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REGULARITY AND AMENABILITY OF THE SECOND DUAL OF WEIGHTED GROUP ALGEBRAS

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Abstract

For a wide variety of Banach algebras A (containing the group algebras $L^1(G), M(G)$ and $A(G)$) the Arens regularity of A^{**} is equivalent to that A, and the amenability of A^{**} is equivalent to the amenability and regularity of A. In this paper, among other things, we show that this variety contains the weighted group algebras $L^1(G, w)$ and $M(G, w)$.

Keywords : *Arens product, Weighted group algebra, Amenability*

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1. Introduction

Over fifty years ago, Arens in his elaborate work [A], pointed out that, for every Banach algebra A, there exist two (Arens) products $\circ$ and $\diamond$ on the second dual A^{**}, extending the product of A. If these two products coincide on A^{**}, then A is said to be (Arens) regular. For further details on the properties of Arens products see the survey article [D-H]. It is readily verified that the regularity of A^{**} (equipped with either $\circ$ or $\diamond$) implies that of A; therefore $(A^{**}, \circ)$ is regular if and only if $(A^{**}, \diamond)$ is regular. However it has been shown in [Y3] that there exists a regular Banach algebra whose second dual is not regular; for a more simple example of such a Banach algebra see [P]. Every C^*-algebra is regular [S], and its second dual is a von Neumann algebra, and so is regular. As a consequence of Young's result [Y1], (which asserts $L^1(G)$ is regular if and only if G is finite) the regularity of $L^1(G)^{**}$ is equivalent to the regularity of $L^1(G)$. For a commutative, semisimple, completely continuous and weakly sequentially complete Banach algebra A whose dual A^* is a von Neumann algebra (for instance, for the Fourier algebra $A(G)$), it has been shown in [U2] that the regularity of A^{**} is equivalent to that of A.

A Banach algebra A is said to be amenable (resp. weakly amenable) if every continuous derivation $D : A \to X^*$ (resp. $D : A \to A^*$) is inner for every Banach $A-$module X. It has been shown in [Go] (see also [G-L-W]) that, if either $(A^{**}, \circ)$ or $(A^{**}, \diamond)$ is amenable then so is A. However, for an infinite amenable group G, $L^1(G)$ is amenable, but $L^1(G)^{**}$ is not; indeed in [G-L-W] they showed that $L^1(G)^{**}$ is amenable if and only if G is finite. For the Fourier algebra $A(G)$ it is known that, $A(G)^{**}$ is amenable if and only if G is finite, see [Gra]. Although, one can use the earlier result of Forrest and Runde [F-R], to give a simple proof for the latter fact; (indeed, if $A(G)^{**}$ is amenable then so is $A(G)$, and the main result of [F-R] implies that, G has an abelian subgroup H of finite index. It induces an epimorphism from $A(G)^{**}$ on $A(H)^{**}$, in particular $A(H)^{**} = L^1(\hat{H})^{**}$ is amenable. It follows by [G-L-W], that $\hat{H}$ is finite, and so G is finite.)

If A is commutative or if it possesses a continuous involution then as it is shown in [G-L], the amenability (resp. weak amenability) of $(A^{**}, \circ)$ is equivalent to that of $(A^{**}, \diamond)$. It seems still not known if there exists a Banach algebra A for which the amenability of $(A^{**}, \circ)$ is not equivalent to that of $(A^{**}, \diamond)$.

The main theme of this paper is to investigate the regularity and amenability of the second dual of the weighted group algebras $L^1(G,w)$ and $M(G,w)$.

