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EXISTENCE RESULT FOR STRONGLY NONLINEAR ELLIPTIC EQUATIONS IN ORLICZ–SOBOLEV SPACES

A. YOUSSFI
UNIVERSITÉ SIDI MOHAMMED BEN ABDALLAH, MAROC

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Abstract

In this paper, we prove the existence of solutions for some strongly nonlinear Dirichlet problems whose model is the following

$$-\mathrm{div}(\overline{M}^{-1}M(|\nabla u|)\frac{\nabla u}{|\nabla u|}) + uM(|\nabla u|) = f - \mathrm{div}F \quad \text{in} \quad \mathcal{D}'(\Omega),$$

where Ω is an open bounded subset of $I\!R^N, N \geq 2$.

We emphasize that no Δ_2-condition is required for the N-function M.

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1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$, and let M be an N-function. Consider the following Dirichlet problem

$$A(u) + H(x, u, \nabla u) = f, \tag{1.1}$$

where

$$A(u) := -\operatorname{div} a(x, u, \nabla u)$$

is a Leray-Lions type operator defined on its domain $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$ and H is a nonlinearity assumed to satisfy the natural growth condition

$$|H(x, s, \xi)| \leq b(|s|)(h(x) + M(|\xi|)) \tag{1.2}$$

Recently, a large number of papers was devoted to the existence of solutions of (1.1). In the variational framework, that is $f \in W^{-1}E_{\overline{M}}(\Omega)$, an existence result was proved in [8] when H depends only on x and u and satisfy the following sign condition

$$H(x, s)s \geq 0,$$

and in [2] when M satisfies the Δ_2-condition and H depends also on ∇u and satisfies

$$H(x, s, \xi)s \geq 0. \tag{1.3}$$

The result in [2] was generalized in [7] to N-functions without Δ_2-condition.

In the case where $f \in L^1(\Omega)$, problem (1.1) was solved in [3] under the so-called coercivity condition

$$|H(x, s, \xi)| \geq \beta M(|\xi|) \qquad \text{for } |s| \geq \text{some } \tau \tag{1.4}$$

and in [5] assuming the sign condition (1.3) but the result was restricted to N-functions satisfying the Δ_2-condition (see bellow). The result contained in [5] was then extended in [6] to N-functions without assuming the Δ_2-condition. The solution u given in this case is such that its truncated function $T_k(u)$ belongs to the energy space $W_0^1L_M(\Omega)$ for all $k > 0$, but not the function u it self.

Our main goal in this paper, is to prove the existence of a solution in $W_0^1L_M(\Omega)$ for problems of the kind of (1.1) when the source term has the form $f - \mathrm{div}F$ with $f \in L^1(\Omega)$ and $|F| \in E_{\overline{M}}(\Omega)$, without any restriction on the N-function M.

The paper is organized as follows, after giving a background in section 2, in section 3 we list the basic assumptions and our main result which will be proved in six steeps in section 4.

2. Prerequisites

2.1 Let $M : I\!R^+ \to I\!R^+$ be an N-function, ie. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$. The N-function conjugate to M is defined as $\overline{M}(t) = \sup\{st - M(t), s \geq 0\}$. We recall the Young's inequality: for all $s, t \geq 0$,

$$st \leq \overline{M}(s) + M(t).$$

If for some $k > 0$,

$$M(2t) \leq kM(t) \quad \text{for all } t \geq 0, \tag{2.1}$$

we said that M satisfies the Δ_2-condition, and if (2.1) holds only for $t \geq$ some t_0, then M is said to satisfy the Δ_2-condition near infinity.
We will extend these N-functions into even functions on all $I\!R$.
Let P and Q be two N-functions. the notation $P \ll Q$ means that P grows essentially less rapidly than Q, i.e.

$$\text{for all } \epsilon > 0, \quad \frac{P(t)}{Q(\epsilon t)} \to 0 \text{ as } t \to \infty,$$

that is the case if and only if

$$\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0 \quad \text{as } t \to \infty.$$

2.2 Let Ω be an open subset of $I\!R^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))dx < \infty \quad \text{(resp. } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < \infty \text{ for some } \lambda > 0).$$

Endowed with the Luxemburg norm

$$\|u\|_M = \inf\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < \infty\},$$

$L_M(\Omega)$ is a Banach space and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The Orlicz norm is defined on $L_M(\Omega)$ by

$$\|u\|_{(M)} = \sup \int_\Omega u(x)v(x)dx,$$

where the supremum is taken over all functions $v \in L_{\overline{M}}(\Omega)$ such that $\|v\|_{\overline{M}} \le 1$.
The two norms $\|.\|_M$ and $\|.\|_{(M)}$ are equivalent (see [13]).
The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
2.3 The Orlicz-Sobolev space $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$).
It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
The space $W_0^1E_M(\Omega)$ is defined as the norm closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
We say that a sequence $\{u_n\}$ converges to u for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \to 0 \quad \text{for all } |\alpha| \le 1,$$

this implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [9, Lemma 6]).
If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then the modular convergence coincides with norm convergence (see [13, Theorem 9.4]).

Recall that the norm $\|Du\|_M$ defined on $W_0^1L_M(\Omega)$ is equivalent to $\|u\|_{1,M}$ (see [10]).
Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open Ω has the segment property then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [10]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined. For an exhaustive treatments one can see for example [1, 13].

2.4 We will use the following lemma, (see [6]), which concerns operators of Nemytskii Type in Orlicz spaces. It is slightly different from the analogous one given in [13].

Lemma 2.1. *Let Ω be an open subset of $I\!R^N$ with finite measure. let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times I\!R \to I\!R$ be a Carathéodory function such that, for a.e.$x \in \Omega$ and for all $s \in I\!R$,*

$$|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x, u(x))$, is strongly continuous from $\mathcal{P}(E_M, \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We will use the following lemma which can be found in [12],

Lemma 2.2. *If $\{f_n\} \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in Ω, f_n, $f \geq 0$ a.e. in Ω and $\displaystyle\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$, then $f_n \to f$ strongly in $L^1(\Omega)$.*

We also use the technical lemma:

