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# PERIODIC SOLUTIONS IN THE SINGULAR LOGARITHMIC POTENTIAL 

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#### Abstract

We consider the singular logarithmic potential $U=\ln \sqrt{x^{2} / b^{2}+y^{2}}$, a potential which plays an important role in the modelling of triaxial systems, such as elliptical galaxies or bars in the centres of galaxy discs. Using properties of the central field in the axis-symmetric case we obtain periodic solutions which are symmetric with respect to the origin for weak anisotropies. Also we generalize our result in order to include more general perturbations of the logarithmic potential.


Mathematics Subject Classification : 34C25, 34 C14.

Key words : Periodic solutions, logarithmic potential, symmetry.

## 1. Introduction

The study of the orbital structure of the logarithmic potential was initially motivated by the need to construct self-consistent models of galaxies (see [13] or [12] and references therein). This potential (in the plane) is given by $U=\ln \sqrt{\frac{x^{2}}{b^{2}}+y^{2}}$. If $b=1$ it means that the galaxy is axis-symmetric.

The main motivations of our work arose first by the following affirmation made by the authors of [12]: "the analytical description in general nonaxis symmetric case remains still open, several problems requires further investigation", and second because this problem plays an important role in the modelling of elliptical galaxies or bars in the centres of galaxy discs. In order to give new information about the dynamic of this problem, this paper is devoted to obtain periodic solutions in the case of weak anisotropy.

The non-axis symmetric logarithmic problem is a one parameter Hamiltonian system with two degrees of freedom whose anisotropic singular potential is given by

$$
\begin{equation*}
U=U(x, y ; b)=\ln \sqrt{\frac{x^{2}}{b^{2}}+y^{2}} \tag{1.1}
\end{equation*}
$$

so that the equation of motion can be expressed as

$$
\begin{align*}
& \ddot{x}=-\frac{x}{x^{2}+b^{2} y^{2}}, \\
& \ddot{y}=-\frac{b^{2} y}{x^{2}+b^{2} y^{2}} . \tag{1.2}
\end{align*}
$$

Then it determines a conservative system with Hamiltonian given by

$$
\begin{equation*}
H=H\left(x, x, p_{x}, p_{y}\right)=\frac{1}{2}\left(p_{x}^{2}+p_{y}^{2}\right)+\ln \sqrt{\frac{x^{2}}{b^{2}}+y^{2}}, \tag{1.3}
\end{equation*}
$$

where $\mathbf{q}=(x, y) \in \mathbb{R}^{2} \backslash\{(0,0)\}$ (position) are the generalized coordinates and $\mathbf{p}=\left(p_{x}, p_{y}\right) \in \mathbb{R}^{2}$ are the momenta (velocity). The Hamiltonian system associated to (1.3) is given by

$$
\begin{array}{ll}
\dot{x}=p_{x}, & \dot{p}_{x}=-\frac{\partial U}{\partial x} \\
\dot{y}=p_{y}, & \dot{p}_{y}=-\frac{\partial U}{\partial y} . \tag{1.4}
\end{array}
$$

Now consider weak anisotropies, i.e., choose the parameter $b$ close to 1. Introducing the notation $\epsilon=b-1$ with $\epsilon \ll 1$ we can expand the Hamiltonian (1.3) in powers of $\epsilon$ and obtain

$$
\begin{align*}
H=H\left(x, y, p_{x}, p_{y}\right)= & \frac{1}{2}\left(p_{x}^{2}+p_{y}^{2}\right)+\frac{1}{2} \ln \left(x^{2}+y^{2}\right)-\frac{x^{2}}{x^{2}+y^{2}} \epsilon+\frac{x^{2} y^{2}}{\left(x^{2}+y^{2}\right)^{2}} \epsilon^{2}+\cdots \\
& \equiv H_{0}+\epsilon h(x, y), \tag{1.5}
\end{align*}
$$

where

$$
\begin{equation*}
H_{0}=H_{0}(x, y)=\frac{1}{2}\left(p_{x}^{2}+p_{y}^{2}\right)+\frac{1}{2} \ln \left(x^{2}+y^{2}\right) \tag{1.6}
\end{equation*}
$$

is the Hamiltonian associated to the axis-symmetric case $(b=1)$ and

$$
h(x, y)= \begin{cases}\frac{H\left(x, y, p_{x}, p_{y}\right)-H_{0}}{\epsilon}, & \text { if } \epsilon \neq 0,  \tag{1.7}\\ 0, & \text { if } \epsilon=0\end{cases}
$$

which is analytic in a neighborhood of $\epsilon=0$ for $x^{2}+y^{2}>0$ and it is invariant under reflections on the $x$-axis and $y$-axis. We will call the system associated to $H_{0}$, which is given by

