q)$, for any Q-tensor field $\mathcal{K}$ of type $(0,r)$ or $(1,r)$, and any $s_1, ..., s_r \in \Gamma(Q)$.

Let $\mathcal{K}$ be a Q-tensor field of type $(0,r)$ or $(1,r)$. The *covariant derivative* of $\mathcal{K}$ is defined by

$$\begin{aligned}\nabla\mathcal{K}(X; s_1, ..., s_r) &= (\nabla_X\mathcal{K})(s_1, ..., s_r) \\ &= \nabla_X\mathcal{K}(s_1, ..., s_r)) - \sum_{i=1}^{r} \mathcal{K}(s_1, .., \nabla_X s_i, .., s_r),\end{aligned}$$

for $s_i \in \Gamma(Q)$ and $X \in \Gamma(TM)$. It's clear that $\nabla_X\mathcal{K} = \Theta(X)\mathcal{K}$ for all $X \in \Gamma(E)$.

The tensor $\mathcal{K}$ is called *holonomy invariant* (or *parallel along the leaves* of $\mathcal{F}$) if

$$\Theta(X)\mathcal{K} = \nabla_X\mathcal{K} = 0 \qquad \text{for all } X \in \Gamma(E).$$

In this case $\nabla\mathcal{K}$ is a well defined Q-tensor field of type $(0, r+1)$ or $(1, r+1)$ given by

$$\nabla\mathcal{K}(t, .) = \nabla_{\sigma(t)}\mathcal{K}(.).$$

If $\mathcal{K}$ is holonomy invariant, the *second covariant differential* $\nabla^2\mathcal{K}$ is defined by

$$\nabla^2\mathcal{K}(X; t, .) = \nabla_X(\nabla\mathcal{K})(t, .)$$

for all $t \in \Gamma(Q)$ and $X \in \Gamma(TM)$.

Let $m \geq 1$. If $\nabla^{m-1}\mathcal{K}$ is well-defined and is holonomy invariant, then the m^{th} covariant differential $\nabla^m\mathcal{K}$ is an $(r+m)$ Q-tensor field defined by

$$(\nabla^m\mathcal{K})(s; s_1, ., s_m) = \nabla_{\sigma(s)}(\nabla^{m-1}\mathcal{K})(s_1, ., s_m).$$

Proposition 1. *Let $\mathcal{K}$ be an holonomy invariant Q-tensor field of type $(0, r)$ or $(1, r)$ on M. The following formulas hold.*

$$\nabla^2\mathcal{K}(X; t, .) = \nabla_X(\nabla_{\sigma(t)}\mathcal{K}) - \nabla_{\sigma(\nabla_X t)}\mathcal{K}, \tag{2.1}$$

for all $t \in \Gamma(Q)$ and $X \in \Gamma(TM)$.

$$\nabla^2\mathcal{K}(X; t, .) = R_\nabla(X, \sigma(t)).\mathcal{K} \tag{2.2}$$

for all $t \in \Gamma(Q)$ and $X \in \Gamma(E)$.

Proof. i) Let $s_1, ..., s_r \in \Gamma(Q)$ and $X \in \Gamma(TM)$. Observe that

$$\begin{array}{rcl} \nabla^2\mathcal{K}(X, t, s_1, ..., s_r) & = & \nabla_X(\nabla\mathcal{K})(t, s_1, ..., s_r) \\ & = & \nabla_X(\nabla\mathcal{K}(t, s_1, ..., s_r)) \\ & - & \sum\limits_{1\leq i\leq r}(\nabla_{\sigma(t)}\mathcal{K})(s_1, ..., \nabla_X s_i, ..., s_r) \\ & - & (\nabla_{\sigma(\nabla_X t)}\mathcal{K})(s_1, ..., s_r). \end{array}$$

On the other hand,

$$\begin{array}{rcl} \nabla_X(\nabla\mathcal{K}(t, s_1, ..., s_r)) & = & \nabla_X((\nabla_{\sigma(t)}\mathcal{K})(s_1, ..., s_r)) \\ & = & (\nabla_X(\nabla_{\sigma(t)}\mathcal{K}))(s_1, ..., s_r) \\ & + & \sum\limits_{1\leq i\leq r}(\nabla_{\sigma(t)}\mathcal{K})(s_1, ..., \nabla_X s_i, ..., s_r). \end{array}$$

An elimination process leads then to the formula (2.1).

ii) Let $t \in \Gamma(Q)$ and $X \in \Gamma(E)$. Since

$$R_\nabla(X, \sigma(t)) = \nabla_X\nabla_{\sigma(t)} - \nabla_{\sigma(t)}\nabla_X - \nabla_{[X,\sigma(t)]} \quad \text{and} \quad \nabla_X\mathcal{K} = 0,$$

we obtain

$$\nabla^2\mathcal{K}(X; t, .) = R_\nabla(X, \sigma(t))\mathcal{K} + \nabla_Y\mathcal{K},$$

where $Y = [X, \sigma(t)] - \sigma(\nabla_X t) \in \Gamma(E)$. The formula (2.2) follows. □

2.3. Transverse vector field

Let $X \in \mathcal{V}(\mathcal{F})$. We say that $\pi(X)$ is *transverse conformal* if $\Theta(X)g_Q = f.g_Q$, where f is a basic function on M. Moreover, if f is a constant function (resp., $f = 0$), $\pi(X)$ is said to be *transverse homothetic* (resp., *transverse Killing*). The section $\pi(X)$ is said to be *transverse almost homothetic* if the normal bundle Q can be decomposed into a direct sum $Q = Q_1 + ... + Q_k$ of pairwise orthogonal subbundles such that $\Theta(X)g_{Q_i} = c_i.g_{Q_i}$, where g_{Q_i}

is the restriction of the metric g_Q to the subbundle Q_i and c_i is a real constant. Finally, $\pi(X)$ is referred to us a *transverse affine* section if X preserves the connection ∇ in Q, that is, $\Theta(X)\nabla = 0$.

2.4. Riemannian foliation and holonomy group

From now on, we suppose $\mathcal{F}$ to be a g_M-Riemannian foliation (i. e., the metric g_M is bundle-like in the sense of Reinhart). So the induced metric g_Q is holonomy invariant.

Recall from [11] that R_∇ is basic, that is, $i_X R_\nabla = 0$ and $\Theta(X)R_\nabla = 0$ for all $X \in \Gamma(E)$. Hence the curvature tensor R_∇ induces an holonomy invariant Q-tensor field R of type $(1,3)$ defined by

$$R(s,t)\eta = R_\nabla(\sigma(s),\sigma(t))\eta$$

for all $s,t,\eta \in \Gamma(Q)$. If we put

$$\nabla^2_{\sigma(s)\sigma(t)} = \nabla_{\sigma(s)}\nabla_{\sigma(t)} - \nabla_{\sigma(\nabla_{\sigma(s)}t)}, \tag{2.3}$$

then we have

$$R(s,t) = \nabla^2_{\sigma(s)\sigma(t)} - \nabla^2_{\sigma(t)\sigma(s)} - \nabla_Z, \tag{2.4}$$

where $Z = [\sigma(s),\sigma(t)] - \sigma\circ\pi[\sigma(s),\sigma(t)] \in \Gamma(E)$.