2. Preliminaries

Throughout this paper, G is a locally compact (topological) group, and w is a weight on G; (which is a continuous function $w : G \to (0,\infty)$ with $w(xy) \le w(x)w(y)$, for all $x, y \in G$) , for convenience we shall assume that $w(e) = 1$, where e is the identity of G. We define $\Omega : G \times G \to (0,1]$ by $\Omega(x,y) = w(xy)/w(x)w(y)$.
A function $h : X \times Y \to \mathbf{C}$ is said to be $0-$cluster if $\lim_n \lim_m h(x_n, y_m) = 0 = \lim_m \lim_m h(x_n, y_m)$ for every two sequences $\{x_n\} \subseteq X$ and $\{y_m\} \subseteq Y$ of distinct points, provided the involved limits exist.

We define w^* on G by $w^*(x) = w(x)w(x^{-1}), (x \in G)$. It can be simply verified that w^* is also a weight on G; moreover w^* is bounded on G if and only if w is semi-multiplicative (that is, there exists $c > 0$ such that $cw(x)w(y) \le w(xy)$, for all $x, y \in G$). Therefore, Ω can not be $0-$cluster when w^* is bounded.

Define $L^1(G,w), L^\infty(G,w), G_0(G,w)$ and $LUC(G,w)$ as follows:

$$\begin{aligned} L^1(G,w) &= \{f : fw \in L^1(G)\}, \\ L^\infty(G,w) &= \{f : f/w \in L^\infty(G)\}, \\ C_0(G,w) &= \{f : f/w \in C_0(G)\}\ , and \\ LUC(G,w) &= \{f : f/w \in LUC(G)\}. \end{aligned}$$

We norm these spaces in such a way the multiplication or division by w becomes an isometry between the non-weighted and the corresponding weighted spaces (whose norm will denote by $\|\cdot\|_w$). Thus the non-weighted and the corresponding weighted spaces are isometrically isomorphic as Banach spaces, but quite different as Banach algebras. Recall the inclusion relations of non-weighted cases of these spaces and the fact that $L^1(G)^* = L^\infty(G)$, we have:

$$C_0(G,w) \subseteq LUC(G,w) \subseteq L^\infty(G,w) = L^1(G,w)^*.$$

We refer the reader to [R2], for more study of different subalgebras of $L^\infty(G,w)$, and their equalities.

We define $M(G,w)$ such that $M(G,w)$ becomes isometric isomorphic to the Banach space $C_0(G,w)^*$. For this sake, let $M^+(G,w)$ be the set of all positive regular measures on G for which μw is again a positive regular measure on G; where $d(\mu w) = w d\mu$. Define an equivalence relation on $M^+(G,w)\times M^+(G,w)$ by $(\mu_1,\nu_1)\sim(\mu_2,\nu_2)$ if and only if $\mu_1+\nu_2=\mu_2+\nu_1$. Now define $M(G,w)$ by

$$M(G,w)=\{[\mu,\nu]:\mu,\nu\in M^+(G,w)\},$$

where $[\mu,\nu]$ is the equivalence class of (μ,ν). For a full discussion on $M(G,w)$ from this point of view and the fact that $C_0(G,w)^*=M(G,w)$ see [R1] and also [B].

It should be remarked that, if w is multiplicative (i.e. $w(xy)=w(x)w(y)$, for all $x,y\in G$, or equivalently, w is a positive character on G) then $L^1(G,w)\cong L^1(G)$ and $M(G,w)\cong M(G)$ as Banach algebras. Indeed it can be readily verified that $f\to fw$ and $[\mu,\nu]\to \mu w-\nu w$ are algebra isomorphism from $L^1(G,w)$ on $L^1(G)$ and $M(G,w)$ on $M(G)$, respectively.

As a ground reference for the second dual of weighted group algebras, one may refer to [D-L].

3. Main Results

We start with the next lemma.