$$
\begin{align*}
& \ddot{x}=-\frac{x}{x^{2}+y^{2}}, \\
& \ddot{y}=-\frac{y}{x^{2}+y^{2}}, \tag{1.8}
\end{align*}
$$

as the unperturbed system associated to (2).
In this paper we study the problem of existence of periodic solutions of the Hamiltonian system associated to $H$ defined in (3) for weak anisotropies, i.e., $\epsilon \ll 1$. More specifically, using properties of the central field in the axis-symmetric case (i.e., $b=1$ ) and perturbation theory we obtain periodic solution which are symmetric with respect to the origin, also for weak anisotropies (this is achieved in Theorem 1). Some references concerning the work on the anisotropic Kepler problem and Manev problem are [9], [14] and [11], [15] respectively.

The paper is organized as follows. In section 2 we study the symmetries of the singular logarithmic potential. In section 3, we prove Theorem 1. The key of the proof is to analyze properties of the solutions of the central field generated by the logarithmic potential which are sufficiently close to the circular orbit. To conclude the proof of the theorem we need to use the result which affirms that the solutions of the differential equations depend uniformly and analytically on all the data of the problem, and also the symmetries of the problem under reflections with respect to the $x$ and $y$ axis. This kind of argument was first used in [2] or [3], when the perturbation
in the logarithmic case is only invariant under reflections with respect to the $x$-axis. In our work the periodic solutions obtained are more restrictive because they present two symmetries, namely, reflection with respect the $x$ and $y$ axes. On the other hand, the author in [15] proved the existence of periodic solutions with fixed period using an interesting adaptation of the Poincaré continuation method. Also, we generalize our result in order to include more general perturbations of the logarithmic potential. We emphasize that the periodic solutions obtained (by different methods) in [15] and Theorem 1 for each $|\epsilon|<\epsilon_{0}$ fixed, are not the same, because they are obtained by different initial conditions.

## 2. Symmetries of the logarithmic potential

To find periodic solution orbits in the singular logarithmic potential it is important to know the symmetries of the system (1.2). Let

$$
\begin{equation*}
X=X\left(x, y, p_{x}, p_{y}\right)=\left(p_{x}, p_{y},-\frac{x}{x^{2}+b^{2} y^{2}},-\frac{y}{x^{2}+b^{2} y^{2}}\right), \tag{2.1}
\end{equation*}
$$

be the vector field associated to the Hamiltonian system (1.4). In order to make clear our presentation, we will remember some important definitions and preliminary results adapted to our problem.

Definition 1. Let $\Psi: \mathbb{R}^{4} \rightarrow \mathbb{R}^{4}$ be a linear application. It is said that the vector field $X$ in (2.1) is $\Psi$-invariant (or $\Psi$ is a symmetry for $X$ ) if $X \circ \Psi=\Psi \circ X$.

Lemma 1. Assume that the system (1.4) admit a $\Psi$-symmetry (as in Definition 1). If $\varphi(t)=\varphi(t, \mathbf{q}, \mathbf{p})$ is a solution of (1.4) then $\psi(t)=\Psi \circ \varphi(t)$ is also a solution of (1.4).

Proof: Derivating with respect to $t$ and since $\Psi$ is linear, we have that

$$
\frac{d \psi(t)}{d t}=\Psi \frac{d \varphi(t)}{d t}=\Psi \circ X(\varphi(t))=X \circ \Psi(\varphi(t))=X(\psi(t)) .
$$

Since the equations of motion (1.2) are invariant under reflections with respect to the $x$ and $y$ axes, we have the following result

Proposition 1. The following linear applications:

$$
\begin{align*}
S_{1}: & \left(x, y, p_{x}, p_{y}\right) \longrightarrow\left(x,-y, p_{x},-p_{y}\right) \\
S_{2}: & \left(x, y, p_{x}, p_{y}\right) \longrightarrow\left(-x, y,-p_{x}, p_{y}\right)  \tag{2.2}\\
S_{3}: & \left(x, y, p_{x}, p_{y}\right) \longrightarrow\left(-x,-y,-p_{x},-p_{y}\right)
\end{align*}
$$

are $S_{i}$-invariant ( $i=1,2,3$ ) for the vector field (1.4).
Another important kind of discrete symmetries are the called reversible, which we define next.

Definition 2. It is said that the system (1.4) is reversible (or $\Phi$-reversible) if there is a linear invertible transformation $\Phi: \mathbb{R}^{4} \rightarrow \mathbb{R}^{4}$ such that $X \circ \Psi=$ $-\Psi \circ X$.