Lemma 2.3. *Let x and y be two nonnegative real numbers and let*

$$\phi(s) = se^{\theta s^2},$$

with $\theta = \frac{y^2}{4x^2}$. Then

$$x\phi'(s) - y|\phi(s)| \geq \frac{x}{2}, \qquad \forall s \in I\!R.$$

3. Assumptions and main result

Let Ω be an open bounded subset of $I\!R^N$, $N \geq 2$, with the segment property and let M and P be two N-functions such that $P \ll M$.
Let $A : \mathcal{D}(\mathcal{A}) \subset W_0^1 L_M(\Omega) \longrightarrow W^{-1} L_{\overline{M}}(\Omega)$ be a mapping (non everywhere defined) given by

$$A(u) := -\mathrm{div}\, a(x, u, \nabla u)$$

where $a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function (i.e., $a(x, \cdot, \cdot)$ is continuous on $I\!R \times I\!R^N$ for almost every x in Ω and $a(\cdot, s, \xi)$ is measurable on Ω for every (s, ξ) in $I\!R \times I\!R^N$) satisfying for a.e. $x \in \Omega$, and for all $s \in I\!R$ and all ξ, $\eta \in I\!R^N$, $\xi \neq \eta$,

$$|a(x, s, \xi)| \leq a_0(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_1 \overline{M}^{-1} M(k_2|\xi|) \tag{3.1}$$

where $a_0(x)$ belongs to $E_{\overline{M}}(\Omega)$ and k_1, k_2 to $I\!R_+^*$,

$$(a(x, s, \xi) - a(x, s, \eta)) \cdot (\xi - \eta) > 0 \tag{3.2}$$

$$a(x, s, \xi) \cdot \xi \geq M(|\xi|) \tag{3.3}$$

Furthermore, let $H : \Omega \times I\!R \times I\!R^N \to I\!R$ be a Carathéodory function such that

$$|H(x, s, \xi)| \leq b(|s|)(M(|\xi|) + h(x)) \tag{3.4}$$

for almost $x \in \Omega$ and for all $s \in I\!R$, $\xi \in I\!R^N$, with b a real valued positive increasing continuous function and h a nonnegative function in $L^1(\Omega)$, and

$$H(x, s, \xi)\mathrm{sgn(s)} \geq M(|\xi|) \tag{3.5}$$

for a.e. $x \in \Omega$, for every $\xi \in I\!R^N$ and for every $s \in I\!R$ such that $|s| \geq \sigma$, where σ is a positive real number. Consider the following Dirichlet problem:

$$\begin{cases} A(u) + H(x, u, \nabla u) = f - \mathrm{div}(F) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.6}$$

We shall prove the following existence result:

Theorem 3.1. *Assume that $f \in L^1(\Omega)$, $|F| \in E_{\overline{M}}(\Omega)$ and (3.1)-(3.5) hold true, then there exists at least a function u solution of (3.6) in the sense that $u \in W_0^1 L_M(\Omega)$, $H(x,u,\nabla u) \in L^1(\Omega)$ and*

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla T_k(u-v)dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)dx$$
$$= \int_\Omega fT_k(u-v)dx + \int_\Omega F\cdot \nabla T_k(u-v)dx$$

for every $v \in W_0^1 L_M(\Omega)\cap L^\infty(\Omega)$ and every $k \geq \sigma$.

Remark 3.1.

1. *We can replace assumptions (3.3), (3.4) and (3.5) by the following ones:*

(3.3)′
$$a(x,s,\xi)\cdot\xi \geq \alpha M\left(\frac{|\xi|}{\lambda}\right)$$

with $\alpha, \lambda > 0$ and

(3.4)′
$$|H(x,s,\xi)| \leq b(|s|)\left(M\left(\frac{|\xi|}{\mu}\right) + h(x)\right)$$

with $0 < \lambda \leq \mu$ and

(3.5)′
$$H(x,s,\xi)\mathrm{sgn(s)} \geq \beta M\left(\frac{|\xi|}{\tau}\right)$$

with $0 < \tau \leq \lambda$ and $\beta > 0$.

2. *A consequence of (3.3) and the continuity of a with respect to ξ, is that, for almost every x in Ω and s in $I\!R$,*

$$a(x,s,0) = 0.$$

3. *Note that assumption (3.5) gives a sign condition on H only near infinity.*
4. *In (3.4) we can assume only that b is positive and continuous.*

Remark 3.2. *The solution of (3.6) given by theorem 3.1 belongs to $W_0^1 L_M(\Omega)$ even if $F = 0$, this regularity is due to assumption (3.5).*

4. Proof of theorem 3.1

Let $\{f_n\}$ be a sequence of $L^\infty(\Omega)$ functions that converges strongly to f in $L^1(\Omega)$.

Let n in $I\!N$ and let

$$H_n(x,s,\xi) = \frac{H(x,s,\xi)}{1+\frac{1}{n}|H(x,s,\xi)|}.$$

It's easy to see that $|H_n(x,s,\xi)| \leq n$, $|H_n(x,s,\xi)| \leq |H(x,s,\xi)|$ and $H_n(x,s,\xi)\mathrm{sgn(s)} \geq 0$ for $|s| \geq \sigma$. Since H_n is bounded for fixed n, there exists, (see [11, Propositions 1 and 5]), a function u_n in $W_0^1L_M(\Omega)$ solution of

$$\begin{cases} A(u_n) + H_n(x,u_n,\nabla u_n) = f_n - \mathrm{div}F & \text{in} \quad \Omega, \\ \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

in the sense

(3.7)

$$\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla v dx + \int_\Omega H_n(x,u_n,\nabla u_n) v dx = \int_\Omega f_n v dx + \int_\Omega F\cdot\nabla v dx$$

for every $v \in W_0^1L_M(\Omega)$.

Step1: Estimation in $W_0^1L_M(\Omega)$.

For $k>0$, we denote by T_k the usual truncation at level k defined by

$$T_k(s) = \max(-k, \min(k,s))$$

for all $s \in I\!R$. Let us choose

$$v = \phi(T_\sigma(u_n))$$

as test function in (3.7), where σ is given by (3.5), ϕ is the function in lemma 2.3 and b is the function in (3.4). Using (3.3) and the Young's inequality, we obtain

$$\int_\Omega M(|\nabla T_\sigma(u_n)|)\phi'(T_\sigma(u_n))dx + \int_\Omega H_n(x,u_n,\nabla u_n)\phi(T_\sigma(u_n))dx$$
$$\leq \phi(\sigma)\|f_n\|_{L^1(\Omega)} + \phi'(\sigma)\int_\Omega \overline{M}(2|F|)dx + \frac{1}{2}\int_\Omega M(|\nabla T_\sigma(u_n)|)\phi'(T_\sigma(u_n))dx.$$