Proposition 2. *Let $\mathcal{K}$ be a Q-tensor field of type $(0,r)$ or $(1,r)$. If $\mathcal{K}$ is holonomy invariant, then for all $m \geq 1$, $\nabla^m\mathcal{K}$ is holonomy invariant well-defined Q-tensor field of type $(0,r+m)$ or $(1,r+m)$.*

Proof. Suppose that for $m \geq 1$ the Q-tensor fields $\mathcal{K},...,\nabla^{m-1}\mathcal{K}$ are holonomy invariant. Let $X \in \Gamma(E)$ and $s \in \Gamma(Q)$. Since $i_X R_\nabla = 0$, it follows from the formula (2.2) that

$$\nabla_X(\nabla^m\mathcal{K})(t,.) = R_\nabla(X,\sigma(t))\nabla^{m-1}\mathcal{K} = 0.$$

Accordingly, $\nabla^m\mathcal{K}$ is also holonomy invariant. $\square$

As an immediate consequence of Proposition 2, we get the following.

Corollary 1. *For all $m \in \mathbf{N}$, $\nabla^m R$ is a holonomy invariant well-defined Q-tensor field of type $(1,m+3)$.*

The proof of the next Bianchi identities are routine and therefore omitted.

Proposition 3. *The curvature tensor R satisfies the first Bianchi identity*

$$R(s,t)\eta + R(t,\eta)s + R(\eta,s)t = 0 \tag{2.5}$$

and satisfies the second Bianchi identity

$$(\nabla_{\sigma(\eta)}R)(s,t) + (\nabla_{\sigma(s)}R)(t,\eta) + (\nabla_{\sigma(t)}R)(\eta,s) = 0. \tag{2.6}$$

Let $x \in M$ and let $\mathcal{C}(x)$ be the space of all loops at x. For each $\tau \in \mathcal{C}(x)$, the parallel transport along τ is an isometry of Q_x. The set of all such isometries of Q_x is the *holonomy group* $\Psi(x)$ of ∇ with reference point x. Let $\mathcal{C}^0(x)$ be the subset of $\mathcal{C}(x)$ consisting of loops which are homotopic to zero. The subgroup of $\Psi(x)$ consisting of the parallel transport along $\tau \in \mathcal{C}^0(x)$ is the *restricted holonomy group* $\Psi^0(x)$ of ∇ with reference point x. We say that $\mathcal{F}$ is *irreducible (reducible)* if the action of $\Psi(x)$ on Q_x is irreducible (reducible) see [3]. It is obvious that if $\mathcal{F}$ is irreducible then the normal bundle Q does not have a connection invariant proper subbundle. We say that $\mathcal{F}$ is without *Euclidian part* if $\Psi(x)$ has no non zero fixed vector.

As in [7], we may get quickly that if M is analytic and $\mathcal{F}$ is an analytic Riemannian foliation, then the restricted holonomy group $\Psi^0(x)$ is completely determined by the values of all successive covariant differentials $\nabla^m R$, $m = 0, 1, 2, ...$, at the point x.

Example 1. *(i) If M is closed and $\mathcal{F}$ is a Lie $\mathbf{R}^q$-foliation of codimension $q \geq 2$, then $\mathcal{F}$ is defined by independent closed one-formes $\omega_1, ..., \omega_q$ [6]. So $\mathcal{F}$ has Euclidian part and hence it is reducible.*

(ii) If M is closed with $\pi_1(M)$ abelian and $\mathcal{F}$ is a one-dimensional Euclidean foliation (i.e. Riemannian tranversally affine), then $\mathcal{F}$ is reducible [4].

In the sequel we assume that M is a closed manifold and $\mathcal{F}$ is a codimension two Euclidean foliation. In this case, we have the following equivalences

$\mathcal{F}$ is a Lie $\mathbf{R}^2$-foliation $\Longleftrightarrow$ $\mathcal{F}$ has Euclidean part $\Longleftrightarrow$ $\mathcal{F}$ is reducible.

For the following results, see [1].

(iii) If all the leaves of $\mathcal{F}$ are simply connected, then $\mathcal{F}$ is reducible.

(iv) If $\dim M = 3$ and $\mathcal{F}$ is without Euclidian part, then $\mathcal{F}$ has a compact leaf.

(v) *If $\pi_1(M)$ is abelian, then either $\mathcal{F}$ is reducible, or $\mathcal{F}$ has a compact leaf L such that $i_* : \pi_1(L) \longrightarrow \pi_1(M)$ is an isomorphism.*

(vi) *If $dimM = 4$ and $\pi_1(M)$ abelian with $\pi_1(M) \neq \mathbf{Z} \times \mathbf{Z}$, then $\mathcal{F}$ is reducible.*

(vii) *If $\mathcal{F}$ is $SO(2)\mathbf{R}^2/SO(2)$-foliation with trivial normal bundle, that is defined by independent one-forms ω_1, ω_2 satisfying*

$$d\omega_1 = \frac{1}{2}\omega_2 \wedge \omega_3 \ , \ d\omega_2 = -\frac{1}{2}\omega_1 \wedge \omega_3 \ , \ d\omega_3 = 0.$$

Then $\mathcal{F}$ is irreductible.

(viii) *We say that $\mathcal{F}$ is Riemannian tranversally almost parallelisable foliation [C] if there is a G-reduction P of the normal frame bundle $F(Q)$ compatible with the foliation, where G is a discrete Lie subgroup of the orthogonal group $O(q, \mathbf{R})$. If $\tilde{M}$ is a connected component of P, then the bundle projection $p : \tilde{M} \longrightarrow M$ is a connected covering space with G as the group of deck transformations such that $\widetilde{\mathcal{F}} = p^{-1}(\mathcal{F})$ is tranversally parallelisable (e-foliation) [5]. Moreover, if $\mathcal{F}$ is two codimentional, then $\widetilde{\mathcal{F}}$ is a Lie $\mathbf{R}^2$-foliation. On the other hand $\mathcal{F}$ is tranversally almost parallelisable, if and only if the basic connection ∇ is flat (i.e. $R = 0$) [7]. Moreover, if ∇ is complete then the universal cover of M is a product $\widehat{L} \times \mathbf{R}^q$, where $\widehat{L}$ is (common) universal cover of the leaves of $\mathcal{F}$ [2].*