In this case we have a direct and important consequence

Lemma 2. Assume that the system (1.4) admits a $\Phi$-reversible symmetry (as in Definition 2). If $\varphi(t)=\varphi(t, \mathbf{q}, \mathbf{p}))$ is a solution of (1.4), then $\phi(t)=$ $\Phi \circ \varphi(-t)$ is also a solution of (1.4).

Proof: It is a direct consequence from the above definition.
Since our system (1.2) is a mechanical system, we can prove easily the following proposition.

Proposition 2. The following linear applications:

$$
\begin{array}{ll}
S_{0}: & \left(x, y, p_{x}, p_{y}\right) \longrightarrow\left(x, y,-p_{x},-p_{y}\right) \\
S_{4}: & \left(x, y, p_{x}, p_{y}\right) \longrightarrow\left(-x, y, p_{x},-p_{y}\right) \\
S_{5}: & \left(x, y, p_{x}, p_{y}\right) \longrightarrow\left(x,-y,-p_{x}, p_{y}\right)  \tag{2.3}\\
S_{6}: & \left(x, y, p_{x}, p_{y}\right) \longrightarrow\left(-x,-y, p_{x}, p_{y}\right),
\end{array}
$$

are $S_{i}$-reversible ( $i=0,4,5,6$ ) for the vector field (1.4).

The symmetry $S_{0}$ is the usual symmetry with respect to the zero velocity curve (i.e., the zero velocity curve is defined by the points $(\mathbf{q}, \mathbf{p})=(\mathbf{q}, \mathbf{0}) \in$ $\mathbb{R}^{4}$ such that $\left.H(\mathbf{q}, \mathbf{p})=h\right)$, which is presented in all the Hamiltonian systems where the Hamiltonian function can be written as kinetic energy plus potential energy.

Using the previous notation, we define:

Definition 3. For $i \in\{0,1,2,3,4,5,6\}$ the solution $\varphi(t)$ of (1.4) is called $S_{i}$-symmetric if and only if $S_{i}(\varphi(t))=\varphi(t)$.

The previous symmetries in Proposition 1 and Proposition 2 are very useful to find symmetric periodic orbits, especially by means of the continuation method, as we will show in the next sections.

Some important properties of the symmetric orbit, are expressed in the following lemma, whose proof is an immediate consequence of the Existence and Uniqueness Theorem for an ordinary differential equation.

Lemma 3. (i) The solution $\varphi(t)=(\mathbf{q}(t), \mathbf{p}(t))=\left(x(t), y(t), p_{x}(t), p_{y}(t)\right)$ is $S_{0}$-symmetric if and only if it has a point on the zero velocity curve (i.e., there is a time $t_{0}$ such that $\left.p_{x}\left(t_{0}\right)=p_{y}\left(t_{0}\right)=0\right)$.
(ii) For $i=1$ (resp. $i=2$ ) the solution $\varphi(t)$ is $S_{i}$-symmetric if and only if it lies on the $x$-axis for all time $t \in \mathbb{R}$ (resp. $y$-axis).
(iii) For $i=4$ (resp. $i=5$ ) the solution $\varphi(t)=(\mathbf{q}(t), \mathbf{p}(t))$ is $S_{i}$-symmetric if and only if it crosses the $y$-axis (resp. $x$-axis) orthogonally, i.e., there is a time $t_{0}$ such that $x\left(t_{0}\right)=0$ and $p_{y}\left(t_{0}\right)=0$ (resp. there is a time $t_{0}$ such that $y\left(t_{0}\right)=0$ and $\left.p_{x}\left(t_{0}\right)=0\right)$.

Proof: First, we will prove the item (i). In order to prove the necessary condition, as the solution $\varphi(t)=S_{0}(\varphi(t))$, it follows that $p_{x}(t)=-p_{x}(-t)$ and $p_{y}(t)=-p_{y}(-t)$ for all $t \in \mathbb{R}$. Thus, at $t=0$ we have $p_{x}(0)=0$ and $p_{y}(0)=0$, i.e., the orbit is on the zero velocity curve. To prove the sufficient condition, first we note that since the Hamiltonian system (1.4) is autonomous we can assume without loss of generality that there is a time $t_{0}=0$ such that $p_{x}(0)=0$ and $p_{y}(0)=0$. As by Proposition 1 , $\bar{\varphi}(t)=S_{0}(\varphi(t))$ is also a solution of (1.4) and since $\bar{\varphi}(0)=\varphi(0)$, then by the Existence and Uniqueness Theorem for an ordinary differential equation it follows that $\bar{\varphi}(t)=\varphi(t)$ for all $t \in \mathbb{R}$.