Since $\{f_n\}$ is bounded in $L^1(\Omega)$, there exists a constant c not depending on n such that

$$\frac{1}{2}\int_\Omega M(|\nabla T_\sigma(u_n)|)\phi'(T_\sigma(u_n))dx + \int_\Omega H_n(x,u_n,\nabla u_n)\phi(T_\sigma(u_n))dx$$
$$\leq c(\phi(\sigma)+\phi'(\sigma)),$$

which we can write, since H_n enjoys the same properties of H,

$$\frac{1}{2}\int_\Omega M(|\nabla T_\sigma(u_n)|)\phi'(T_\sigma(u_n))dx + \int_{\{|u_n|<\sigma\}} H_n(x,u_n,\nabla u_n)\phi(T_\sigma(u_n))dx$$
$$+ \int_{\{|u_n|\geq\sigma\}} H_n(x,u_n,\nabla u_n)\phi(T_\sigma(u_n))dx$$
$$\leq c(\phi(\sigma)+\phi'(\sigma)).$$

By (3.4) we have

$$\int_{\{|u_n|<\sigma\}} H_n(x,u_n,\nabla u_n)\phi(T_\sigma(u_n))dx$$
$$\leq b(\sigma)\left(\int_\Omega M(|\nabla T_\sigma(u_n)|)\phi(T_\sigma(u_n))dx + \phi(\sigma)\|h\|_{L^1(\Omega)}\right),$$

while using(3.5), we get

$$\int_{\{|u_n|\geq\sigma\}} H_n(x,u_n,\nabla u_n)\phi(T_\sigma(u_n))dx \geq \phi(\sigma)\int_{\{|u_n|\geq\sigma\}} M(|\nabla u_n|)dx.$$

Hence, we obtain

$$\int_\Omega M(|\nabla T_\sigma(u_n)|)\left(\tfrac{1}{2}\phi'(T_\sigma(u_n)) - b(\sigma)|\phi(T_\sigma(u_n))|\right)dx$$

$$+, \phi(\sigma)\int_{\{|u_n|\geq\sigma\}} M(|\nabla u_n|)dx \leq c(\phi(\sigma)+\phi'(\sigma)) + b(\sigma)\phi(\sigma)\|h\|_{L^1(\Omega)}.$$

Then, lemma 2.3 with the choice $x = \frac{1}{2}$ and $y = b(\sigma)$, yields

$$\frac{1}{4}\int_\Omega M(|\nabla T_\sigma(u_n)|)dx \,+\, \phi(\sigma)\int_{\{|u_n|\geq\sigma\}} M(|\nabla u_n|)dx$$

$$\leq\; c(\phi(\sigma)+\phi'(\sigma)) + b(\sigma)\phi(\sigma)\|h\|_{L^1(\Omega)},$$

which implies that

$$\int_\Omega M(|\nabla u_n|)dx \,\leq\, c_0, \tag{3.8}$$

where c_0 is a constant not depending on n. Thus $\{u_n\}$ is bounded in $W_0^1 L_M(\Omega)$, and consequently there exist a function u in $W_0^1 L_M(\Omega)$ and a subsequence still denoted by $\{u_n\}$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^1 L_M(\Omega) \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}) \tag{3.9}$$

and

$$u_n \to u \quad \text{in} \quad E_M(\Omega) \quad \text{strongly} \quad \text{and} \quad \text{a.e. in } \Omega. \tag{3.10}$$

Step2: $\{a(x, T_k(u_n), \nabla T_k(u_n))\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$ for all $k \geq \sigma$.
We will use the Orlicz norm. For that, let $\psi \in (L_M(\Omega))^N$ with $\|\psi\|_M \leq 1$.
For all $k \geq \sigma$, we write using (3.2)

$$\int_\Omega \left(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \frac{\psi}{k_2})\right)\cdot\left(\nabla T_k(u_n) - \frac{\psi}{k_2}\right)dx \geq 0,$$

so that

(3.11)

$$\begin{aligned}\frac{1}{k_2}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\cdot\psi dx \quad &\leq \quad \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(u_n)dx \\ &\quad - \int_\Omega a(x, T_k(u_n), \frac{\psi}{k_2})\cdot\nabla T_k(u_n)dx \\ &\quad + \frac{1}{k_2}\int_\Omega a(x, T_k(u_n), \frac{\psi}{k_2})\cdot\psi dx.\end{aligned}$$

To estimate the first term in the right, we take $v = T_k(u_n)$ as test function in (3.7) and then use the Young's inequality, the fact that $H_n(x, u_n, \nabla u_n)T_k(u_n) \geq 0$ on the set $\{|u_n| \geq k\}$ and (3.3), to obtain

$$\frac{1}{2}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)dx + \int_{\{|u_n|\leq k\}} H_n(x, u_n, \nabla u_n)T_k(u_n)dx$$

$$\leq k\|f_n\|_{L^1(\Omega)} + \int_\Omega \overline{M}(2|F|)dx.$$

Assumption (3.4) yields

$$\begin{aligned}\left|\int_{\{|u_n|\leq k\}} H_n(x, u_n, \nabla u_n)T_k(u_n)dx\right| &\leq kb(k)\left(\int_\Omega M(|\nabla u_n|)dx + \|h\|_{L^1(\Omega)}\right)\\ &\leq kb(k)(c_0 + \|h\|_{L^1(\Omega)}),\end{aligned}$$

where c_0 is the constant in (3.8). Hence, since $\{f_n\}$ is bounded in $L^1(\Omega)$, we deduce that

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)dx \leq \lambda_k,$$

with λ_k a constant depending on k. By the Young's inequality, (3.11) becomes

$$\begin{aligned}\frac{1}{k_2}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \psi\, dx &\leq \lambda_k + (1+2k_1)\int_\Omega M(|\nabla T_k(u_n)|)dx\\ &+ (1+\frac{1}{k_2})(1+2k_1)\int_\Omega \overline{M}\left(\frac{|a(x, T_k(u_n), \frac{\psi}{k_2})|}{1+2k_1}\right)dx\\ &+ \frac{1+2k_1}{k_2}\int_\Omega M(|\psi|)dx.\end{aligned}$$

By virtue of (3.1) and the convexity of $\overline{M}$, we get

$$\int_\Omega \overline{M}\left(\frac{|a(x,T_k(u_n),\dfrac{\psi}{k_2})|}{1+2k_1}\right)dx \leq \frac{1}{1+2k_1}\left(\int_\Omega (|a_0(x)|)dx + k_1\overline{M}^{-1}M(kk_2)|\Omega|\right) + \frac{k_1}{1+2k_1}\int_\Omega M(|\psi|)dx.$$

We conclude that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\psi\,dx \;\leq\; c_k$$

for all $\psi \in (L_M(\Omega))^N$ with $\|\psi\|_M \leq 1$, this means that

$$\|a(x,T_k(u_n),\nabla T_k(u_n))\|_{(\overline{M})} \;\leq\; c_k, \tag{3.12}$$

for every $k \geq \sigma$.

