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QUASI - MACKEY TOPOLOGY

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Abstract

Let E_1, E_2 be Hausdorff locally convex spaces with E_2 quasi-complete, and $T : E_1 \to E_2$ a continuous linear map. Then T maps bounded sets of E_1 into relatively weakly compact subsets of E_2 if and only if T is continuous with quasi-Mackey topology on E_1. If E_1 has quasi-Mackey topology and E_2 is quasi-complete, then a sequentially continuous linear map $T : E_1 \to E_2$ is an unconditionally converging operator.

Keywords : *quasi - Mackey topology, weakly unconditionally Cauchy, unconditionally converging operators.*

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1. Introduction and Notation

In this paper, for locally convex spaces, the notations are results of ([5]) are used. All vector spaces are over the field of real numbers. N and R will stand, respectively, for the set of natural numbers and real numbers. For a locally convex space E, E' and E'' will denote its dual and bidual respectively. The topology induced on E by $(E'', \tau(E'', E'))$ will be called quasi-Mackey ([3]) (in [4], for a Banach space E, this tology is called the "Right topology").

In [4], using quasi-Mackey topology, some interesting results about weak compactness of linear mappings between Banach spaces are proved. Also a connection is established between quasi-Mackey topology and unconditionally convergent operators. In this paper, we extend these results to locally convex spaces. Our methods of proofs are different from [4].

2. Main Results

Theorem 1. *E_1, E_2 are two Hausdorff locally conves spaces, $T : E_1 \to E_2$ a continuous linear map such that $T(B)$ is relatively weakly compact in E_2 for every bounded set B in E_1. Then*
(i) the adjoint map $T' : (E_2', \tau(E_2', E_2)) \to (E_1', \beta(E_1', E_1))$ is continuous;
(ii) the adjoiint map of T' in (i), $T'' : (E_1'', \tau(E_1'', E_1)) \to (E_2, \tau(E_2, E_2'))$ is also continuous. As a result, the mapping T, with quasi-Mackey topology on E_1 and the given topology on E_2, is continuous.

Conversely suppose E_2 is quasi-complete, E_1 has quasi-Mackey topology and $T : E_1 \to E_2$ is a continuous linear map. Then T maps bounded subsets of E_1 into relatively weakly compact subsets of E_2.

Proof. To prove the continuity of $T' : (E_2', \tau(E_2', E_2)) \to (E_1', \beta(E_1', E_1))$, take a net $f_\alpha \to 0$ in $(E_2', \tau(E_2', E_2)$ and a absolutely convex, bounded set $B \subset E_1$. By hypothesis, the convex set $T(B)$ is relatively weakly compact in E_2 which means $f_\alpha \to 0$ uniformly on $T(B)$. So $T'(f_\alpha) \to 0$ in $(E_1', \beta(E_1', E_1))$. This proves (i). (ii) follows from ([5], Theorem 7.4, p.158).

Now we come to the converse. With given assumptions, E_1 is a dense subspace of $(E_1'', \tau(E_1'', E_1'))$ with induced topology. This means its completion $\widetilde{E}_1 \supset E_1''$. Let $\widetilde{T} : \widetilde{E}_1 \to \widetilde{E}_2$ be the unique continuous linear extension of T. Thus we have a continuous linear mapping $\widetilde{T}_{|E_1''} : \left(E_1'', \tau\left(E_1'', E_1'\right)\right) \to \widetilde{E}_2$ and so this mapping remains continous when weak topologies are assigned on both sides ([5], Theorem 7.4, p.158). Since for any bounded set

$B \subset E_1$, it closure in $(E_1'', \tau(E_1'', E_1'))$ is weakly compact and E_2 is quasi-complete, we get $T(B)$ is relatively weakly compact in E_2. This proves the result.

Before the next theorem, we make some comments about unconditionally converging operators.

Let $\sum x_n$ be a series in E. We denote by I the collection of all finite subsets of N; I is a directed set (by inclusion). For each $\alpha \in I$, put $s_\alpha = \sum_{i \in \alpha} x_i$. The series is said to be unconditionally convergent if the net $\{s_\alpha\}$ converges in E ([5], p. 120); The series will be called unconditionally Cauchy if this net in Cauchy. The following lemma is easily verified. The proof is omitted.

Lemma 2. *Let E be a Hausdorff locally convex space and $\sum x_n$ be a series in E. Then*
(i) if the series in unconditionally Cauchy in weak topology then $\sum |f(x_n)| < \infty$, $\forall f \in E'$
(ii) if the series is not unconditionally Cauchy in E, then, there is a continuous semi-norm $\|.\|$ on E, a $c > 0$, and a sequence $\{\alpha_n\} \subset I$ such that $\sup(\alpha_n) < \inf(\alpha_{n+1})$, $\forall n$ and $\|y_n\| > c$, $\forall n$ where $y_n = \sum_{i \in \alpha_n} x_i$.

We denote by $2^N = \{0,1\}^N$ all subsets of N.

Lemma 3. *Let E be a Hausdorff locally convex space with E' its dual. If a $\mu : 2^N \to E$ is countably additive in $(E, \sigma(E, E')$, then it also countably additive in $(E, \tau(E, E')$.*

Proof. This is well-known Orlicz-pettis theorem; for normed spaces E, this result is proved in ([1], p. 22); it has straight extension to locally convex spaces.

For locally convex spaces E_1 and E_2, a linear operator $T : E_1 \to E_2$ will be called unconditionally Cauchy if for any series $\sum x_n$, which is weakly unconditionally Cauchy in E_1, $\sum T(x_n)$, is unconditionally Cauchy in E_2 ([2]).

Theorem 4. *E_1, E_2 are two Hausdorff locally convex spaces with E_1 having quasi-Mackey topology and E_2 being quasi-complete. $T : E_1 \to E_2$ a sequentially continuos linear map. Then T is an unconditionally converging operator.*

Proof. Assume a series $\sum x_i$ in E_1 be unconditionally weakly Cauchy but $\sum T(x_i)$ is not an unconditionally Cauchy. In the notations of Lemma 2, there is a continuous semi-norm $\|.\|$ on E_2 and a $c > 0$, $\|T(y_n)\| > c$, $\forall n$. From the definition of y_n, $\sum y_n$ is also unconditionally weakly Cauchy in E_1. For notational convenience, we denote y_n again by x_n; thus $\|T(x_n)\| > c$, $\forall n$. In the notations introduced before Lemma 2, $B = \{s_\alpha : \alpha \in I\}$ is a bounded subset of E_1 and so its closure B in $(E_1'', \sigma(E_1'', E_1'))$ is compact. Using this and the the fact that $\sum x_i$ is unconditionally weakly Cauchy, we get that for any subset $M \subset N$, $s_M = \sum_{i\in M} x_i$ is convergent in $(E_1'', \sigma(E_1'', E_1'))$. This means the measure $\mu : 2^N \to (E_1'', \sigma(E_1'', E_1'))$, $\mu(M) = s_M$ is countably additive. By Lemma 3, it is countably additive in $\tau(E_1'', E_1')$. This means $x_n \to 0$ in E_1; since T is sequentially continuos, $T(x_n) \to 0$ in E_2. This is a contradiction. Thus $\sum T(x_i)$ is unconditionally Cauchy in E_2; since E_2 is quasi-complete, we get that $\sum T(x_i)$ is unconditionally convergent in E_2. This proves that T is an unconditionally converging operator.

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