3. Some computational results

Let $X \in \mathcal{V}(\mathcal{F})$, $Y \in \Gamma(TM)$ and $s \in \Gamma(Q)$, so we have

$$\begin{aligned} (\Theta(X)\nabla)(Y,s) &= \Theta(X)(\nabla(Y,s)) - \nabla(\Theta(X)Y,s) - \nabla(Y,\Theta(X)s) \\ &= \Theta(X)\nabla_Y s - \nabla_{[X,Y]}s - \nabla_Y\Theta(X)s \\ &= [\Theta(X),\nabla_Y]s - \nabla_{[X,Y]}s. \end{aligned} \tag{3.1}$$

Then the operator $K = \Theta(X)\nabla \in Hom(\Gamma(TM), End(\Gamma(Q))$ is defined by

$$K(Y) = [\Theta(X), \nabla_Y] - \nabla_{[X,Y]} \quad \text{for all } Y \in \Gamma(TM),$$

measures the deviation of $\nu = \pi(X)$ of being transverse affine (i.e. X preserves the connection ∇). Since

$$[\Theta(X), \Theta(Y)] - \Theta([X,Y]) = 0$$

for all $Y \in \Gamma(E)$, see [9], then K is a semi-basic form in the sense that $i_Y K = 0$. Consequently, for $s \in \Gamma(Q)$, the operator

$$K(s) = [\Theta(X), \nabla_{\sigma(s)}] - \nabla_{[X,\sigma(s)]} \tag{3.2}$$

is a well defined endomorphism on $\Gamma(Q)$, that is K is a $Q-$tensor field of type $(1,2)$ on M. On the other hand, since $T_\nabla = 0$, then we have

$$\Theta\,(X) = A_X + \nabla_X \tag{3.3}$$

where A_X is a Q-tensor field of type (1,1) on M defined by

$$A_X s = -\nabla_{\sigma(s)}\nu. \tag{3.4}$$

Proposition 4. *We have the following properties,*

i) A_X is holonomy invariant,
ii) K is holonomy invariant.

Proof. First, we remark that from relations (3.2) and (3.3), we have for $s \in \Gamma(Q)$

$$K(s) = R_\nabla(X, \sigma(s)) + [A_X, \nabla_{\sigma(s)}] = R(\nu, s) - \nabla_{\sigma(s)}(A_X). \tag{3.5}$$

i) Let $Y \in \Gamma(E)$ and $s \in \Gamma(Q)$; since $\nabla_Y s = \Theta(Y)s$, then by formula (3.4), we get

$$\begin{aligned} (\Theta(Y)A_X)(s) &= \Theta(Y).A_X(s) - A_X(\Theta(Y)s) \\ &= (-\Theta(Y)\nabla_{\sigma(s)} + \nabla_{[Y,\sigma(s)]})(\nu) \\ &= (R(s,\pi(Y)) - \nabla_{\sigma(s)}\nabla_Y)(\nu) = 0 \end{aligned} \tag{3.6}$$

because $\pi(Y) = 0$ and $\nabla_Y \nu = \pi[Y, X] = 0$.

ii) Let $Y \in \Gamma(E)$ and $s \in \Gamma(Q)$; since A_X is holonomy invariant, so by formula (3.5), we have

$$\begin{aligned} K(\Theta(Y)s) &= R(\nu, \pi[Y, \sigma(s)]) - \nabla_{\sigma\circ\pi([Y,\sigma(s)])}(A_X) \\ &= R(\nu, \pi[Y, \sigma(s)]) - \nabla_{[Y,\sigma(s)]}(A_X). \end{aligned} \tag{3.7}$$

On the other hand, since the curvature tensor R is holonomy invariant (section 2.4) and $X \in V(\mathcal{F})$, then

$$\begin{aligned} \Theta(Y)(K(s)) &= (\Theta(Y)R)(\nu, s) + R(\pi[Y, X], s) \\ &+ R(\nu, \pi[Y, \sigma(s)]) - \Theta(Y)(\nabla_{\sigma(s)}(A_X)) \\ &= R(\nu, \pi[Y, \sigma(s)]) - \Theta(Y)(\nabla_{\sigma(s)}(A_X)). \end{aligned} \tag{3.8}$$

Now let's estimate the term $\Theta(Y)(\nabla_{\sigma(s)}(A_X))$. Since $\nabla_Y(A_X) = 0$ and $\Theta(Y)t = \nabla_Y t$ for $t \in \Gamma(Q)$, then

$$\begin{aligned} \Theta(Y)(\nabla_{\sigma(s)}(A_X))(t) &= \nabla_Y \nabla_{\sigma(s)}(A_X)(t) - \nabla_{\sigma(s)}(A_X)(\nabla_Y t) \\ &= \nabla_Y \nabla_{\sigma(s)} A_X(t) - \nabla_Y A_X(\nabla_{\sigma(s)} t) \\ &- \nabla_{\sigma(s)} A_X(\nabla_Y t) + A_X(\nabla_{\sigma(s)} \nabla_Y t) \\ &= \nabla_Y \nabla_{\sigma(s)} A_X(t) - A_X(\nabla_Y \nabla_{\sigma(s)} t) \\ &- \nabla_{\sigma(s)} \nabla_Y A_X(t) + A_X(\nabla_{\sigma(s)} \nabla_Y t) \\ &= (\nabla_Y \nabla_{\sigma(s)} - \nabla_{\sigma(s)} \nabla_Y)(A_X)(t). \end{aligned} \tag{3.9}$$

It follows from (3.7), (3.8), (3.9) and (3.6) that

$$\begin{aligned} (\Theta(Y)K)(s) &= \Theta(Y)(K(s)) - K(\Theta(Y)s) \\ &= \nabla_{[Y, \sigma(s)]}(A_X) - \Theta(Y)(\nabla_{\sigma(s)}(A_X)) \\ &= R_\nabla(\sigma(s), Y)(A_X) = 0 \end{aligned}$$

because $i_Y R_\nabla = 0$. □

Proposition 5. *The Q-tensor field K has the following properties.*

i) for any $s, t \in \Gamma(Q)$ and $Y \in \Gamma(M)$ we have

$$(\nabla_Y K)(s)(t) = (\nabla_Y K)(t)(s), \tag{3.10}$$

ii) for any $s, t \in \Gamma(Q)$ we have

$$\nabla_{\sigma(s)} K)(t) - (\nabla_{\sigma(t)} K)(s) = (\Theta(X)R)(s, t), \tag{3.11}$$

iii) if ν is a transverse conformal, then for $s \in \Gamma(Q)$ we have

$$(\nabla_Y K)(s) g_Q = (\nabla_Y \omega)(s).g_Q \tag{3.12}$$

where $\omega = -df \circ \sigma \in \Gamma(Q^*)$.