Step3: Almost everywhere convergence of the gradients.

Since the function u belongs to $W_0^1L_M(\Omega)$, there exists a sequence $\{v_j\} \subset \mathcal{D}(\Omega)$, (see [9]), which converges to u for the modular convergence in $W_0^1L_M(\Omega)$ and a.e. in Ω.
For $m \geq k \geq \sigma$, we define the function ρ_m by

$$\rho_m(s) = \begin{cases} 1 & \text{if} \quad |s| \leq m \\ m+1-|s| & \text{if} \quad m \leq |s| \leq m+1 \\ 0 & \text{if} \quad |s| \geq m+1. \end{cases}$$

Let $\theta_n^j = T_k(u_n) - T_k(vj)$, $\theta^j = T_k(u) - T_k(v_j)$ and $z_{n,m}^j = \phi(\theta_n^j)\rho_m(u_n)$ where ϕ is the function in lemma 2.3.

In what follows, we denote by $\epsilon_i(n,j)$, $(i \in I\!N)$, various sequences of real numbers which tend to 0 when n and $j \to \infty$ respectively, i.e.

$$\lim_{j\to\infty}\lim_{n\to\infty}\epsilon_i(n,j) = 0.$$

We use $z^j_{n,m} \in W^1_0 L_M(\Omega)$ as test function in (3.7) to get

(3.13)
$$< A(u_n), z^j_{n,m} > + \int_\Omega H_n(x, u_n, \nabla u_n) z^j_{n,m} dx = \int_\Omega f_n z^j_{n,m} dx + \int_\Omega F \cdot \nabla z^j_{n,m} dx.$$

In view of (3.10), we have $z^j_{n,m} \to \phi(\theta^j)\rho_m(u)$ weakly in $L^\infty(\Omega)$ for $\sigma^*(L^\infty, L^1)$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int_\Omega f_n z^j_{n,m} dx = \int_\Omega f \phi(\theta^j)\rho_m(u) dx,$$

and since $\phi(\theta^j) \to 0$ weakly in $L^\infty(\Omega)$ for $\sigma(L^\infty, L^1)$ as $j \to \infty$, we have

$$\lim_{j\to\infty} \int_\Omega f \phi(\theta^j)\rho_m(u) dx = 0,$$

hence, we obtain

$$\int_\Omega f_n z^j_{n,m} dx = \epsilon_0(n, j).$$

Thanks to (3.8) and (3.10), we have as $n \to \infty$

$$z^j_{n,m} \rightharpoonup \phi(\theta^j)\rho_m(u) \quad \text{ in } W^1_0 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}),$$

which implies that

$$\lim_{n\to\infty} \int_\Omega F \cdot \nabla z^j_{n,m} dx = \int_\Omega F \cdot \nabla\theta^j \phi'(\theta^j)\rho_m(u) dx + \int_\Omega F \cdot \nabla u \phi(\theta^j)\rho'_m(u) dx.$$

On the one hand, by Lebesgue's theorem we get

$$\lim_{j\to\infty} \int_\Omega F \cdot \nabla u \phi(\theta^j)\rho'_m(u) dx = 0,$$

on the other hand, we write

$$\begin{aligned} \int_\Omega F \cdot \nabla\theta^j \phi'(\theta^j)\rho_m(u) dx &= \int_\Omega F \cdot \nabla T_k(u) \phi'(\theta^j)\rho_m(u) dx \\ &\quad - \int_\Omega F \cdot \nabla T_k(v_j) \phi'(\theta^j)\rho_m(u) dx, \end{aligned}$$

so that, by Lebesgue's theorem one has

$$\lim_{j\to\infty}\int_\Omega F\cdot\nabla T_k(u)\phi'(\theta^j)\rho_m(u)dx=\int_\Omega F\cdot\nabla T_k(u)\rho_m(u)dx,$$

Let $\lambda>0$ such that $M\left(\frac{|\nabla v_j-\nabla u|}{\lambda}\right)\to 0$ strongly in $L^1(\Omega)$ as $j\to\infty$ and $M\left(\frac{|\nabla u|}{\lambda}\right)\in L^1(\Omega)$, the convexity of the N-function M allows us to have

$$\begin{aligned}&M\left(\frac{|\nabla T_k(v_j)\phi'(\theta^j)\rho_m(u)-\nabla T_k(u)\rho_m(u)|}{4\lambda\phi'(2k)}\right)\\ &\le\frac{1}{4}M\left(\frac{|\nabla v_j-\nabla u|}{\lambda}\right)+\frac{1}{4}\left(1+\frac{1}{\phi'(2k)}\right)M\left(\frac{|\nabla u|}{\lambda}\right).\end{aligned}$$

Then, by using the modular convergence of $\{\nabla v_j\}$ in $(L_M(\Omega))^N$ and Vitali's theorem, we obtain

$$\nabla T_k(v_j)\phi'(\theta^j)\rho_m(u)\to\nabla T_k(u)\rho_m(u)\quad\text{ in }\ (L_M(\Omega))^N$$

for the modular convergence, and then

$$\lim_{j\to\infty}\int_\Omega F\cdot\nabla T_k(v_j)\phi'(\theta^j)\rho_m(u)dx=\int_\Omega F\cdot\nabla T_k(u)\rho_m(u)dx.$$

We have proved that

$$\int_\Omega F\cdot\nabla z^j_{n,m}dx=\epsilon_1(n,j).$$

Since $H_n(x,u_n,\nabla u_n)z^j_{n,m}\ge 0$ on the set $\{|u_n|>k\}$ and $\rho_m(u_n)=1$ on the set $\{|u_n|\le k\}$, we have

$$< A(u_n),z^j_{n,m}>+\int_{\{|u_n|\le k\}}H_n(x,u_n,\nabla u_n)\phi(\theta^j_n)dx\le\epsilon_2(n,j)\tag{3.14}$$