The item (ii) for $i=1$ (the other case is analogous) follows observing that $S_{1}(\varphi(t))=\varphi(t)$ if and only if $y(t)=0$ and $p_{y}(t)=0$ for all $t \in \mathbb{R}$.

To prove item (iii) for $i=4$ (the same arguments are true for $i=5$ ), assuming that $\varphi(t)=S_{4}(\varphi(t))$ it follows that $x(t)=-x(-t)$ and $p_{y}(t)=$ $-p_{y}(-t)$ for all $t \in \mathbb{R}$. Thus, at $t=0$ we have $x(0)=0$ and $p_{y}(0)=0$, i.e., the orbit crosses the $y$-axis orthogonally. To prove the sufficient condition, first we note that since the Hamiltonian system (1.4) is autonomous we can assume without loss of generality that there is time $t_{0}=0$ such that $x(0)=$ 0 and $p_{y}(0)=0$. As by Proposition $1 \bar{\varphi}(t)=S_{4}(\varphi(t))$ is also a solution of (1.4) and since $\bar{\varphi}(0)=\varphi(0)$, then by the Existence and Uniqueness Theorem for an ordinary differential equation it follows that $\bar{\varphi}(t)=\varphi(t)$ for all $t \in \mathbb{R}$.

The properties of the $S_{i}$-symmetric orbits were first studied by Birkhoff [4] for the restricted three body problem and later by many authors. We now introduce an useful technique to obtain symmetric periodic orbits in our problem.

Proposition 3. (i) For $i=4$ (resp. $i=5$ ) the solution $\varphi(t)$ is a $S_{i^{-}}$ symmetric periodic orbit if and only if it crosses the $y$-axis (resp. $x$-axis) orthogonally in exactly two distinct points.
(ii) An orbit $\varphi(t)$ is a $S_{4}$ and $S_{5}$-symmetric periodic orbit (duple symmetric) if and only if it crosses the $x$-axis and the $y$-axis orthogonally.
(iii) An orbit $\varphi(t)$ is a $S_{0}$-symmetric periodic orbit if and only if it meets the zero velocity curves at two distinct points.

Proof: In order to clarify the ideas of the proof we will prove only the item (i) for $i=4$. The proof in the other cases are similar. For $i=4$, considering $\varphi(t)$ a $P$-symmetric periodic solution of (1.4) we have that $S_{4}(\varphi(t+P))=\varphi(t)$, so at $t=-P / 2$ it follows that $x(P / 2)=0$ and $p_{y}(P / 2)=0$. Thus, since at $t=0, x(0)=0$ and $p_{y}(0)=0$, the orbit crosses the $y$-axis orthogonally in exactly two distinct points. The proof in the other direction is as follows. Assuming that the solution $\varphi(t)$ satisfies $x(0)=p_{y}(0)=0$ and $x(T)=p_{y}(T)=0$ and $x(t) \neq 0$ for $t \in(0, T)$, it is enough to define

$$
\bar{\varphi}(t)=\left\{\begin{array}{lc}
\varphi(t), & 0 \leq t \leq T \\
S_{4}(\varphi(2 T-t)), & T<t \leq 2 T .
\end{array}\right.
$$

Since $S_{4}(\varphi(T))=\varphi(T)$ and $S_{4}(\varphi(0))=\varphi(0)$ by the Existence and Uniqueness Theorem for an ordinary differential equation it follows that $\bar{\varphi}(t+2 T)=\bar{\varphi}(t)$ for all $t \in \mathbb{R}$ and it is $S_{4}$-symmetric.

Remarks. 1) Item (i) says that we need to construct only the half of one orbit that crosses the $x$-axis (resp. $y$-axis) orthogonally at two distinct points to get one symmetric periodic solution with respect to the $x$-axis (resp. $y$-axis).
2) Item (ii) says that we need to construct only a quarter of one orbit that crosses the $x$-axis and the $y$-axis orthogonally to get one symmetric periodic solution with respect to the $x$-axis and $y$-axis.

## 3. Existence of $S_{4}$ and $S_{5}$ symmetric periodic orbits

In this section we will prove the existence of $S_{4}$ and $S_{5}$ symmetric periodic orbits of the system (1.2) with $b$ close to 1 . Initially we will study the unperturbed problem, i.e., the problem (1.8).