iv) for all $s \in \Gamma(Q)$ we have

$$K(s)R = \Theta(X)(\nabla R)(s) - \nabla_{\sigma(s)} \Theta(X)R. \tag{3.13}$$

Proof. i) Let $s, t \in \Gamma(Q)$, according to (3.4) and (2.3) we have

$$\begin{aligned} \nabla_{\sigma(s)}(A_X)(t) &= \nabla_{\sigma(s)} A_X(t) - A_X(\nabla_{\sigma(s)} t) \\ &= -\nabla_{\sigma(s)} \nabla_{\sigma(t)} \nu + \nabla_{\sigma(\nabla_{\sigma(s)} t)} \nu \\ &= -\nabla^2_{\sigma(s)\sigma(t)} \nu. \end{aligned} \tag{3.14}$$

Consequently, from (2.4) we obtain

$$\nabla_{\sigma(t)}(A_X)(s) - \nabla_{\sigma(s)}(A_X)(t) = R(s,t)\nu - \nabla_Z \nu = R(s,t)\nu, \tag{3.15}$$

because $X \in V(\mathcal{F})$. It follows from (3.5), (3.15) and the first Bianchi identity (2.5) that

$$K(s)(t) - K(t)(s) = R(\nu, s)t + R(t, \nu)s + R(s,t)\nu = 0. \tag{3.16}$$

By taking the covariant differential ∇_Y of (3.16) and using (3.16) again, we obtain the relation (3.10).

ii) Let $s, t \in \Gamma(Q)$, we take the covariant differential of (3.5) and use (3.4) to obtain

$$(\nabla_{\sigma(s)} K)(t) = (\nabla_{\sigma(s)} R)(\nu, t) - R(A_X(s), t) - \nabla^2_{\sigma(s),\sigma(t)}(A_X). \tag{3.17}$$

Since A_X is holonomy invariant, then we have

$$\begin{aligned} \nabla^2_{\sigma(s),\sigma(t)} A_X - \nabla^2_{\sigma(t),\sigma(s)} A_X = R(s,t)(A_X) + \nabla_Z(A_X) \\ = R(s,t)(A_X). \end{aligned} \tag{3.18}$$

On the other side let $\nu \in \Gamma(Q)$, so

$$\begin{aligned} (A_X R)(s,t)\nu &= A_X(R(s,t))\nu - R(A_X(s),t)\nu - R(s, A_X(t))\nu \\ &= A_X(R(s,t)\nu) - R(s,t)(A_X(\nu)) \\ &- R(A_X(s),t)\nu - R(s, A_X(t))\nu \\ &= -(R(s,t)(A_X)(\nu) + R(A_X(s),t)\nu + R(s, A_X(t))\nu) \end{aligned} \tag{3.19}$$

So by virtu of (3.17), (3.18), (3.19), (3.3) and the second Bianchi identity (2.6) we get

$$\begin{aligned}(\nabla_{\sigma(s)}K)(t)-(\nabla_{\sigma(t)}K)(s) &= \nabla_{\sigma(s)}R(\nu,t)+\nabla_{\sigma(t)}R(s,\nu)\\ &- R(s,t)(A_X)-R(A_X(s),t)-R(s,A_X(t))\\ &= \nabla_{\sigma(s)}R(\nu,t)+\nabla_{\sigma(t)}R(s,\nu)+(A_XR)(s,t)\\ &= (\Theta(X)R)(s,t).\end{aligned}$$

Hence we have the formula (3.11).

iii) Let $s\in\Gamma(Q)$, since g_Q is holonomy invariant, then $\nabla g_Q=0$ and we have

$$K(s)g_Q=-\nabla_{\sigma(s)}\Theta(X)g_Q=-\nabla_{\sigma(s)}(f.g_Q)=\omega(s)g_Q \tag{3.20}$$

where $\omega=-df\circ\sigma$. Now we take ∇_Y of (3.20) and obtain the relation (3.12).

iv) Let $s\in\Gamma(Q)$, since we have

$$\Theta(X)(\nabla R)(s)=\Theta(X)(\nabla_{\sigma(s)}R)-\nabla_{[X,\sigma(s)]}R,$$

then

$$\begin{aligned}K(s).R &= \Theta(X)\nabla_{\sigma(s)}R-\nabla_{\sigma(s)}\Theta(X)R-\nabla_{[X,\sigma(s)]}R\\ &= \Theta(X)(\nabla R)(s)-\nabla_{\sigma(s)}\Theta(X)R.\end{aligned}$$

□

4. Transverse affine vector field and harmonic Riemannian foliation

In this section we generalize some classical results on Riemannian manifolds to the Riemannian foliation case.

4.1. Harmonic foliation

For unexplained notation and terminology, we refer the reader to [8.11].

Let $(E_i)_{1\leq i\leq n}$ be a local orthonormal frame of TM such that $E_i\in\Gamma(E)$ for $0\leq i\leq p$, and $E_i\in\Gamma(E^{\perp})$ for $p+1\leq i\leq n$, where $p+q=n$. For $1\leq i\leq q$ let $e_i=\pi(E_{p+i})$, so $\sigma(e_i)=E_{p+i}$ and the family $(e_i)_{0\leq i\leq q}$ is a local orthonormal frame of $\Gamma(Q)$. Let $X\in\Gamma(TM)$ and $s\in\Gamma(Q)$, the classical divergence operator with respect to the connection ∇^M is defined by $divX=\sum\limits_{i=1}^{n}g(\nabla_{E_i}X,E_i)$. Similarly, the *transverse divergence* operator div_∇ with respect to ∇ is defined by

$$div_{\nabla} s = \sum_{i=1}^{q} g(\nabla_{\sigma(e_i)} s, e_i).$$

The tension field τ of the foliation $\mathcal{F}$ is defined by

$$\tau = -\pi(\sum_{i=1}^{p} \nabla_{E_i} E_i).$$

It's easily seen that the following equation

$$div\sigma(s) = div_{\nabla} s - g_Q(\tau, s). \tag{4.1}$$

holds for all $s \in \Gamma(Q)$.

Proposition 6. *Let $X \in \mathcal{V}(\mathcal{F})$ and $\nu = \pi(X)$, then $div_{\nabla}\nu$ is a basic function. The following equation*

$$div_{\nabla}(div_{\nabla}\nu.\nu) = (div_{\nabla}\nu)^2 + \nabla_X div_{\nabla}\nu. \tag{4.2}$$

holds.