Now, we will evaluate the first term of the left-hand side of (3.14) by writing

$$
\begin{aligned}
&< A(u_n), z^j_{n,m} > \\
&= \int_\Omega a(x,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j))\phi'(\theta^j_n)\rho_m(u_n)dx \\
&+ \int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n \phi(\theta^j_n)\rho'_m(u_n)dx \\
&= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(v_j))\phi'(\theta^j_n)dx \\
&- \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\cdot\nabla T_k(v_j)\phi'(\theta^j_n)\rho_m(u_n)dx \\
&+ \int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n \phi(\theta^j_n)\rho'_m(u_n)dx,
\end{aligned}
$$

and then

$$
\begin{aligned}
&< A(u_n), z^j_{n,m} > \\
&= \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j)) \\
&\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j)\phi'(\theta^j_n)dx \\
&+ \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^s_j))\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j)\phi'(\theta^j_n)dx \qquad (3.15)\\
&- \int_{\Omega\setminus\Omega^s_j} a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\phi'(\theta^j_n)dx \\
&- \int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\phi'(\theta^j_n)\rho_m(u_n)dx \\
&+ \int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla u_n \phi(\theta^j_n)\rho'_m(u_n)dx,
\end{aligned}
$$

where by χ^s_j, $s>0$, we denote the characteristic function of the subset

$$\Omega^s_j = \{x\in\Omega : |\nabla T_k(v_j)| \le s\}.$$

For fixed m and s, we will pass to the limit in n and then in j in the second, third, fourth and five terms in the right side of (3.15). Starting with the second term , we have

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) \cdot (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\phi'(\theta_n^j)dx$$

$$\to \int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)) \cdot (\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s)\phi'(\theta^j)dx$$

as $n \to \infty$, since by lemma 2.1 one has

$$a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s))\phi'(\theta_n^j) \to a(x, T_k(u), \nabla T_k(v_j)\chi_j^s))\phi'(\theta^j)$$

strongly in $(E_{\overline{M}}(\Omega))^N$ as $n \to \infty$, while

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$$

weakly in $(L_M(\Omega))^N$ by (3.8). Let χ^s denote the characteristic function of the subset

$$\Omega^s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}.$$

As $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(E_M(\Omega))^N$ as $j \to \infty$, one has

$$\int_\Omega a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)) \cdot (\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s)\phi'(\theta^j)dx \to 0$$

as $j \to \infty$. Then

(3.16)
$$\int_\Omega a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)) \cdot (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\phi'(\theta_n^j)dx = \epsilon_3(n, j).$$

For the third term of (3.15), by virtue of (3.12) there exist a subsequence still indexed again by n and a function l_k in $(L_{\overline{M}}(\Omega))^N$ with $k \geq \sigma$ such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup l_k \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

Then, since $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_j^s} \in (E_M(\Omega))^N$, we obtain

$$-\int_{\Omega\setminus\Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(v_j)\phi'(\theta_n^j)dx \to -\int_{\Omega\setminus\Omega_j^s} l_k \cdot \nabla T_k(v_j)\phi'(\theta^j)dx$$

as $n \to \infty$. The modular convergence of $\{v_j\}$ allows us to have

$$-\int_{\Omega\setminus\Omega_j^s} l_k \cdot \nabla T_k(v_j)\phi'(\theta^j)dx \to -\int_{\Omega\setminus\Omega^s} l_k \cdot \nabla T_k(u)dx$$

as $j \to \infty$. This, proves that

$$-\int_{\Omega\setminus\Omega_j^s} a(x, T_k(u_n), \nabla T_k(u_n))\cdot\nabla T_k(v_j)\phi'(\theta_n^j)dx = -\int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx+\epsilon_4(n,j). \tag{3.17}$$

As regards the fourth term, observe that $\rho_m(u_n) = 0$ on the subset $\{|u_n| \geq m+1\}$, so we have

$$-\int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n) \cdot \nabla T_k(v_j)\phi'(\theta_n^j)\rho_m(u_n)dx$$

$$= -\int_{\{|u_n|>k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \cdot \nabla T_k(v_j)\phi'(\theta_n^j)\rho_m(u_n)dx.$$

As above, we obtain

$$-\int_{\{|u_n|>k\}} a(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \cdot \nabla T_k(v_j)\phi'(\theta_n^j)\rho_m(u_n)dx$$

$$= -\int_{\{|u|>k\}} l_{m+1} \cdot \nabla T_k(u)\rho_m(u)dx + \epsilon_5(n,j).$$

Observing that $\nabla T_k(u) = 0$ on the subset $\{|u| > k\}$, one has

$$-\int_{\{|u_n|>k\}} a(x, u_n, \nabla u_n) \cdot \nabla T_k(v_j)\phi'(\theta_n^j)\rho_m(u_n)dx = \epsilon_5(n,j) \tag{3.18}$$

For the last term of (3.15), we have

$$\left|\int_{\Omega} a(x, u_n, \nabla u_n) \cdot \nabla u_n \phi(\theta_n^j)\rho_m'(u_n)dx\right|$$

$$= \left|\int_{\{m\leq|u_n|\leq m+1\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \phi(\theta_n^j)\rho_m'(u_n)dx\right|$$

$$\leq \phi(2k)\int_{\{m\leq|u_n|\leq m+1\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx.$$

To estimate the last term of the previous inequality, we test by $T_1(u_n - T_m(u_n)) \in W_0^1 L_M(\Omega)$ in (3.7), to get

$$\begin{aligned}\int_{\{m\le |u_n|\le m+1\}} & a(x,u_n,\nabla u_n)\cdot \nabla u_n dx \\ & + \int_{\{|u_n|\ge m\}} H_n(x,u_n,\nabla u_n) T_1(u_n - T_m(u_n))dx \\ & = \int_\Omega f_n T_1(u_n - T_m(u_n))dx + \int_{\{m\le |u_n|\le m+1\}} F\cdot \nabla u_n dx.\end{aligned}$$

Using the fact that $H_n(x,u_n,\nabla u_n)T_1(u_n - T_m(u_n)) \ge 0$ on the subset $\{|u_n| \ge m\}$ and the Young's inequality, we get

$$\begin{aligned}&\frac{1}{2}\int_{\{m\le |u_n|\le m+1\}} a(x,u_n,\nabla u_n)\cdot \nabla u_n dx \\ &\le \int_{\{|u_n|\ge m\}} |f_n|dx + \int_{\{m\le |u_n|\le m+1\}} \overline{M}(|F|)dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\left|\int_\Omega a(x,u_n,\nabla u_n)\cdot \nabla u_n \phi(\theta_n^j)\rho_m'(u_n)dx\right| \\ &\le 2\phi(2k)\left(\int_{\{|u_n|\ge m\}} |f_n|dx + \int_{\{m\le |u_n|\le m+1\}} \overline{M}(|F|)dx\right).\end{aligned} \tag{3.19}$$

