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A NOTE ON KKT-INVEXITY IN NONSMOOTH CONTINUOUS-TIME OPTIMIZATION

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Abstract

We introduce the notion of KKT-invexity for nonsmooth continuous-time nonlinear optimization problems and prove that this notion is a necessary and sufficient condition for every KKT solution to be a global optimal solution.

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1. INTRODUCTION

We regard the continuous-time nonlinear programming problem

$$\left.\begin{aligned} &\text{minimize} \quad \phi(x) = \int_0^T f(t, x(t))dt, \\ &\text{subject to} \quad g(t, x(t)) \leq 0 \text{ a.e. in } [0, T],\ x \in X. \end{aligned}\right\} \qquad (CNP)$$

Here X is a nonempty open convex subset of the Banach space $L^n_\infty[0,T]$, $\phi : X \to \mathbf{R}$, $f(t, x(t)) = \xi(x)(t)$, $g(t, x(t)) = \gamma(x)(t)$, $\xi : X \to \Lambda^1_1[0,T]$ and $\gamma : X \to \Lambda^m_1[0,T]$, where $L^n_\infty[0,T]$ denotes the space of all n-dimensional vector-valued Lebesgue measurable functions defined on the compact interval $[0,T] \subset \mathbf{R}$, which are essentially bounded, with norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1\leq j\leq n} \operatorname{ess\,sup}\{|x_j(t)|,\ 0 \leq t \leq T\},$$

where for each $t \in [0,T]$, $x_j(t)$ is the jth component of $x(t) \in \mathbf{R}^n$ and $\Lambda^m_1[0,T]$ denotes the space of all m-dimensional vector functions which are essentially bounded and Lebesgue measurable, defined on $[0,T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1\leq j\leq m} \int_0^T |y_j(t)|dt.$$

The continuous problem was first investigated in 1953 by Bellman in [1]. He studied a type of optimization problem, which is now known as a continuous-time linear problem. After that, various authors have studied more general continuous-time problems, regarding, for example, nonlinear problems. In [9], Zalmai obtained Karush-Kuhn-Tucker conditions of optimality. The results by Zalmai are natural generalizations of the KKT conditions in finite dimension. The nonsmooth problem was considered, for instance, in Brandão et al. [2] and Rojas-Medar et al. [7]. A good list of references about continuous-time problems can be found in [9].

The notion of invexity was introduced in [4] by Hanson. This concept, which generalize convexity, is important on getting sufficient conditions of optimality. In the work [5], Martin relaxed invexity. He introduced the notion of KKT-invexity (in fact he called it KT-invexity), which is (like invexity) a sufficient condition for a KKT point to be a global minimizer. But what is interesting in the Martin's result is that KKT-invexity is also a necessary condition of optimality. Martin showed that every KKT point is a global minimizer if and only if the problem is KKT-invex. In [6], de Oliveira and Rojas-Medar obtained a similar result for the continuous-time problem, but with smooth functions. In this work we generalize the result of de Oliveira and Rojas-Medar for the nonsmooth case.

2. ASSUMPTIONS AND NOTATION

Let V be an open convex subset of $\mathbf{R}^n$ containing the set $\{x(t) \in \mathbf{R}^n : x \in X,\ t \in [0,T]\}$.

We assume that f and g_i (the ith component of g), $i \in I = \{1, 2, \ldots, m\}$, are real functions defined on $V \times [0,T]$.

The functions $t \mapsto f(t, x(t))$ and $t \mapsto g(t, x(t))$ are assumed to be Lebesgue measurable and integrable for all $x \in X$.

We assume that, given $a \in V$, there exist an $\varepsilon > 0$ and a positive number k such that for all $t \in [0,T]$, and for all $x, y \in a + \varepsilon B$ (B denotes the unit ball of $\mathbf{R}^n$) we have $|f(t,x) - f(t,y)| \leq k\|x - y\|$. Similar hypotheses are assumed for $g_i,\ i \in I$. Hence, $f(t,\cdot)$ and $g_i(t,\cdot),\ i \in I$, are locally Lipschitz on V throughout $[0,T]$.

Let $x \in X$ and $h \in L^n_\infty[0,T]$. We denote by $\phi^\circ(x;h)$ and $g_i^\circ(t, x(t); h(t)),\ i \in I$, the Clarke generalized directional derivative of ϕ and $g_i,\ i \in I$, at x on the direction h, respectively. See Clarke [3] for more details.