Proof. i) Let $Y \in \Gamma(E)$, first we note the following two points,

1) since $\nabla_Y \nu = \pi([Y, X]) = 0$ and $i_Y R_{\nabla} = 0$, for all $1 \leq i \leq q$, we have

$$\nabla_Y \nabla_{\sigma(e_i)} \nu = \nabla_{[Y, \sigma(e_i)]} \nu,$$

2) Since $\nabla_Y e_i = \sum_{j=1}^{q} g_Q(\nabla_Y e_i, e_j) e_j = -\sum_{j=1}^{q} g_Q(e_i, \nabla_Y e_j) e_j$ for all $1 \leq i \leq q$, hence

$$\begin{aligned} \sum_{i=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nu, \nabla_Y e_i) &= \sum_{i,j=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nu, e_j) g_Q(\nabla_Y e_i, e_j) \\ &= -\sum_{j=1}^{q} g_Q(\nabla_{\sigma(\nabla_Y e_j)}\nu, e_j). \end{aligned}$$

Consequently

$$\begin{aligned} \nabla_Y div_\nabla \nu &= \sum_{i=1}^{q} g_Q(\nabla_Y \nabla_{\sigma(e_i)}\nu, e_i) + \sum_{i=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nu, \nabla_Y e_i) \\ &= \sum_{j=1}^{q} g_Q(\nabla_{[Y,\sigma(e_j)]}\nu, e_j) - \sum_{j=1}^{q} g_Q(\nabla_{\sigma(\nabla_Y e_j)}\nu, e_j) = 0, \end{aligned}$$

because $([Y, \sigma(e_j)] - \sigma(\nabla_Y e_j) \in \Gamma(E)$. Consequently, the transverse divergence is a basic function.

ii) Let f be a basic function, the relation (4.2) follows from

$$div_\nabla f\nu = X.f + f div_\nabla \nu.$$

□

Definition 1. *The foliation $\mathcal{F}$ is harmonic or minimal if all the leaves of $\mathcal{F}$ are minimal submanifolds.*

Proposition 7. [8] *The foliation $\mathcal{F}$ is harmonic if and only if $\tau = 0$.*

4.2. Some integral formulas

First, we give some notations that are needed in the sequel. Let $X \in \mathcal{V}(\mathcal{F})$ and put $\nu = \pi(X)$,

$$tr\nabla^2 = \sum_{i=1}^{q} \nabla^2_{\sigma(e_i),\sigma(e_i)}, \tag{4.3}$$

$$|A_X|^2 = \sum_{i=1}^{q} g_Q(A_X(e_i), A_X(e_i)) = \sum_{i=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nu, \nabla_{\sigma(e_i)}\nu), \tag{4.4}$$

$$trA_X^2 = \sum_{i=1}^{q} g_Q(A_X^2(e_i), e_i), \tag{4.5}$$

and

$$|\Theta(X)g_Q|^2 = \sum_{1=i,j}^{q} (\Theta(X)g_Q(e_i, e_j))^2. \tag{4.6}$$

Proposition 8. *Let $X \in \mathcal{V}(\mathcal{F})$, $\nu = \pi(X)$ and $\xi \in \Gamma(Q)$ such that*

$$g_Q(\xi, s) = g_Q(\nabla_{\sigma(s)}\nu, \nu)$$

for all $s \in \Gamma(Q)$. Then

(i) ξ is parallel along the leaves, and

(ii) the relations

$$g_Q(tr\nabla^2\nu, \nu) = div_\nabla\xi - |A_X|^2 \tag{4.7}$$

$$|\Theta(X)g_Q|^2 = 2|A_X|^2 + 2trA_X^2. \tag{4.8}$$

hold.

Proof. i) Let $Y \in \Gamma(E)$ and $s \in \Gamma(Q)$. Since $\nabla_Y\nu = 0$ and $i_Y R_\nabla = 0$, we get

$$\begin{aligned} g_Q(\nabla_Y\xi, s) &= \nabla_Y g_Q(\nabla_{\sigma(s)}, \nu) - g_Q(\xi, \nabla_Y s) \\ &= g_Q(\nabla_Y\nabla_{\sigma(s)}\nu, \nu) + g_Q(\nabla_{\sigma(s)}, \nabla_Y\nu) - g_Q(\nabla_{[Y,\sigma(s)]}\nu, \nu) \\ &= g_Q(R_\nabla(Y, \sigma(s))\nu, \nu) = 0. \end{aligned}$$

ii) Now we prove the relation (4.7). Indeed,

$$\begin{aligned} g_Q(tr\nabla^2\nu, \nu) &= \sum_{i=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nabla_{\sigma(e_i)}\nu, \nu) - \sum_{i=1}^{q} g_Q(\nabla_{\sigma(\nabla_{\sigma(e_i)}e_i)}\nu, \nu) \\ &= -\sum_{i=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nu, \nabla_{\sigma(e_i)}\nu) + \sum_{i=1}^{q} \nabla_{\sigma(e_i)} g_Q(\nabla_{\sigma(e_i)}\nu, \nu) \\ &- \sum_{i=1}^{q} g_Q(\nabla_{\sigma(\nabla_{\sigma(e_i)}e_i)}\nu, \nu) \\ &= -|A_X|^2 + \sum_{i=1}^{q} \nabla_{\sigma(e_i)} g_Q(\xi, e_i) - \sum_{i=1}^{q} g_Q(\xi, \nabla_{\sigma(e_i)}e_i) \\ &= -|A_X|^2 + div_\nabla\xi. \end{aligned}$$

Prove the second relation (4.8). If $1 \le i \le q$ then

$$A_X(e_i) = \sum_{j=1}^{q} g_Q(A_X(e_i), e_j)e_j.$$

Therefore the relations (4.4) and (4.5) can be reformulated as follows

$$|A_X|^2 = \sum_{i,j=1}^{q} (g_Q(A_X(e_i), e_j))^2, \quad trA_X^2 = \sum_{i,j=1}^{q} g_Q(A_X(e_i), e_j) g_Q(A_X(e_j), e_i).$$

On the other hand, for $1 \leq i, j \leq q$, we have

$$(\Theta(X)g_Q)(e_i, e_j) = g_Q(A_X(e_i), e_j) + g_Q(e_i, A_X(e_j)).$$

So the relation (4.6) becomes

$$\begin{aligned} |\Theta(X)g_Q|^2 &= 2\sum_{i,j=1}^{q} (g_Q(A_X(e_i), e_j))^2 \\ &+ 2\sum_{i,j=1}^{q} g_Q(A_X(e_i), e_j) g_Q(A_X(e_j), e_i) \\ &= 2|A_X|^2 + 2trA_X^2 \end{aligned}$$

and we are done. □

Let $s, t \in \Gamma(Q)$. The Ricci curvature Ric with respect to ∇ is the symmetric bilinear form on Q given by

$$Ric(s,t) = \sum_{1 \leq i \leq q} g_Q(R(s, e_i)e_i, t) = \sum_{1 \leq i \leq q} g_Q(R(e_i, s)t, e_i).$$

As R and g_Q are holonomy, Ric is also holonomy invariant.

Proposition 9. *Let $X \in \mathcal{V}(\mathcal{F})$ and $\nu = \pi(X)$. The equations*

$$Ric(\nu, \nu) + trA_X^2 + \nabla_X div_\nabla \nu = div_\nabla \nabla_X \nu, \tag{4.9}$$

and

$$Ric(\nu, \nu) + trA_X^2 - (div_\nabla \nu)^2 = div_\nabla (\nabla_X \nu - (div_\nabla \nu)\nu). \tag{4.10}$$

hold.