From (3.16), (3.17), (3.18) and (3.19) we obtain

$$\begin{aligned}< A(u_n), z_{n,m}^j > \ \ge & \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)) \\ & \cdot(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\phi'(\theta_n^j)dx \\ & -2\phi(2k)\left(\int_{\{|u_n|\ge m\}} |f_n|dx + \int_{\{m\le |u_n|\le m+1\}} \overline{M}(|F|)dx\right) \\ & - \int_{\Omega\setminus\Omega^s} l_k\cdot \nabla T_k(u)dx + \epsilon_6(n,j).\end{aligned} \tag{3.20}$$

Now, we turn to second term of the left hand side of (3.14). We have

$$\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} H_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\right| \\
&\qquad = \left|\int_{\{|u_n|\le k\}} H_n(x,T_k(u_n),\nabla T_k(u_n))\phi(\theta_n^j)dx\right| \\
&\qquad \le b(k)\int_\Omega M(|\nabla T_k(u_n)|)|\phi(\theta_n^j)|dx + b(k)\int_\Omega h(x)|\phi(\theta_n^j)|dx \\
&\qquad \le b(k)\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|\phi(\theta_n^j)|dx + \epsilon_7(n,j).
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\int_{\{|u_n|\le k\}} H_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx\right| \\
&\le b(k)\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big) \\
&\qquad \cdot\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)|\phi(\theta_n^j)|dx \\
&\qquad + b(k)\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)|\phi(\theta_n^j)|dx \\
&\qquad + b(k)\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\chi_j^s|\phi(\theta_n^j)|dx + \epsilon_7(n,j).
\end{aligned} \tag{3.21}$$

We proceed as above to get

$$b(k)\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s))\cdot(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)|\phi(\theta_n^j)|dx = \epsilon_8(n,j)$$

and

$$b(k)\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\chi_j^s|\phi(\theta_n^j)|dx = \epsilon_9(n,j).$$

Hence, we have

$$
\begin{aligned}
&\left| \int_{\{|u_n|\le k\}} H_n(x,u_n,\nabla u_n)\phi(\theta_n^j)dx \right| \\
&\le b(k)\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big) \\
&\qquad \cdot \Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\,|\phi(\theta_n^j)|dx \\
&\qquad + \ \epsilon_{10}(n,j).
\end{aligned} \tag{3.22}
$$

Combining (3.14), (3.20) and (3.22), we get

$$
\begin{aligned}
&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big) \\
&\qquad \cdot \Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\left(\phi'(\theta_n^j) - b(k)|\phi(\theta_n^j)|\right)dx \\
&\le \int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx \\
&+2\phi(2k)\left(\int_{\{|u_n|\ge m\}}|f_n|dx + \int_{\{m\le|u_n|\le m+1\}}\overline{M}(|F|)dx\right) + \epsilon_{11}(n,j).
\end{aligned}
$$

Then, lemma 2.3 with $x=1$ and $y=b(k)$, yields

$$
\begin{aligned}
&\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big) \\
&\cdot \Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\,dx \\
&\le 2\int_{\Omega\setminus\Omega^s} l_k\cdot\nabla T_k(u)dx \\
&+4\phi(2k)\left(\int_{\{|u_n|\ge m\}}|f_n|dx + \int_{\{m\le|u_n|\le m+1\}}\overline{M}(|F|)dx\right) + \epsilon_{11}(n,j).
\end{aligned} \tag{3.23}
$$

On the other hand

$$\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s))$$

$$\cdot (\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\, dx$$

$$= \int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big)$$

$$\cdot \Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\, dx$$

$$+ \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s) dx$$

$$- \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s) \cdot (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx$$

$$+ \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s) \cdot (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx.$$

We shall pass to the limit in n and then in j in the last three terms of the right hand side of the above equality. By similar arguments as in (3.15) and (3.21), we obtain

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n)) \cdot (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s) dx = \epsilon_{12}(n,j)$$

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s) \cdot (\nabla T_k(u_n) - \nabla T_k(u)\chi^s) dx = \epsilon_{13}(n,j)$$

and

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s) \cdot (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx = \epsilon_{14}(n,j), \tag{3.24}$$

so that

$$
\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)) \\
&\quad \cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)dx \\
&= \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)) \\
&\quad \cdot(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)dx \\
&\quad + \epsilon_{15}(n,j).
\end{aligned} \tag{3.25}
$$

Let $r \leq s$, we use (3.2), (3.25) and (3.23) to get

$$
\begin{aligned}
0 &\leq \int_{\Omega^r} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))) \\
&\quad \cdot(\nabla T_k(u_n) - \nabla T_k(u))dx \\
&\leq \int_{\Omega^s} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))) \\
&\quad \cdot(\nabla T_k(u_n) - \nabla T_k(u))dx \\
&= \int_{\Omega^s} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)) \\
&\quad \cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)dx \\
&\leq \int_{\Omega} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)) \\
&\quad \cdot(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)dx
\end{aligned}
$$

$$= \int_\Omega (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s))$$
$$\cdot (\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) dx + \epsilon_{15}(n, j)$$
$$\leq 2 \int_{\Omega \setminus \Omega^s} l_k \cdot \nabla T_k(u) dx$$
$$+ 4\phi(2k) \left(\int_{\{|u_n| \geq m\}} |f_n| dx + \int_{\{m \leq |u_n| \leq m+1\}} \overline{M}(|F|) dx \right) + \epsilon_{16}(n, j),$$

Which gives by passing to the limit sup over n and then over j

$$0 \leq \limsup_{n \to \infty} \int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))$$
$$\cdot (\nabla T_k(u_n) - \nabla T_k(u)) dx$$

$$\leq 2 \int_{\Omega \setminus \Omega^s} l_k \cdot \nabla T_k(u) dx + 4\phi(2k) \left(\int_{\{|u| \geq m\}} |f| dx + \int_{\{m \leq |u| \leq m+1\}} \overline{M}(|F|) dx \right).$$

Letting s and then $m \to \infty$, taking into account that

$l_k \nabla T_k(u) \in L^1(\Omega),\ f \in L^1(\Omega),\ \overline{M}(|F|) \in L^1(\Omega),\ |\Omega \setminus \Omega^s| \to 0,$

$\{|u| \geq m\} \to 0$, and $\{m \leq |u| \leq m+1\} \to 0$, one has

$$\int_{\Omega^r} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) dx \to 0$$

as $n \to \infty$. As in [4], we deduce that there exists a subsequence of $\{u_n\}$ still indexed again by n such that

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } \Omega. \tag{3.26}$$

Thus, by (3.12) and (3.26) we have

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x, T_k(u), \nabla T_k(u)) \qquad \text{weakly in } (L_{\overline{M}}(\Omega))^N \tag{3.27}$$

for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ and for all $k \geq \sigma$.