### 3.1. Analysis of the logarithmic potential

Initially we will summarize some important properties of the unperturbed problem (1.8) (see [1] or [5]) which is given by

$$
\begin{equation*}
\ddot{\mathbf{q}}=-\frac{\mathbf{q}}{\|\mathbf{q}\|^{\mathbf{2}}} \tag{3.1}
\end{equation*}
$$

where $\mathbf{q}=(x, y)$, that will be essential to get our main result in this section. We will denote by $\mathbf{p}=\dot{\mathbf{q}}$ the momentum of the system. Using polar coordinates $(\rho, \theta)$ the equation (3.1) assumes the form

$$
\begin{align*}
\ddot{\rho} & =-\frac{1}{\rho}+\frac{c^{2}}{\rho^{3}}  \tag{3.2}\\
\rho^{2} \dot{\theta} & =c .
\end{align*}
$$

The energy conserved along the motion will be denoted by

$$
\begin{equation*}
H=\frac{\dot{\rho}^{2}}{2}+\frac{c^{2}}{2 \rho^{2}}+\ln \rho \equiv h . \tag{3.3}
\end{equation*}
$$

Observing the graph of the effective potential energy in Figure 1


Figure 1: Graph of the effective potential energy.

$$
\begin{equation*}
V_{e f f}(\rho)=\frac{c^{2}}{2 \rho^{2}}+\ln \rho \tag{3.4}
\end{equation*}
$$

and using the relation

$$
\begin{equation*}
\dot{\rho}= \pm \sqrt{2\left[h-V_{e f f}(\rho)\right]} \tag{3.5}
\end{equation*}
$$

it follows that the phase portrait of the problem given by the first equation in (3.2) is given by the Figure 2.


Figure 2: Phase portrait of (13)

Easily we verify that the circular orbit given by $\rho(t)=a$ (where $a=$ $c=\frac{1}{\omega}$ ) is an equilibrium position of (3.1) which is stable in the sense of Liapunov, because it corresponds to a minimum of the effective potential, and therefore, the solutions of (3.1) which are close to the circular are closed in the variables $(\rho, \dot{\rho})$ (but not necessarily periodic in the original variables $(x, y)$, in fact, it is necessary the commensurability between $2 \pi$ and the period of $\rho(t))$. The period $T$ of these oscillations is

$$
T \simeq \frac{2 a \pi}{\sqrt{2}}
$$

(where $\simeq$ means sufficiently close) since the eigenvalues of the linear part associated to the periodic orbit $\mathbf{r}^{0}(t)$ are given by $\pm i \frac{\sqrt{2}}{a}$. Let us now move on to deal directly with the orbit. For an almost circular orbit, we have seen above that $\rho$ oscillates around the value $a$. It is known that the apsidal angle, that is the variation of the polar angle $\theta$ during the time in which $\rho$ oscillates from a minimum ( $\rho_{\text {min }}$, pericenter) to the next maximum ( $\rho_{\max }$, apocenter) is close to $\tau=\frac{2 \pi}{\omega}$. Now, we will enunciate the following important result.

Lemma 4. The angle between the successive pericenter and apocenter for an orbit close to the circular $\mathbf{r}^{0}(t)$ is $\Phi \simeq \frac{\pi}{\sqrt{2}}$. In particular, $\frac{\pi}{2}<\Phi<\pi$.

Proof: It is necessary to evaluate the apsidal angle of the polar angle $\theta$ during the time in which $\rho$ oscillates from a minimum to the next maximum. The time is obviously close to $\frac{T}{2}$ (see Figure 2). To evaluate the angle swept out in this time we shall have to multiply by a mean angular velocity. Since $\dot{\theta}=\frac{c}{\rho^{2}}$ and $\rho$ oscillates around $a$, we shall assume the mean angular velocity to be

$$
\overline{\dot{\theta}}=\frac{c}{a^{2}}=\frac{1}{a} .
$$

The apsidal angle will therefore be

$$
\Phi \simeq \frac{1}{2} T \bar{\theta}=\frac{a \pi}{\sqrt{2}} \frac{1}{a}=\frac{\pi}{\sqrt{2}}
$$

Since the circular orbit can be written as $\mathbf{r}^{0}(t)=a e^{i \omega t}$ we have

$$
\mathbf{r}^{0}(0)=(a, 0) \equiv a, \quad \dot{\mathbf{r}}^{0}(0)=(0, a \omega) \equiv i a \omega
$$

considering these vectors as complex numbers. Let the one-parametric family of solutions of (3.1) with $\mu \in(-1,1)$ be given by
$\mathbf{q}_{\mu}(t)=\left(x_{\mu}(t), y_{\mu}(t)\right) \quad$ such that $\quad \mathbf{q}_{\mu}(0)=a \quad$ and $\quad \mathbf{p}_{\mu}=\dot{\mathbf{q}}_{\mu}(0)=i(1+\mu) a \omega$,
i.e., they are solution of (3.1) with the same initial position as the circular orbit $\mathbf{r}^{0}(t)$ and initial velocity with the same direction that the circular orbit. We will introduce the notation