Proof. i) We prove the relation (4.9). First we notice that

$$A_X^2(s) = -A_X(\nabla_{\sigma(s)}\nu) = \nabla_{\sigma(\nabla_{\sigma(s)}\nu)}\nu \quad \text{for all } s \in \Gamma\,(Q)$$

So $trA_X^2 = \sum_{i=1}^{q} g_Q(\nabla_{\sigma(\nabla_{\sigma(e_i)}\nu)}\nu, e_i)$. Now, set

$$P = \sum_{i=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nu, \nabla_X e_i) \text{ and } S = \sum_{j=1}^{q} g_Q(\nabla_\sigma(\nabla_X e_j)\nu, e_j).$$

Observe that

$$P = \sum_{i,j=1}^{q} g_Q(\nabla_{\sigma(e_i)}\nu, e_j) g_Q(\nabla_X e_i, ej)$$

$$S = \sum_{i,j=1}^{q} g_Q(\nabla_X e_j, e_i) g_Q(\nabla_{\sigma(e_i)}\nu, e_j).$$

We derive that $Ric(\nu,\nu) = B + C$, where

$$B = \sum_{i=1}^{q} g_Q(\nabla^2_{\sigma(e_i),X}\nu, e_i) = div\nabla_X\nu - trA_X^2,$$

and

$$\begin{aligned} C &= \sum_{i=1}^{q} g_Q(\nabla^2_{X,\sigma(e_i)}\nu, e_i) \\ &= \sum_{i=1}^{q} \nabla_X g_Q(\nabla_{\sigma(e_i)}\nu, e_i) - (P+S) \\ &= \nabla_X div\nu - (P+S), \end{aligned}$$

But $P + S = 0$ because $g_Q(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) = 0$.

ii) The formula (4.9) together with the identity (4.2) leads straightforwardly to the relation (4.10). $\square$

Proposition 10. *Let $\mathcal{F}$ be an harmonic g_M-Riemannian foliation on a closed Riemannian manifold (M, g_M), and let X be an infinitesimal automorphism of $\mathcal{F}$. Then the following integral formulas*

$$\int_M (Ric(\nu,\nu) + trA_X^2 - (div_\nabla\nu)^2) d_M = 0 \tag{4.11}$$

and

$$\int_M (Ric(\nu,\nu) + g_Q(tr\nabla^2\nu,\nu) + \frac{1}{2}|\Theta(X)g_Q|^2 - (div_\nabla\nu)^2) d_M = 0 \tag{4.12}$$

hold, where $\nu = \pi(X)$ and d_M is a volume form of M.

Proof. i) Since $\mathcal{F}$ is harmonic, we get

$$div\sigma(\nabla_X \nu - (div_\nabla \nu)\nu) = div_\nabla(\nabla_X \nu - (div_\nabla \nu)\nu).$$

So, the first integral formula follows from the relation (4.10) and the Green theorem.

ii) According to (4.7) and (4.8), we obtain

$$tr A_X^2 = g_Q(tr\nabla^2 \nu, \nu) + \frac{1}{2}|\Theta(X)g_Q|^2 - div_\nabla \xi.$$

Using once more the relation (4.10) and the Green theorem, we obtain the second integral formula. □

4.3. Transverse affine vector field case

In [9] the authors show that on a Riemannian manifold endowed with a g_M-Riemannian foliation any transverse Killing field is transverse affine. In this section we show the converse.

Proposition 11. *Let $X \in \mathcal{V}(\mathcal{F})$. If $\nu = \pi(X)$ is a transverse affine vector field, then the following relations*

$$Ric(\nu, \nu) + tr A_X^2 = div_\nabla \nabla_X \nu \tag{4.13}$$

and

$$Ric(\nu, \nu) + g_Q(tr\nabla^2 \nu, \nu) = 0 \tag{4.14}$$

occur.

Proof. i) Let $s \in \Gamma(Q)$. From (3.4) and (3.5) it follows

$$\begin{aligned} A_X^2 s &= -A_X \circ \nabla_{\sigma(s)} \nu \\ &= -\nabla_{\sigma(s)} \circ A_X \nu + (\nabla_{\sigma(s)} A_X)\nu \\ &= \nabla_{\sigma(s)} \nabla_X s + R(\nu, s)\nu. \end{aligned}$$

Therefore,

$$tr A_X^2 = \sum_{i=1}^{q} g_Q(A_X^2(e_i), e_i) = div_\nabla(\nabla_X \nu) - Ric(\nu, \nu).$$

ii) In view of (3.14) and (3.5), we may write

$$R(\nu, s)t + \nabla^2_{\sigma(s),\sigma(t)}\nu = R(\nu, s)t - \nabla_{\sigma(s)}(A_X)t = K(s)t = 0.$$

Thus,

$$\sum_{i=1}^{q} g_Q((R(\nu, e_i)e_i + \nabla^2_{\sigma(e_i),\sigma(e_i)}\nu, \nu) = Ric(\nu, \nu) + g_Q(tr\nabla^2\nu, \nu) = 0$$

and the proof is complete. □

Now we arrive to the main result of this section

Theorem 3. Let $\mathcal{F}$ be an harmonic g_M-Riemannian foliation on a closed Riemannian manifold (M, g_M), and let X be an infinitesimal automorphism of $\mathcal{F}$. If $\pi(X)$ is transverse affine then $\pi(X)$ is transverse Killing.

Proof. From the first integral formula (4.11) and the relation (4.13) it follows that $\int_M (div_\nabla \nu)^2 d_M = 0$. Hence $div_\nabla \nu = 0$. Now by the second integral formula (4.12) and the relation (4.14), we obtain $\int_M |\Theta(X)g_Q|^2 d_M = 0$. Whence $\Theta(X)g_Q = 0$ and $\pi(X)$ is thus transverse Killing. □

We end this section by the following result

Proposition 12. *Let $\mathcal{F}$ be a g_M-Riemannian foliation on a closed Riemannian manifold (M, g_M), and let X be an infinitesimal automorphism of $\mathcal{F}$. If $\pi(X)$ is transverse homothetic, then $\pi(X)$ is transverse affine.*

Proof. The local flow φ_t generated by X maps leaves into leaves. Let Φ_t be the induced flow on TM. Then Φ_t sends E to itself, and thus induces a local flow $\widetilde{\Phi}_t$ of bundle maps of Q over φ_t, i.e., making the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\Phi_t} & Q \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi_t} & M \end{array}$$

commutative. If $\pi(X)$ is transverse homothetic, then $\widetilde{\Phi}_t^* g_Q = ae^{tc} g_Q$ for all t and for some constants $a > 0$ and $c \in \mathbf{R}$. By the uniqueness theorem for the metric and torsion free connection of a Riemannian foliation [11], the connection associated to g_Q and $ae^{tc} g_Q$ are the same for all t. This proves that X preserves the connection. □

5. Preliminaries Theorems

Theorem 4. Let $\mathcal{F}$ be an irreductible g_M-Riemannian foliation of codimension ≥ 3 on a Riemannian manifold, and let X be an infinitesimal automorphism of $\mathcal{F}$ such that $\pi(X)$ is a transverse conformal vector field and $\Theta(X)R = 0$. Then $\pi(X)$ is transverse homothetic.