Step4: Modular convergence of the truncations.

Going back to (3.23), we write

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx$$

$$\le \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\chi_j^s dx$$

$$+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\cdot(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s)dx$$

$$+4\phi(2k)\left(\int_{\{|u_n|\ge m\}}|f_n|dx+\int_{\{m\le|u_n|\le m+1\}}\overline{M}(|F|)dx\right)$$

$$+2\int_{\Omega\setminus\Omega^s}a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dx+\epsilon_{11}(n,j).$$

and by (3.24) we get

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx$$

$$\le \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(v_j)\chi_j^s dx$$

$$+4\phi(2k)\left(\int_{\{|u_n|\ge m\}}|f_n|dx+\int_{\{m\le|u_n|\le m+1\}}\overline{M}(|F|)dx\right)$$

$$+2\int_{\Omega\setminus\Omega^s}a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dx+\epsilon_{17}(n,j).$$

We pass now to the limit sup over n in both sides of this inequality, to obtain

$$\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dx$$

$$\le \int_\Omega a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(v_j)\chi_j^s dx$$

$$+4\phi(2k)\left(\int_{\{|u|\ge m\}}|f|dx+\int_{\{m\le|u|\le m+1\}}\overline{M}(|F|)dx\right)$$

$$+2\int_{\Omega\setminus\Omega^s}a(x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dx+\lim_{n\to\infty}\epsilon_{17}(n,j),$$

in which, we pass to the limit in j to get

$$\begin{aligned}
&\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx \\
&\leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) \chi^s dx \\
&\quad + 4\phi(2k) \left(\int_{\{|u|\geq m\}} |f| dx + \int_{\{m\leq |u|\leq m+1\}} \overline{M}(|F|) dx \right) \\
&\quad + 2 \int_{\Omega\setminus\Omega^s} a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) dx,
\end{aligned}$$

letting s and then $m \to \infty$, one has

$$\limsup_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx \leq \int_\Omega a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) dx$$

On the other hand, by Fatou's lemma, we have

$$\int_\Omega a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) dx \leq \liminf_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx.$$

It follows that

$$\lim_{n\to\infty} \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx = \int_\Omega a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) dx.$$

By lemma 2.2 we conclude that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) \to a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) \tag{3.28}$$

strongly in $L^1(\Omega)$, $\forall k \geq \sigma$. The convexity of the N-function M and (3.3) allow us to have

$$\begin{aligned}
&M\left(\frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) \\
&\leq \frac{1}{2} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) + \frac{1}{2} a(x, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u).
\end{aligned}$$

Then, by (3.28) we get

$$\lim_{|E|\to 0} \sup_n \int_E M\left(\frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2}\right) dx = 0.$$

So that, by Vitali's theorem one has

$$T_k(u_n) \to T_k(u) \quad \text{in } W_0^1 L_M(\Omega)$$

for the modular convergence, for all $k \geq \sigma$.

Step5: Equi-integrability of the nonlinearities.
As a consequence of (3.10) and (3.26), one has

$$H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \quad \text{a.e. in } \Omega.$$

We shall prove that the sequence $\{H_n(x, u_n, \nabla u_n)\}$ is uniformly equi-integrable in Ω.
Let E be a measurable subset of Ω, for all $m \geq \sigma$, we have

$$\int_E |H_n(x, u_n, \nabla u_n)| dx = \int_{E \cap \{|u_n| \leq m\}} |H_n(x, u_n, \nabla u_n)| dx$$

$$+ \int_{E \cap \{|u_n| > m\}} |H_n(x, u_n, \nabla u_n)| dx.$$

On the one hand, the use of $T_1(u_n - T_{m-1}(u_n))$ as test function in (3.7), the Young's inequality and (3.3) led to

$$\int_\Omega H_n(x, u_n, \nabla u_n) T_1(u_n - T_{m-1}(u_n)) dx \leq \int_{\{|u_n| \leq m-1\}} |f_n| dx$$

$$+ \int_{\{m-1 \leq |u_n| \leq m\}} \overline{M}(2|F|) dx.$$

Then, assumption (3.5) gives

$$\int_{\{|u_n| \geq m\}} |H_n(x, u_n, \nabla u_n)| dx \leq \int_{\{|u_n| \leq m-1\}} |f_n| dx + \int_{\{m-1 \leq |u_n| \leq m\}} \overline{M}(2|F|) dx.$$

For all $\epsilon > 0$, one can find an $m = m(\epsilon) > 1$ such that

$$\sup_n \int_{\{|u_n| \geq m\}} |H_n(x, u_n, \nabla u_n)| dx \leq \frac{\epsilon}{2}.$$

On the other hand, we use (3.3) and (3.4) to get

$$\begin{aligned}
\int_{E\cap\{|u_n|\leq m\}} |H_n(x,u_n,\nabla u_n)|dx &\leq \int_E |H_n(x,T_m(u_n),\nabla T_m(u_n))|dx \\
&\leq b(m)\left(\int_E M(|\nabla T_m(u_n)|)dx + \int_E h(x)dx\right) \\
&\leq b(m)\int_E a(x,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)dx \\
&\quad +b(m)\int_E h(x)dx.
\end{aligned}$$

We use the fact that from (3.28) the sequence $\{a(x,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)\}$ is equi-integrable and that $h\in L^1(\Omega)$ to obtain

$$\lim_{|E|\to 0}\sup_n \int_{E\cap\{|u_n|\leq m\}} |H_n(x,u_n,\nabla u_n)|dx = 0,$$

where $|E|$ denotes the Lebesgue measure of the subset E. Consequently

$$\lim_{|E|\to 0}\sup_n \int_E |H_n(x,u_n,\nabla u_n)|dx = 0.$$

This proves that the sequence $\{H_n(x,u_n,\nabla u_n)\}$ is uniformly equi-integrable in Ω. By Vitali's theorem, we conclude that $H(x,u,\nabla u)\in L^1(\Omega)$ and

$$H_n(x,u_n,\nabla u_n)\to H(x,u,\nabla u) \tag{3.29}$$

strongly in $L^1(\Omega)$.