Let $\mathbf{F}$ be the set of all feasible solutions of (CNP) (which we suppose nonempty), i.e.,

$$\mathbf{F} = \{x \in X : g(t, x(t)) \leq 0 \text{ a.e. in } [0,T]\}.$$

Given $x \in \mathbf{F}$, we denote by $A_i(x)$ the subset of $[0,T]$ where the ith constraint is active, i.e.,

$$A_i(x) = \{t \in [0,T] : g_i(t, x(t)) = 0\}.$$

3. INVEX CHARACTERIZATION OF KKT SOLUTIONS

In [5] Martin introduced the notion of KKT-invexity for mathematical programming problems and proved that every KKT point is a global minimizer if and only if the problem is KKT-invex. In this section we extend this concept for (CNP) and get a similar result.

Definition 3.1. *We say that (CNP) is Karush-Kuhn-Tucker invex (or KKT-invex) if there exists a function $\eta : [0,T] \times V \times V \to \mathbf{R}^n$ such that*

$\eta(t, x(t), y(t)) \in L^n_\infty[0, T]$,

$$\phi(x) - \phi(y) \geq \phi^\circ(y; \eta(x, y)), \tag{3.1}$$

and

$$- g_i^\circ(t, y(t); \eta(t, x(t), y(t)) \geq 0 \text{ a.e. } in\ A_i(y),\ i \in I, \tag{3.2}$$

for all $x, y \in \mathbf{F}$.

Remark 3.2. *By* $\eta(x, y)$ *in (3.1) we mean the map from* $X \times X$ *into* $L^n_\infty[0, T]$ *given by* $\eta(x, y)(t) = \eta(t, x(t), y(t))$.

Remark 3.3. *The definition of invexity differs from the KKT-invexity one by the requirement that* $g_i(t, x(t)) - g_i(t, y(t)) \geq g_i^\circ(t, y(t); \eta(t, x(t), y(t))$ a. e. *in* $A_i(y),\ i \in I$, *instead of (3.2).*

Definition 3.4. *We say that* $y \in \mathbf{F}$ *is a Karush-Kuhn-Tucker solution (or KKT solution) of (CNP) if there exist* $\lambda_i \in L_\infty[0, T],\ i \in I$, *such that*

$$\phi^\circ(y; h) + \int_0^T \sum_{i \in I} \lambda_i(t) g_i^\circ(t, y(t); h(t)) dt \geq 0\ \forall\ h \in L^n_\infty[0, T], \tag{3.3}$$

$$\lambda_i(t) g_i(t, y(t)) = 0 \text{ a.e. } in\ [0, T],\ i \in I, \tag{3.4}$$

$$\lambda_i(t) \geq 0 \text{ a.e. } in\ [0, T],\ i \in I. \tag{3.5}$$

Definition 3.5. *We say that* $y \in \mathbf{F}$ *is a global optimal solution of (CNP) if* $\phi(x) \geq \phi(y)$ *for all* $x \in \mathbf{F}$.

In the next example we study a KKT-invex problem which is not an invex one, where hold the property that every KKT solution is a global optimal solution. So, this example shows that invexity, despite being sufficient, is not a necessary condition to hold such property.

Example 3.6. *Let us consider the following nonlinear continuous-time problem:*

$$\begin{array}{ll} \textit{minimize} & \phi(x) = \displaystyle\int_0^2 f(x(t)) dt \\ \textit{subject to} & g(x(t)) \leq 0 \text{ a.e. } in\ [0, 2],\ x \in L_\infty[0, 2], \end{array}$$

where $f, g : \mathbf{R} \to \mathbf{R}$ *are given respectively by*

$$f(x) = \begin{cases} 1 - \exp(-x) \text{ if } x \geq 0, \\ -x^2 \text{ if } x < 0 \end{cases}$$

and $g(x) = -x$. *Let* $x, h \in \mathbf{R}$. *Is is easy to see that* f *is Clarke regular (see [3]) and*

$$f^{\circ}(x; h) = \begin{cases} \exp(-x)h \text{ if } x > 0, \\ -2xh \text{ if } x < 0, \\ h \text{ if } x = 0 \text{ and } h \geq 0, \\ 0 \text{ if } x = 0 \text{ and } h < 0. \end{cases}$$

Also, $g^{\circ}(x; h) = -h$ *for all* $x, h \in \mathbf{R}$.