Proof. There exists a basic function f on M such that $\Theta(X)g_Q = f.g_Q$. Let $\omega = df \circ \sigma$. We claim that ω is parallel, i.e. $\nabla\omega = 0$. To this end, we argue in two steps.

1) The form ω is parallel along the leaves, that is $\nabla_Y\omega = 0$ for all $Y \in \Gamma(\mathcal{F})$. Indeed, if $s \in \Gamma(Q)$ then

$$\begin{aligned}(\nabla_Y\omega)(s) &= Y(\sigma(s)f) - \omega(\nabla_Y s) = Y(\sigma(s)f) - df \circ \sigma(\pi[Y, \sigma(s)]) \\ &= Y(\sigma(s)f) - [Y, \sigma(s)]f = Y(\sigma(s)f) - Y(\sigma(s)f) = 0,\end{aligned}$$

where we use the fact that f is basic.

2) We prove now that $\nabla_Y\omega = 0$ for all $Y \in \Gamma(E^\perp)$. Let $s, t \in \Gamma(Q)$ and observe that $\Theta(X)R = 0$ and (3.11) yield that $(\nabla_{\sigma(s)}K)(t) = (\nabla_{\sigma(t)}K)(s)$. Moreover, (3.12) implies that $(\nabla_{\sigma(s)}\omega)(t) = (\nabla_{\sigma(t)}\omega)(s)$. So, it is sufficient to prove that $(\nabla_{\sigma(s)}\omega)(s) = 0$ for $s \in \Gamma(Q)$. Let $(e_i)_{1\leq i\leq q}$ be a local orthonormal frame of Q via the metric g_Q and let $i \neq j$. From (3.12), it follows that

$$((\nabla_{\sigma(e_j)}K)(e_i)g_Q)(e_i, e_j) = (\nabla_{\sigma(e_j)}\omega)(e_i).g_Q(e_i, e_j) = 0. \tag{5.1}$$

Using (3.10), (3.11) and (5.1), we obtain

$$\begin{aligned}(\nabla_{\sigma(e_i)}\omega)(e_i) &= (\nabla_{\sigma(e_i)}\omega)(e_i).g_Q(e_j, e_j) = ((\nabla_{\sigma(e_i)}K)(e_i).g_Q)(e_j, e_j) \\ &= -2g_Q((\nabla_{\sigma(e_i)}K)(e_i)(e_j), e_j) = -2g_Q((\nabla_{\sigma(e_i)}K)(e_j)(e_i), e_j) \\ &= -2g_Q((\nabla_{\sigma(e_j)}K)(e_i)(e_i), e_j) = 2g_Q(e_i, \nabla_{\sigma(e_j)}K(e_i)(e_j)) \\ &= 2g_Q(e_i, \nabla_{\sigma(e_j)}K(e_j)(e_i)) = -(\nabla_{\sigma(e_j)}\omega)(e_j).\end{aligned} \tag{5.2}$$

Since $q \geq 3$, we get $\nabla_{\sigma(e_i)}\omega(e_i) = 0$ for all i. Now, pick an arbitrary vector e_i and show that $(\nabla_{\sigma(e_i)}\omega)(e_j) = 0$ for all $j = 1, ..., q$. Let $0 \leq j \leq q$ and construct a new local orthonormal frame $(f_k)_{1\leq i\leq q}$ by putting

i) $f_k = e_k$ for $k \neq i, j$,

ii)$f_i = \frac{e_i+e_j}{\sqrt{2}}$,

ii) $f_j = \frac{e_i-e_j}{\sqrt{2}}$.

By similar computations as previously used in (5.2), we have $(\nabla_{\sigma(f_k)}\omega)(f_k) = 0$ for all $k = 1, ..., q$. In particular $(\nabla_{\sigma(f_i)}\omega)(f_i) = 0$, so $(\nabla_{\sigma(e_i)}\omega)(e_j) = 0$, as required.

Finally, since $\nabla\omega = 0$, $Ker\omega$ is a non trivial subbundle of Q which is invariant by the connection holonomy group. Consequently, $\omega = 0$ because $\mathcal{F}$ is irreducible. Thus f is a constant and the proof is finished. □

Now if $\mathcal{F}$ is a codimension two foliation, then $Ric = \lambda g_Q$ where λ is a basic function and we have $Ric = 0$ if and only if $R = 0$. Moreover, if $\mathcal{F}$ is irreducible, then ∇ is not flat and so λ is not identically zero.

Theorem 5. Let $\mathcal{F}$ be an irreducible analytic g_M-Riemannian foliation of codimension 2 on a Riemannian manifold, and let X be an infinitesimal automorphism of $\mathcal{F}$ such that

$$\Theta(X)Ric = 0 \quad \text{and} \quad \Theta(X)(\nabla Ric) = 0. \tag{5.3}$$

Then $\pi(X)$ is transverse homothetic.

Proof. Let $s \in \Gamma(Q)$. The relation (3.13) and the hypothesis (5.3) yields that

$$K(s)(Ric) = \Theta(X)(\nabla Ric)(s) - \nabla_{\sigma(s)}(\Theta(X)Ric) = 0.$$

Since $Ric = \lambda.g_Q$, where λ is a basic function which is not identically zero, we get

$$0 = K(s)(\lambda.g_Q) = \lambda.K(s)g_Q) = \lambda.\nabla_Y\Theta(X)g_Q. \tag{5.4}$$

But λ is analytic in the normal coordinate because $\mathcal{F}$ is analytic. So the zero leaves of λ are isolated in the set of the leaves of $\mathcal{F}$. It follows from (5.4) that $\nabla(\Theta(X)g_Q) = 0$. As $\mathcal{F}$ is irreducible and the symmetric tensor $\Theta(X)g_Q$ is invariant by the connection holonomy group, [7] Appendix 5 leads to $\Theta(X)g_Q = c.g_Q$, where c is a real constant. □

Theorem 6. Let $\mathcal{F}$ be an irreducible analytic g_M- Riemannian foliation of codimension ≥ 2 on an analytic Riemannian manifold, and let X be an infinitesimal automorphism of $\mathcal{F}$ such that $\Theta(X)\nabla^m R = 0$, for $m \in \mathbf{N}$. Then $\pi(X)$ is transverse homothetic.