$$
\rho_{\mu}(t)=\left\|\mathbf{q}_{\mu}(t)\right\| .
$$

By the hypothesis it follows that

$$
\dot{\rho}_{\mu}(0)=0,
$$

and by (3.2) it is clear that

$$
\ddot{\rho}_{\mu}(0)=\frac{(2+\mu) \mu}{a} .
$$

Therefore for $\mu \in(-1,1)$

$$
s g\left(\ddot{\rho}_{\mu}(0)\right)=s g(\mu)
$$

(where $s g(s)$ means the sign of the real number $s$ ). In conclusion, $\rho_{\mu}(0)$ is a maximum or minimum, hence it is an apocenter or pericenter depending if $\mu<0$ or $\mu>0$, respectively.

As in the previous section $\Phi(t,(\mathbf{q}, \mathbf{p}), 0)=(\mathbf{q}(t,(\mathbf{q}, \mathbf{p}), 0), \mathbf{p}(t,(\mathbf{q}, \mathbf{p}), 0)$ will denote the flow of the Hamiltonian system associated to (3.1). Since the solution $\mathbf{r}^{0}(t)$ crosses the positive $y$-axis transversally (in fact, orthogonally) at time $\frac{\tau}{4}=\frac{\pi}{2 \omega}$, by the Continuous Dependence of Initial Conditions and Parameters Theorem (see [10]) there exists $\delta>0(\delta<1)$ such that for $|\mu|<\delta, \Phi\left(t,\left(\mathbf{q}_{\mu}, \mathbf{p}_{\mu}, 0\right)=\left(\mathbf{q}_{\mu}(t), \mathbf{p}_{\mu}(t)\right)\right.$ is defined in $\left[0, \frac{\tau}{4}\right]$ and intersects the positive $y$-axis in a single point. Moreover, the time-function $t_{\mu} \equiv$ $t\left(\left(\mathbf{q}_{\mu}, \mathbf{p}_{\mu}\right), \mu\right) \in\left[0, \frac{\tau}{4}\right]$ defined by the first time that $\mathbf{q}_{\mu}\left(t_{\mu}\right)$ intersects the positive $y$-axis is continuous.

For $0<\mu<\delta$ (resp. $-\delta<\mu<0$ ) $\mathbf{q}_{\mu}(0)$ is a pericenter (resp. apocenter) of the solution $\mathbf{q}_{\mu}(t)$ of the unperturbed problem (3.1), and since an apocenter (resp. pericenter) corresponds to critical points of $\rho_{\mu}(t)$ and they are given by consecutive times, we obtain:

Lemma 5. The function $\rho_{\mu}(t)$ for $0<\mu<\delta$ (resp. $-\delta<\mu<0$ ) is an increasing function (resp. decreasing) for $t \in\left(0, t_{\mu}\right]$.

Using Lemma 4 and the previous lemma, it is clear that Figure 3 shows the behavior of the solution $\rho_{\mu}(t)$ of (3.2) for $\mu$ sufficiently small.


Figure 3 : Particular solutions $\rho_{\mu}(t)$ of (3.2)
By the previous result it follows that $\dot{\rho}_{\mu}(t)>0$ for $0<\mu<\delta$ (resp. $\dot{\rho}_{\mu}(t)<0$ for $\left.-\delta<\mu<0\right)$ and $\left.t \in\left(0, t_{\mu}\left(\mathbf{q}_{\mu}, \mathbf{p}_{\mu}\right), \mu\right)\right]$. Since $y_{\mu}=\rho_{\mu}(t) \sin \left(\theta_{\mu}(t)\right)$ and $\theta_{\mu}\left(t_{\mu}\right)=\frac{\pi}{2}$, it is clear that

$$
\begin{gathered}
\dot{y}_{\mu}\left(t_{\mu}\right)=\dot{\rho}_{\mu}\left(t_{\mu}\right)>0 \quad(\text { resp. }<0) \quad \text { if } 0<\mu<\delta \\
\text { (resp. }-\delta<\mu<0) .
\end{gathered}
$$

Summarizing, the following lemma has been proved.

Lemma 6. There exists $\delta>0$ such that $\dot{y}_{\mu}\left(t_{\mu}\right)>0$ (resp. $\left.\dot{y}_{\mu}\left(t_{\mu}\right)<0\right)$ for $\mu \in(0, \delta)$ (resp. for $\mu \in(-\delta, 0)$ ).