Step6: Passage to the limit.
Let $v\in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$. By [9, Lemma 4], there exists a sequence $\{v_j\}\subset\mathcal{D}(\Omega)$ such that $\|v_j\|_\infty\leq (N+1)\|v\|_\infty$ and

$$v_j\to v \quad \text{ in } W_0^1L_M(\Omega)$$

for the modular convergence and a.e. in Ω. Let $k\geq\sigma$. We go back to approximate equations (3.7) and use $T_k(u_n-v_j)$ as test function to obtain
(3.30)

$$\begin{aligned}
&\int_\Omega a(x,T_t(u_n),\nabla T_t(u_n))\cdot\nabla T_k(u_n-v_j)dx + \int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n-v_j)dx \\
&= \int_\Omega f_nT_k(u_n-v_j)dx + \int_\Omega F\cdot\nabla T_k(u_n-v_j)dx,
\end{aligned}$$

where $t = k + (N+1)\|v\|_\infty$. The first term in the left-hand side of (7.3.23) is written as

$$\int_\Omega a(x, T_t(u_n), \nabla T_t(u_n)) \cdot \nabla T_k(u_n - v_j)dx$$

$$= \int_{\{|u_n - v_j| < k\}} a(x, T_t(u_n), \nabla T_t(u_n)) \cdot \nabla T_t(u_n)dx$$

$$- \int_{\{|u_n - v_j| < k\}} a(x, T_t(u_n), \nabla T_t(u_n)) \cdot \nabla v_j dx$$

Thus, by (3.27) and (3.28) we obtain

$$\int_\Omega a(x, T_t(u_n), \nabla T_t(u_n)) \cdot \nabla T_k(u_n - v_j)dx \to \int_\Omega a(x, T_t(u), \nabla T_t(u)) \cdot \nabla T_k(u - v_j)dx$$

as $n \to \infty$. Since

$$T_k(u_n - v_j) \to T_k(u - v_j) \quad \text{in } L^\infty(\Omega) \text{ for } \sigma^*(L^\infty, L^1),$$

we use (3.29) and the fact that $f_n \to f$ strongly in $L^1(\Omega)$ as $n \to \infty$, to obtain

$$\int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - v_j)dx \to \int_\Omega H(x, u, \nabla u) T_k(u - v_j)dx,$$

$$\int_\Omega f_n T_k(u_n - v_j)dx \to \int_\Omega f T_k(u - v_j)dx,$$

as $n \to \infty$. For the last term in the right-hand side of (3.30) we write

$$\int_\Omega F \cdot \nabla T_k(u_n - v_j)dx = \int_{\{|u_n - v_j| < k\}} F \cdot \nabla u_n dx - \int_{\{|u_n - v_j| < k\}} F \cdot \nabla v_j dx.$$

Hence, by (3.9) we obtain

$$\int_\Omega F \cdot \nabla T_k(u_n - v_j)dx \to \int_\Omega F \cdot \nabla T_k(u - v_j)dx.$$

Therefore, passing to the limit as $n \to \infty$ in (3.30), we get

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla T_k(u - v_j)dx + \int_\Omega H(x, u, \nabla u) T_k(u - v_j)dx$$

$$= \int_\Omega f T_k(u - v_j)dx + \int_\Omega F \cdot \nabla T_k(u - v_j)dx,$$

in which we pass to the limit as $j \to \infty$ to obtain

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla T_k(u-v)dx \,+\, \int_\Omega H(x,u,\nabla u)T_k(u-v)dx$$

$$= \int_\Omega fT_k(u-v)dx \,+\, \int_\Omega F\cdot \nabla T_k(u-v)dx.$$

Which completes the proof of theorem 3.1.

Remark 4.1. *If the N-function M satisfies the Δ_2-condition, the sequence $\{a(x,u_n,\nabla u_n)\}$ will be bounded in $(L_{\overline{M}}(\Omega))^N$. Then, the function u solution of the problem (3.6) is such that: $u \in W_0^1L_M(\Omega)$, $H(x,u,\nabla u) \in L^1(\Omega)$ and*

$$\int_\Omega a(x,u,\nabla u)\cdot \nabla v dx \,+\, \int_\Omega H(x,u,\nabla u)v dx = \int_\Omega fv dx \,+\, \int_\Omega F\cdot\nabla v dx,$$

for every $v \in W_0^1L_M(\Omega)\cap L^\infty(\Omega)$.

Remark 4.2. *We can interpret theorem 3.1 in the following sense: the problem*

$$\begin{cases} u \in W_0^1L_M(\Omega),\ H(x,u,\nabla u) \in L^1(\Omega) \\ -\mathrm{div}\, a(x,u,\nabla u) + H(x,u,\nabla u) = \mu \end{cases}$$

admits a solution if and only if μ belongs to $L^1(\Omega) + W^{-1}L_{\overline{M}}(\Omega)$.

Remark 4.3. *If we replace (3.1) by the more general growth condition*

$$|a(x,s,\xi)| \le b_0(|s|)(a_0(x) + \overline{M}^{-1}M(\tau|\xi|))$$

where $a_0(x)$ belongs to $E_{\overline{M}}(\Omega)$, $\tau > 0$ and b_0 is a positive continuous increasing function, we can adapt the same ideas to prove the existence of solutions for the problem

$$\begin{cases} u \in W_0^1L_M(\Omega),\ H(x,u,\nabla u) \in L^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx \,+\, \int_\Omega H(x,u,\nabla u)T_k(u-v)dx \\ \displaystyle=\int_\Omega fT_k(u-v)dx \,+\, \int_\Omega F\cdot\nabla T_k(u-v)dx \\ \text{for } v \in W_0^1L_M(\Omega)\cap L^\infty(\Omega), \end{cases}$$

by considering the following approximation problems

$$\begin{cases} u_n \in W_0^1L_M(\Omega) \\ -\mathrm{div}\, a(x,T_n(u_n),\nabla u_n) + H_n(x,u_n,\nabla u_n) = f_n - \mathrm{div}F \quad \textit{in } \Omega. \end{cases}$$

As an application of this result, we give

$$-\mathrm{div}((1+|u|)^q \frac{\exp(|\nabla u|^p)-1}{|\nabla u|^2}\nabla u) + u(\exp(|\nabla u|^p)-1) = f - \mathrm{div}F \quad in \ \Omega.$$

with $p > 1$ *and* $q > 0$.

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A. Youssfi
Département de Mathématiques et Informatique
Faculté des Sciences
Dhar-Mahraz
B.P 1796 Atlas
Fès Morocco
Morocco
e-mail: Ahmed.youssfi@caramail.com