Lemma 1. *If G is infinite (discrete) and Ω is $0-$cluster, then $F\circ G = 0 = F\diamond G$, for every $F,G\in l^1(G,w)^{**}\setminus l^1(G,w)$.*

Proof. Since Ω is 0-cluster, the mapping $(x,y)\to(\frac{\phi}{w})(xy)\Omega(x,y)$ is $0-$cluster for every $\phi\in L^\infty(G,w)$. Using the Example 2 in page 312 of [Y2], the mapping $(f,g)\to\phi(f\star g)=\sum\sum(\frac{\phi}{w})(xy)(fw)(x)(gw)(y)\Omega(x,y)$ is $0-$cluster on $l^1(G,w)\times l^1(G,w)$ (the sums are taken on $x,y\in G$). Now for $F,G\in l^1(G,w)^{**}\setminus l^1(G,w)$ there exist two nets $\{f_\alpha\}$ and $\{g_\beta\}$, each consisting of distinct points in $l^1(G,w)$ such that $f_\alpha\longrightarrow F$ and $g_\alpha\longrightarrow G$, in the weak* topology, with

$< F\circ G,\phi >= \lim_\alpha\lim_\beta \phi(f_\alpha\star g_\beta)$ and $< F\diamond G,\phi >= \lim_\beta\lim_\alpha \phi(f_\alpha\star g_\beta)$,

for every $\phi\in L^\infty(G,w)$. One can construct two subsequences $\{f_{\alpha_m}\}$ and $\{g_{\beta_n}\}$ of $\{f_\alpha\}$ and $\{g_\beta\}$, respectively, such that, $< F\circ G,\phi >= \lim_m\lim_n\phi(f_{\alpha_m}\star g_{\beta_n}) = 0 = \lim_n\lim_m\phi(f_{\alpha_m}\star g_{\beta_n}) =< F\diamond G,\phi >$, as required.

Now, we come to the one of the main results.

Theorem 2. *The following statements are equivalent.*

(i) $L^1(G,w)$ *is regular,*

(ii) G *is finite or* G *is discrete and* Ω *is* $0-$*cluster,*

(iii) $L^1(G,w)^{**}$ *is regular.*

Proof. For (i)$\Rightarrow$(ii), suppose that $L^1(G,w)$ be regular. Since $L^1(G,w)$ is weakly sequentially complete and admits a bounded approximate identity, it is unital by theorem 3.3 of [U1]. Therefore G is discrete. If G is infinite, then by corollary 3.8 of [B-R] Ω must be $0-$cluster. For (ii)$\Rightarrow$(iii), if G is finite then $L^1(G,w)$ is reflexive; for the infinite case (iii) follows from Lemma1.

Suppose that G admits a multiplicative weight bounded by w, (for instance, it is the case if either $1 \leq w$ or G is amenable (as a group) , for the latter see Lemma 1 of [W]). Then, there exists a unique multiplicative weight on G which is equivalent to w, provided w^* is bounded. Indeed, $\varphi(x) = \lim_{n\to\infty} w(x^n)^{1/n}$ defines a multiplicative weight on G with $\varphi \leq w \leq c\varphi$, in which $c = \sup_{x\in G} w^*(x)$; see [W] for further details. In particular, $L^1(G,w) = L^1(G,\varphi) \cong L^1(G)$ and $M(G,w) = M(G,\varphi) \cong M(G)$.

An elegant result of [Gro] states $L^1(G,w)$ is amenable if and only G is amenable and w^* is bounded. Therefore, $L^1(G,w)$ is amenable if and only if G is amenable and $L^1(G,w) \cong L^1(G)$. Recently, it has been proved in [D-G-H] that $M(G)$ is amenable if and only if G is amenable and discrete. As a weighted version of this we have;

Proposition 3. $M(G,w)$ *is amenable if and only if* G *is amenable, discrete and* w^* *is bounded.*

Proof. If $M(G,w)$ is amenable, then $L^1(G,w)$ is amenable, therefore G is amenable and w^* is bounded; and so by the discussion just before the proposition, there exists a unique multiplicative weight on G equivalent to w. It implies that, $M(G,w) \cong M(G)$. In particular, $M(G)$ is amenable. By [D-G-H] G must be discrete. Since in the discrete setting $M(G,w) = L^1(G,w)$, the converse follows from [Gro]

As the second main result we have the next which is an extension of Theorem 1.3 of [G-L-W].