### 3.2. Symmetric periodic solution of the perturbed problem

Let $\mathbf{q}(t,(\mathbf{q}, \mathbf{p}), \epsilon)$ be the solution of the perturbed problem

$$
\begin{align*}
& \ddot{x}=-\frac{x}{x^{2} y^{2}}-\epsilon \frac{\partial h}{\partial x}  \tag{3.7}\\
& \ddot{y}=-\frac{y}{x^{2}+y^{2}}-\epsilon \frac{\partial h}{\partial y} .
\end{align*}
$$

We will consider $(x, y) \in \mathbb{R}^{2}$ such that $x^{2}+y^{2}>\nu^{2}$ with $0<\nu<a$, so that $h$ is analytic in this domain.

Henceforth, in this subsection we will consider solutions of (3.7) with initial positions $(x(0), y(0))=(a, 0) \equiv a$. To get the appropriate initial velocities $\mathbf{p}$, we fix $\delta>0$ as before and define the segment in $\mathbb{R}^{2}$ by

$$
I_{\delta}=\{(0, \lambda a \omega) / \lambda \in(1-\delta, 1+\delta)\}
$$

and we consider $\mathbf{p} \in I_{\delta}$.
Now, let the function

$$
l: I_{\delta} \times(-\delta, \delta) \rightarrow \mathbb{R}
$$

be

$$
l(\mathbf{p}, \epsilon)=\dot{y}\left(t_{a}(\mathbf{p}, \epsilon), a, \mathbf{p}, \epsilon\right)
$$

where $t_{a}(\mathbf{p}, \epsilon)$ means the positive minimal time at which the solution intersects the positive $y$-axis. This function is well defined and it is continuous because of the previous results and since the solutions of the first order systems of differential equations associated to (3.7), by the Kamke's theorem (see [10]) depend uniformly analytically on all the data of the problem, i.e., the initial values and the functions that define the differential equation.

As we have seen in Subsection 3.1, for $\epsilon=0$ and by the previous subsection, for each $\mu \in(-\delta, \delta)$ fixed, the initial velocity associated to the solution $\left(\mathbf{q}_{\mu}(t), \mathbf{p}_{\mu}(t)\right)$ is $\mathbf{p}_{\mu}=i(1+\mu) a \omega$, then

$$
\left\|i a \omega-\mathbf{p}_{\mu}\right\|=a \omega|\mu|
$$

and by Lemma 6 we have

$$
l\left(\mathbf{p}_{\mu}, 0\right)>0 \quad \text { for } \quad \mu \in(0, \delta) \quad \text { and } \quad l\left(\mathbf{p}_{\mu}, 0\right)<0 \quad \text { for } \quad \mu \in(-\delta, 0)
$$

For the next argument we only need to consider one (fixed) value of $\mu$, so we will choose

$$
\mathbf{p}_{+}=\mathbf{p}_{\mu} \quad \text { and } \quad \mathbf{p}_{-}=\mathbf{p}_{-\mu} \quad \text { with } \quad \mu \in(0, \delta)
$$

By the continuity of $l$, and taking $\eta=\frac{1}{2} \min \left\{l\left(\mathbf{p}_{+}, 0\right),\left|l\left(\mathbf{p}_{-}, 0\right)\right|\right\}>0$ there exists $\epsilon_{0}<\delta$ such that

$$
\left|l\left(\mathbf{p}_{-}, 0\right)-l\left(\mathbf{p}_{-}, \epsilon\right)\right|<\eta, \quad \text { and } \quad\left|l\left(\mathbf{p}_{+}, 0\right)-l\left(\mathbf{p}_{+}, \epsilon\right)\right|<\eta
$$

for $\epsilon \in\left(-\epsilon_{0}, \epsilon_{0}\right)$. It follows that

$$
\begin{equation*}
l\left(\mathbf{p}_{+}, \epsilon\right)>0 \quad \text { and } \quad l\left(\mathbf{p}_{-}, \epsilon\right)<0, \quad \text { for } \quad \epsilon \in\left(-\epsilon_{0}, \epsilon_{0}\right) . \tag{3.8}
\end{equation*}
$$

Now, we are in position to enunciate the main result of this section.

Theorem 1. There exists $\epsilon_{0}>0$ such that for every $\epsilon \in\left(-\epsilon_{0}, \epsilon_{0}\right)$, there is $\mathbf{p}_{\epsilon}$ so that the solution $\mathbf{q}(t)=\mathbf{q}\left(t,\left(a, \mathbf{p}_{\epsilon}\right), \epsilon\right)$ of (3.7) is periodic and symmetric with respect to the $x$-axis and $y$-axis.

Proof: Our objective is to prove that there exist $\epsilon_{0}>0$ and an initial velocity $\mathbf{p}$ for each value of $\epsilon \in\left(-\epsilon_{0}, \epsilon_{0}\right)$ such that $l(\mathbf{p}, \epsilon)=0$, i.e., the solution intersects orthogonally the $y$-axis. The conclusion of the theorem follows from item (iii) of proposition 3.