Proof. If $\mathcal{F}$ is a codimension two foliation, then the result follows from theorem 5. We assume therefore $\mathcal{F}$ to be a codimension $q \geq 3$ foliation.
Let $\pi : O(Q) \longrightarrow M$ be the orthonormal frame bundle of Q, a principal $O(q)$-bundle. Let Γ be the connection in $\mathcal{O}(Q)$ corresponding to ∇. Since Γ is a real analytic connection in the real analytic principal fiber-bundle $\mathcal{O}(Q)$, the holonomy algebra $\mathcal{G}_x$ (the Lie algebra of $\Psi^0(x)$) is generated by all endomorphisms of the form

$$R(s_1, s_2); (\nabla R)(s_1, s_2, s_3); .....; (\nabla^m R)(s_1, s_2, ..., s_{m+2}) \qquad \text{where } s_i \in Q_x.$$

We have $(A_X)_x$ at a point $x \in M$ belongs to the normalizor $\mathcal{N}(\mathcal{G}_x)$. Indeed, from the assumption $\Theta(X)(\nabla^m R) = 0$ it follows that

$$\begin{aligned}
[A_X, (\nabla^m R)(s_1, s_2, ..., s_{m+2})] &= A_X \circ (\nabla^m R)(s_1, s_2, ..., s_{m+2}) \\
&- (\nabla^m R)(s_1, s_2, ..., s_{m+2}) \circ A_X \\
&= (A_X(\nabla^m R))(s_1, s_2, ..., s_{m+2}) \\
&+ \sum_{i=1}^{m+2} (\nabla^m R)(s_1, ..., A_X(s_i), ..., s_{m+2}) \\
&= -(\nabla^{m+1} R)(\pi(X), s_1, s_2, ..., s_{m+2}) \\
&+ \sum_{i=1}^{m+2} (\nabla^m R)(s_1, .., A_X(s_i), ..., s_{m+2})
\end{aligned}$$

Hence

$$[(A_X)_x, B] \in \mathcal{G}_x, \qquad \text{for} \quad B \in \mathcal{G}_x. \tag{5.5}$$

As ∇ is a metric connection with respect to g_Q, we get

$$B g_Q = [A_X, B].g_Q = 0, \qquad \text{for } B \in \mathcal{G}_x,$$

We derive that $B.\Theta(X).g_Q = 0$. But $\mathcal{G}_x$ is irreductible and then $\Theta(X).g_Q$ is a scalar multiple of the tensor $(g_Q)_x$ at x. This happens actually at every point x of M, so we have $\Theta(X)g_Q = f.g_Q$ where f is a function. We claim that f is basic. To this end, let $Y \in \Gamma(E)$ and observe that

$$\begin{aligned}
(Y.f).g_Q &= \nabla_Y(f.g_Q) = \Theta(Y)\Theta(X)g_Q = [\Theta(Y), \Theta(X)]g_Q \\
&= \Theta([Y, X])g_Q = 0
\end{aligned}$$

because $X \in V(\mathcal{F})$ and g_Q is holonomy invariant. This means that $\pi(X)$ is transverse conformal. Theorem 4 implies that $\pi(X)$ is transverse homothetic. □

6. Proofs of the main Theorems

Obviously, Theorem 1 follows directly from Theorems 6,3 and Proposition 12. The proof of Theorem 2 is now in order.

Proof. Let $x \in M$. In view of [3], we have the direct sum

$$Q_x = (Q_0)_x \oplus (Q_1)_x \oplus ... \oplus (Q_k)_x$$

of mutually orthogonal subspaces invariant under $\Psi(x)$, where $(Q_0)_x$ is the set of vectors in Q_x which are fixed by $\Psi(x)$ and where $(Q_1)_x, ..., (Q_k)_x$ are all irreducible. Since $\mathcal{F}$ is without Euclidean part, $dim(Q_0)_x = 0$. For each $i = 1, .., k$, let $\mathcal{F}_i$ be the foliation of M which is integral to the distribution

$$L_i = E \oplus \sigma(Q_1) \oplus ... \oplus \widehat{\sigma(Q_i)} \oplus ... \oplus \sigma(Q_k),$$

where $\widehat{\sigma(Q_i)}$ indicates that $\sigma(Q_i)$ is omitted. Each $\mathcal{F}_i$ is an irreducible g_M-Riemannian foliation. Indeed, in [3] the authors show that the metric g_Q is a direct sum $g_Q = \bigoplus\limits_{1 \leq i \leq k} g_{Q_i}$ and the restriction of the connection ∇ to each Q_i is the unique torsion free metric connection associated to g_{Q_i}. In other words, for $i = 1, ..., k$ the metric g_{Q_i} is holonomy invariant with respect to the foliation $\mathcal{F}_i$. Now we show that for $i = 1, ..., k$, the vector field X is an infinitesimal automorphism of the foliation $\mathcal{F}_i$, that is $X \in \mathcal{V}(\mathcal{F}_i)$. Let $x \in M$. We know by (5.5) that the endomorphism $(A_X)_x$ lies in the normalizor of the holonomy algebra $\mathcal{G}_x$. Thus the 1-parameter group of linear transformation $\exp tA_X$ of Q_x lies in the normalizor of the holonomy group $\Psi(x)$. By virtue of the uniqueness of the decomposition

$$Q_x = (Q_1)_x + ... + (Q_k)_x,$$

it follows that, for each t and $1 \leq j \leq k$, $(\exp tA_X)(Q_i)_x$ coincides with some $(Q_l)_x$. By continuity, we see that

$$(\exp tA_X)(Q_i)_x = (Q_i)_x \qquad \text{for every } t.$$

This implies that $A_X(Q_j)_x \subset (Q_j)_x$. But Q_j is invariant under the parallel transport, that is $\nabla_X(Q_j) \subset Q_j$. Then (3.3) leads to

$$\Theta(X)(Q_j) \subset Q_j \quad \text{and hence} \quad \Theta(X)(L_i) \subset L_i.$$

Now let ∇_i be the adapted connection to respect the Riemannian foliation $\mathcal{F}_i$. The tensor curvature R_{∇_i} of ∇_i induces a Q_i-tensor field R_i.

Furthermore, it is easy to see that ∇_i (resp. R_i) coincides with the restriction of ∇ (resp. R) to Q_i. Thus $\Theta(X)\nabla_i^m R_i = \Theta(X)\nabla^m R = 0$. By theorem 6, we deduce that $\pi(X)$ is transverse homothetic with respect to the foliation $\mathcal{F}_i$. So $\pi(X)$ is transverse almost homothetic with respect to the foliation $\mathcal{F}$. This completes the proof. $\square$

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Mohamed Ali Chaouch
Faculté des Sciences de Bizerte
Université du 7 Novembre Á Carthage
7021-Zarzouna.
Bizerte -Tunisie.
e-mail : MohamedAli.Chaouch@fsb.rnu.tn

and

Nabila Torki-Hamza
Faculté des Sciences de Bizerte
Université du 7 Novembre Á Carthage
7021-Zarzouna.
Bizerte -Tunisie.
e-mail : Nabila.Torki-Hamza@fsb.rnu.tn