Let $\bar{x}(t) = 0 \in L_{\infty}[0, 2]$ *and* $\bar{\lambda}(t) = 1 \in L_{\infty}[0, 2]$. *We have that*

$$\phi^{\circ}(\bar{x}; h) + \int_0^2 \bar{\lambda}(t) g^{\circ}(\bar{x}(t); h(t)) dt = \int_0^2 [f^{\circ}(0; h(t)) - h(t)] dt \geq 0 \ \forall\ h \in L_{\infty}[0, 2].$$

So $\bar{x} = 0$ *is a KKT solution. Let us suppose that* $y(t) > 0$ *a.e. in* $P \subseteq [0, 2]$ *is another KKT solution, where* P *has positive measure. Then there exists* $\lambda \in L_{\infty}[0, 2]$, $\lambda(t) \geq 0$ *a.e. in* $[0, 2]$, *satisfying*

$$\phi^{\circ}(y; h) + \int_0^2 \lambda(t) g^{\circ}(y(t); h(t)) dt \geq 0 \ \forall\ h \in L_{\infty}[0, 2], \tag{3.6}$$

$$\lambda(t) g(y(t)) = 0 \text{ a.e. in } [0, 2]. \tag{3.7}$$

Let $\hat{h} : [0, 2] \to \mathbf{R}$ *be defined by*

$$\hat{h}(t) = \begin{cases} -1 & \text{if } t \in P, \\ 0 & \text{if } t \notin P. \end{cases}$$

It is clear that $\hat{h} \in L_{\infty}[0, 2]$. *From (3.7) we see that* $\lambda(t) = 0$ *a.e. in* P. *Therefore from (3.6) it comes*

$$\begin{aligned} 0 &\leq \phi^{\circ}(y; \hat{h}) + \int_0^2 \lambda(t) g^{\circ}(y(t); \hat{h}(t)) dt \\ &= \phi^{\circ}(y; \hat{h}) - \int_0^2 \lambda(t) \hat{h}(t) dt = -\int_P \exp(-y(t)) dt, \end{aligned}$$

what is an absurd. Thus $\bar{x} = 0$ *is the only KKT solution of this problem.*

It is clear that $\phi(x) \geq \phi(0)$ *for all* $x \in \mathbf{F} = \{x \in L_{\infty}[0, 2] : x(t) \geq 0$ *a.e. in* $[0, 2]\}$. *Thus every KKT solution is a global optimal solution.*

This problem is not invex. Indeed, if we assume that it is invex we get a contradiction as follows. Suppose that the problem is invex. Then there exist $\eta : [0,2] \times V \times V \to \mathbf{R}$ *such that* $t \mapsto \eta(t, x(t), y(t)) \in L_\infty[0,2]$,

$$\phi(x) - \phi(y) \geq \int_0^2 f^\circ(y(t); \eta(t, x(t), y(t)))dt$$

and

$$-x(t) + y(t) \geq -\eta(t, x(t), y(t)) \text{ a.e. } \textit{in } [0,2]$$

for all $x, y \in L_\infty[0,2]$. *Using the last inequality, it is not difficult to verify that* $f^\circ(y(t); x(t) - y(t)) \leq f^\circ(y(t); \eta(t, x(t), y(t)))$ a.e. *in* $[0,2]$ *for all* $x, y \in L_\infty[0,2]$. *Hence*

$$\begin{aligned} &\phi(x) - \phi(y) - \int_0^2 f^\circ(y(t); x(t) - y(t))dt \\ &\qquad \geq \phi(x) - \phi(y) - \int_0^2 f^\circ(y(t); \eta(t, x(t), y(t)))dt \geq 0 \end{aligned} \tag{3.8}$$

for all $x, y \in L_\infty[0,2]$. *For* $x(t) = 0$ *and* $y(t) = t$ *in* $[0,2]$ *we have*

$$\phi(x) - \phi(y) - \int_0^2 f^\circ(y(t); x(t) - y(t))dt = -4\exp(-2) < 0,$$

which contradicts (3.9).