Using the previous analysis, more explicitly from (3.8), it follows by the Intermediate Value Theorem, that there is $\epsilon_{0}>0$ such that for every $\epsilon \in\left(-\epsilon_{0}, \epsilon_{0}\right)$ there exists $\mathbf{p}_{\epsilon}$ such that $l\left(\mathbf{p}_{\epsilon}, \epsilon\right)=0$.

In Figures 4 and 5 we exhibit numerically, one example of a periodic solution near the circle of radius 1 and one non periodic solution respectively.


Figure 4 : Periodic solution of (2) for $a=1, \epsilon=b-1=0,000001$, $\left(\mathrm{p}_{x}(0), p_{y}(0)\right)=(0,0.9999999)$.


Figure 5 : Solution of (2) for $a=1, \epsilon=b-1=0.000001,\left(p_{x}(0), p_{y}(0)\right)=$ $(0,0.95)$.

Remark We have used similar arguments to those considered in [2] where the author proved the existence of periodic solutions of the system

$$
\ddot{\mathbf{q}}=\mathbf{g}(\mathbf{q}, \epsilon)
$$

where $\mathbf{g}(\mathbf{q}, 0)=-\frac{\mathbf{q}}{\|\mathbf{q}\|^{\alpha+2}}, \quad \alpha \geq 0$, and $\mathbf{g}$ with the particular symmetry given by the reflection with respect to the $x$-axis, while in our approach we have considered two kinds of symmetries simultaneously. Therefore, our Theorem 1 is not included in [2]. It is clear that our result admits the following generalization. Let the following mechanical system

$$
\begin{equation*}
\ddot{\mathbf{q}}=\nabla V(\mathbf{q})+\epsilon \mathbf{g}(\mathbf{q}, \epsilon) \tag{3.9}
\end{equation*}
$$

be such that

- $V(\mathbf{q})=\frac{1}{2} \ln \|\mathbf{q}\| ;$
- $\mathbf{g}(\mathbf{q}, \epsilon)$ is an analytic function;
- $\mathbf{g}(\mathbf{q}, \epsilon)=\nabla G(\mathbf{q}, \epsilon)$ for some analytic real function $G$ which is invariant under the reflections $(x, y) \rightarrow(-x, y)$ and $(x, y) \rightarrow(x,-y)$.
Thus, applying the same argument used to prove Theorem 1 we obtain:
Theorem 2. Under the above conditions, there exists $\epsilon_{0}>0$ such that for every $\epsilon \in\left(-\epsilon_{0}, \epsilon_{0}\right)$, there is $\mathbf{p}_{\epsilon}$ so that the solution $\mathbf{q}(t)=\mathbf{q}\left(t,\left(a, \mathbf{p}_{\epsilon}\right), \epsilon\right)$ of (3.9) is periodic and symmetric with respect the $x$-axis and $y$-axis.


## 4. Conclusions

From the following affirmation made by the authors in [12]: "the analytical description in general non-axis symmetric case remains still open, several problems require further investigation", and due to the practical importance of this problem in the modelling of elliptical galaxies or bars in the centres of galaxy discs, we were induced to attack this problem, in order to obtain periodic solutions of the problem for weak anisotropies.

In fact, in order to get weak anisotropies, i.e., when the parameter $b$ is close to 1 we introduce the notation $\epsilon=b-1$ with $\epsilon \ll 1$, and then we expand the Hamiltonian(1.3) in powers of $\epsilon$, such that now we have one problem which is a perturbation of the central field defined by the potential $U=\ln \|\mathbf{q}\|$.

We have performed an analytical study of the singular logarithmic potential for weak anisotropies. Using important properties of the central field defined by the potential $U=\ln \|\mathbf{q}\|$, we obtain the existence of periodic solutions which are symmetric with respect to the origin. Here, we used the fact that the perturbation function is invariant under reflections with respect to the $x$ and $y$ axes.

We observe that the initial conditions originating periodic solutions obtained by Theorem 1 are given by $\left(x(\epsilon), y(\epsilon)=(a, 0)\right.$ and $\left(p_{x}(\epsilon), p_{y}(\epsilon)\right)=$ $\mathbf{p}(\epsilon)$. On the other hand, from Theorem 1 we only can affirm that the period is close to $\tau$. Also we emphasize that the periodic solutions obtained for this problem by the application of Theorem 2 in [15] and Theorem 1 for each fixed $|\epsilon|<\epsilon_{0}$, are not the same, because they are obtained by different initial conditions. Notice that the methods used in order to get periodic solutions of the logarithmic problem in [15] and the approach given in this paper are different.

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