Theorem 4. *The following statements are equivalent.*

(i) $L^1(G,w)^{**}$ *is amenable,*

(ii) $L^1(G,w)$ *is amenable and regular,*

(iii) $L^1(G,w)$ *is regular and* w^* *is bounded,*
(iv) $L^1(G,w)$ *is reflexive and* w^* *is bounded,*
(v) $L^1(G,w)$ *is a* C^*-*algebra,*
(vi) G *is finite.*

Proof. Trivially (vi) implies the other parts. If $L^1(G,w)^{**}$ is amenable, then so is $L^1(G,w)$, and so $L^1(G,w) \cong L^1(G)$. Now the amenability of $L^1(G)^{**}$ necessitates G must be finite by Theorem 1.3 of [G-L-W]. Thus (i) $\Rightarrow$ (vi) follows. The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii) are obvious. Let $L^1(G,w)$ be regular and w^* be bounded; therefore Ω can not be $0-$cluster and by Theorem 2, G is finite. Assume that $L^1(G,w)$ is a C^*-algebra; then it is regular and so G is discrete. Moreover, the equality $\|\delta_x * \delta_x^*\|_w = \|\delta_x\|_w^2$, for every $x \in G$ implies that $w(x) = \Delta(x)^{1/2}$, for each $x \in G$ (Δ is the modular function of G), and this implies that w is multiplicative and so $\Omega = 1$. Now Theorem 2 implies that G is finite and this completes the proof.

Remarks. (i) The conclusions of Theorems 2 and 4 remains valid if we replace $L^1(G,w)$ by $M(G,w)$.

(ii) For a Banach algebra A if $A^{***} \cdot F = A^* \diamond F$, for every $F \in A^{**}$, (where $< m \cdot F, G > = < m, F \circ G >$ for every $m \in A^{***}, F, G \in A^{**}$) then it is not hard to prove that the regularity of A^{**} is equivalent to that of A; (indeed if A is regular, then for every $f \in A^*$, the mapping $F \to f \diamond F : A^{**} \to A^*$ is weakly compact, and the equality $A^{***} \cdot F = A^* \diamond F$ implies that for every $\Phi \in A^{***}$ the mapping $F \to \Phi \cdot F : A^{**} \to A^{***}$ is weakly compact, which is equivalent to the regularity of A^{**}). Using this fact, one may give a different proof to the Theorem 2.

(iii) For a Banach algebra A with a bounded approximate identity of norm one, A^*A is a closed subspace of A^*, and $A^{**} = (A^*A)^* \oplus (A^*A)^{\perp}$ (as Banach spaces), where $(A^*A)^{\perp} = \{F \in A^{**} : A^{**} \circ F = 0\}$ is a closed ideal of $(A^{**}, \circ)$ and $(A^*A)^*$ is a closed subalgebra of $(A^{**}, \circ)$. These observations together with the Lemma 2.3 of [L-L] imply that; if $(A^{**}, \circ)$ is weakly amenable then so is $(A^*A)^*$. Now for $A = L^1(G,w)$ it has been shown in Proposition 1.3 of [Gro] that $A^*A = LUC(G,w)$. On the other hand, using the methods of Lemma 1.1 of [G-L-L], we have $LUC(G,w)^* = M(G,w) \oplus C_0(G,w)^{\perp}$, and that $M(G,w)$, $C_0(G,w)^{\perp}$ are closed subalgebra and closed ideal of $LUC(G,w)^*$, respectively. Again use Lemma 2.3 of [L-L] the weak amenability of $L^1(G,w)^{**}$ implies that of $M(G,w)$, which is an extension of Proposition 4.14 in [L-L].

(iv) The existing examples support the conjecture that, for a Banach algebra A if A^{**} is amenable then A is regular.

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