Now we show that this problem is KKT-invex. Define $\eta : V \times V \to \mathbf{R}$ *by*

$$\eta(x,y) = \begin{cases} \exp(y)(f(x) - f(y)) \text{ if } y > 0, \\ (-2y)^{-1}(f(x) - f(y)) \text{ if } y < 0, \\ f(x) - f(y) \text{ if } y = 0. \end{cases}$$

Let $x, y \in \mathbf{F}$ *and* $t \in [0,2]$. *We have that*

$$f^\circ(y(t); \eta(x(t), y(t))) = \begin{cases} \exp(-y(t))\eta(x(t), y(t)) = f(x(t)) - f(y(t)) \text{ if } y(t) > 0, \\ -2y(t)\eta(x(t), y(t)) = f(x(t)) - f(y(t)) \text{ if } y(t) < 0, \\ \eta(x(t), y(t)) = f(x(t)) - f(y(t)) \text{ if } y(t) = 0, \end{cases}$$

so that

$$\phi(x) - \phi(y) - \int_0^2 f^\circ(y(t); \eta(x(t), y(t)))dt = 0$$

and for $t \in A(y) = \{t \in [0,2] : y(t) = 0\}$,

$$-g^\circ(y(t); \eta(x(t), y(t))) = 1 - \exp(-x(t)) \geq 0.$$

Therefore this problem is KKT-invex.

Different of the finite dimensional case, here we need of a constraint qualification.

Definition 3.7. *We say that the constraint g satisfies (CQ) at $y \in \mathbf{F}$ if there do not exist $u_i \in L_\infty[0,T],\ u_i \geq 0,\ i \in I$, not all zero, such that*

$$\sum_{i \in I} \int_{A_i(y)} u_i(t) g_i^\circ(t, y(t); h(t)) dt \geq 0 \ \forall\ h \in L_\infty^n[0,T].$$

Lemma 3.8. *Let $y \in \mathbf{F}$ and assume that g satisfies (CQ) at y. If y is not a KKT solution of (CNP) then there exists $h \in L_\infty^n[0,T]$ such that*

$$\phi^\circ(y; h) < 0, \tag{3.9}$$

$$g_i^\circ(t, y(t); h(t)) \leq 0 \ \text{ a.e. in } A_i(y),\ i \in I. \tag{3.10}$$

Proof. If the system in (3.9) and (3.10) does not have a solution, particularly, the system

$$\phi^\circ(y; h) < 0,$$
$$\chi_i(t) g_i^\circ(t, y(t); h(t)) \leq 0 \ \text{ a.e. in } [0,T],\ i \in I,$$

does not have a solution, where $\chi_i : [0,T] \to \mathbf{R}$ is defined for each $i \in I$ by

$$\chi_i(t) = \begin{cases} 1 & \text{if } t \in A_i(y), \\ 0 & \text{if } t \notin A_i(y). \end{cases}$$

It follows from Corollary 3.1 on page 134 of [8], that there exist $u_0 \in \mathbf{R}$ and $u_i \in L_\infty[0,T],\ i \in I$, with $u_0 \geq 0$ and $u_i(t) \geq 0$ a.e. in $[0,T],\ i \in I$, not all zero, such that

$$u_0 \phi^\circ(y; h) + \int_0^T \sum_{i \in I} u_i(t) \chi_i(t) g_i^\circ(t, y(t); h(t)) dt \geq 0 \ \forall\ h \in L_\infty^n[0,T]. \tag{3.11}$$

If $u_0 = 0$ we have a contradiction with the constraint qualification. Therefore $u_0 > 0$. Then dividing the expression in (3.11) by u_0 and defining $\lambda_i = u_i \chi_i / u_0,\ i \in I$, we obtain

$$\phi^\circ(y; h) + \int_0^T \sum_{i \in I} \lambda_i(t) g_i^\circ(t, y(t); h(t)) dt \geq 0 \ \forall\ h \in L_\infty^n[0,T].$$

Thus y is a KKT solution, what contradicts the hypothesis. Hence, there exists $h \in L_\infty^n[0,T]$ satisfying (3.9) and (3.10). $\square$

Theorem 3.9. *Assume that g satisfies (CQ) at each $y \in \mathbf{F}$. Then, every KKT solution of (CNP) is a global optimal solution if and only if (CNP) is KKT-invex.*

Proof. *Necessity.* Suppose that every KKT solution of (CNP) is a global optimal solution. Let $x, y \in \mathbf{F}$.

If $\phi(x) < \phi(y)$, then y is not a global optimal solution, and so, by hypothesis, y is not a KKT solution of (CNP). It follows from Lemma 3.8 that there exists $h \in L_\infty^n[0,T]$ satisfying (3.9) and (3.10). Set

$$\alpha = \phi^\circ(y; h)$$

and

$$\eta(t, x(t), y(t)) = \{\phi(x) - \phi(y)\}\alpha^{-1} h(t).$$

Because of (3.9) we know that

$$\{\phi(x) - \phi(y)\}\alpha^{-1} > 0. \tag{3.12}$$

Hence

$$\phi^\circ(y; \eta(x, y)) = \phi^\circ(y; \{\phi(x) - \phi(y)\}\alpha^{-1} h) = \{\phi(x) - \phi(y)\}\alpha^{-1}\phi^\circ(y; h),$$

and therefore

$$\phi^\circ(y; \eta(x, y)) = \phi(x) - \phi(y). \tag{3.13}$$

From (3.10) and (3.12) we get

$$\begin{aligned} g_i^\circ(t, y(t); \eta(t, x(t), y(t))) &= \{\phi(x) - \phi(y)\}\alpha^{-1} g_i^\circ(t, y(t); h(t)) \\ &\leq 0 \text{ a.e. in } A_i(y), \; i \in I. \end{aligned}$$

Hence

$$-g_i^\circ(t, y(t); \eta(t, x(t), y(t))) \geq 0 \text{ a.e. in } A_i(y), \; i \in I. \tag{3.14}$$

By (3.13) and (3.14) we conclude that for $\phi(x) < \phi(y)$, (CNP) is KKT-invex.

If $\phi(x) \geq \phi(y)$, define $\eta(t, x(t), y(t)) = 0$ a.e. in $[0, T]$. We have that

$$\phi(x) - \phi(y) - \phi^\circ(y; \eta(x, y)) \geq 0 \tag{3.15}$$

and

$$g_i^\circ(t, y(t); \eta(t, x(t), y(t))) = 0 \text{ a.e in } A_i(y), \; i \in I. \tag{3.16}$$

So, from (3.15) and (3.16) we see that (CNP) is KKT-invex.

In the cases above we do not define η for $x, y \notin \mathbf{F}$. But we can take $\eta(t, x(t), y(t)) = 0$ when x or y is not feasible.

Sufficiency. Suppose that (CNP) is KKT-invex. Let $y \in \mathbf{F}$ be a KKT solution of (CNP). It follows from (3.4) that $\lambda_i(t) = 0$ a.e. in $[0, T] \setminus A_i(y)$, $i \in I$. Then by (3.1), (3.2) and (3.5) we have

$$\phi(x) - \phi(y) - \phi^{\circ}(y; \eta(x, y)) - \int_0^T \sum_{i \in I} \lambda_i(t) g_i^{\circ}(t, y(t); \eta(t, x(t), y(t))) dt \geq 0$$

for all $x \in \mathbf{F}$. So, by (3.3) we obtain $\phi(x) \geq \phi(y)$ for all $x \in \mathbf{F}$, that is, y is a global optimal solution of (CNP). □

Remark 3.10. *We observe that the assumption that g satisfies (CQ) in the last theorem is necessary just on proving the "only if" part.*

Remark 3.11. *If $f(t, \cdot)$ and $g(t, \cdot)$ are Clarke regular at y, then (3.3) in Definition 3.4 can be replaced by*

$$0 \in \partial L(y, \lambda),$$

where

$$L(x, \lambda) = \int_0^T \left[f(t, x(t)) + \sum_{i \in I} \lambda_i(t) g_i(t, x(t)) \right] dt.$$

Theorem 3.9, of course, is still valid.

Remark 3.12. *It is well known that a convex function is Clarke regular. In particular, when $f(t, \cdot)$ and $g(t, \cdot)$ are convex at $y \in X$ throughout $[0, T]$ they are regular at y and we have*

$$\partial L(y, \lambda) = \int_0^T \left[\partial f(t, y(t)) + \sum_{i \in I} \lambda_i \partial g_i(t, y(t)) \right] dt. \tag{3.17}$$

An interesting open problem is to know if the relation (3.17) is still true when $f(t, \cdot)$ and $g(t, \cdot)$ are invex at y throughout $[0, T]$. When we have a finite sum instead of an integral, we verified that this is true.

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