Proyecciones
Vol. 27, N$^o$ 3, pp. 289–305, December 2008.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172008000300005

# POSITIVE MATRICES WITH PRESCRIBED SINGULAR VALUES *

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*Received : June 2008. Accepted : October 2008*

**Abstract**

*We consider the problem of constructing positive matrices with prescribed singular values. In particular, we show how to construct an $m \times n$ positive matrix, $m \geq n$, with prescribed singular values $\sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$.*

**AMS classification :** *15A18; 15A29.*

**Keywords :** *Singular values, positive matrices.*

*Supported by Fondecyt 1085125, Chile.

## 1. Introduction

A *singular value decomposition* of a matrix $A \in \mathbf{C}^{m\times n}$ is a factorization $A = U\Sigma V^*$, where $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_r) \in \mathbf{R}^{m\times n}$, $r = \min\{m, n\}$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$ and both $U \in \mathbf{C}^{m\times m}$ and $V \in \mathbf{C}^{n\times n}$ are unitary. The diagonal entries of $\Sigma$ are called *the singular values* of $A$. The columns $\mathbf{u}_j$ of $U$ are called *left singular vectors* of $A$ and the columns $\mathbf{v}_j$ of $V$ are called *right singular vectors* of $A$. Every $A \in \mathbf{C}^{m\times n}$ has a *singular value decomposition* $A = U\Sigma V^*$ and the following relations hold: $A\mathbf{v}_j = \sigma_j \mathbf{u}_j$, $A^*\mathbf{u}_j = \sigma_j \mathbf{v}_j$ and $\mathbf{u}_j^* A\mathbf{v}_j = \sigma_j$. If $A \in \mathbf{R}^{m\times n}$, then $U$ and $V$ may be taken to be real (see [3]).

In this paper we consider the following inverse singular value problem:
*Problem*: Given the real numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, to construct an $m \times n$, $m \geq n$, entrywise positive matrix $A$ with singular values $\sigma_i$, $i = 1, \ldots, n$.

In the case $m = n$, the problem has always a solution if $\sigma_1 > \sigma_2$. We construct such a solution from the following two results, due to Fiedler [2], and related with the *nonnegative inverse eigenvalue problem*:

**Lemma 1.** *[2] Let $\alpha > \beta \geq 0$, $\epsilon > 0$. Then there exists a number $c > 0$ such that the matrix*

$$\begin{pmatrix} \alpha & c \\ c & \beta \end{pmatrix} \quad \textit{has eigenvalues} \quad \alpha + \epsilon, \beta - \epsilon.$$

**Lemma 2.** *[2] Let $A$ be a symmetric $m \times m$ matrix with eigenvalues $\alpha_1, \ldots, \alpha_m$, and let $\mathbf{u}$, $\|\mathbf{u}\| = 1$, be a unit eigenvector corresponding to $\alpha_1$. Let $B$ be a symmetric $n \times n$ matrix with eigenvalues $\beta_1, \ldots, \beta_n$, and let $\mathbf{v}$, $\|\mathbf{v}\| = 1$, be a unit eigenvector corresponding to $\beta_1$. Then for any $\rho$, the matrix*

$$S = \begin{pmatrix} A & \rho\mathbf{u}\mathbf{v}^T \\ \rho\mathbf{v}\mathbf{u}^T & B \end{pmatrix}$$

*has eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2$, where $\gamma_1, \gamma_2$ are eigenvalues of the matrix*

$$\widehat{S} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

To construct a positive matrix, no necessaryly symmetric, with prescribed singular values, we first use the below result, Lemma 4, which shows how to construct an $n \times n$ nonnegative semibordered matrix

$$C = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, \tag{1.1}$$

with prescribed singular values and then, to obtain the required positive matrix, we apply a Brauer [1] type singular value perturbation result given in [4, Corollary 14], which shows how to modify a single singular value of an $n \times n$ matrix without changing any of the remaining singular values.

**Lemma 3.** *The characteristic polynomial of the arrow matrix*

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ b_2 & a_2 & & \\ \vdots & & \ddots & \\ b_n & & & a_n \end{pmatrix}$$

*is*

$$p(\lambda) = (b_1 - \lambda) \prod_{i=2}^{n} (a_i - \lambda) - \sum_{i=2}^{n} b_i^2 \prod_{\substack{k=2 \\ k \neq i}}^{n} (a_k - \lambda).$$

**Proof.** By expanding

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \begin{vmatrix} b_1 - \lambda & b_2 & b_3 & \cdots & b_n \\ b_2 & a_2 - \lambda & & & \\ b_3 & & a_3 - \lambda & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n - \lambda \end{vmatrix} \\ &= (b_1 - \lambda) \prod_{i=2}^{n} (a_i - \lambda) - \sum_{i=2}^{n} b_i^2 \prod_{\substack{k=2 \\ k \neq i}}^{n} (a_k - \lambda). \end{aligned}$$

□

**Lemma 4.** *Given the real numbers $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_n > 0$, then there exists an $n \times n$ nonnegative matrix $C$ of the form (1.1) with singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$.*

**Proof.** Let $\mathrm{A}=\begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix}$.

Then

$$\begin{aligned} B &= AA^T = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix} \begin{pmatrix} b_1 & & & & \\ b_2 & a_2 & & & \\ b_3 & & a_3 & & \\ \vdots & & & \ddots & \\ b_n & & & & a_n \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{n} b_i^2 & a_2b_2 & a_3b_3 & \cdots & a_nb_n \\ a_2b_2 & a_2^2 & & & \\ a_3b_3 & & a_3^2 & & \\ \vdots & & & \ddots & \\ a_nb_n & & & & a_n^2 \end{pmatrix} \end{aligned}$$

is an $n \times n$ arrow symmetric matrix with characteristic polynomial

$$(1.2) \quad p(\lambda) = \left(\sum_{i=1}^{n} b_i^2 - \lambda\right) \prod_{i=2}^{n} \left(a_i^2 - \lambda\right) - \sum_{i=2}^{n} (a_ib_i)^2 \prod_{j=2 j\neq i}^{n} \left(a_j^2 - \lambda\right)$$

and eigenvalues $\sigma_1^2, \sigma_2^2, ..., \sigma_n^2$. We define

$$a_k = \sqrt{\frac{\sigma_{k-1}^2 + \sigma_k^2}{2}}, \ k = 2, 3, ..., n.$$

Then, by taking $\lambda = \dfrac{\sigma_{k-1}^2 + \sigma_k^2}{2}$, $k = 2, 3, ...n$, in (1.2), we have

$$\prod_{i=2}^{n} \left(a_i^2 - \frac{\sigma_{k-1}^2 + \sigma_k^2}{2}\right) = 0$$

and

$$-\left(a_k b_k\right)^2 \prod_{\substack{j=2\\ j\neq k}}^{n}\left(a_j^2-\frac{\sigma_{k-1}^2+\sigma_k^2}{2}\right)=\prod_{i=1}^{n}\left(\frac{\sigma_{k-1}^2+\sigma_k^2}{2}-\sigma_i^2\right).$$

Hence,

$$b_k=\sqrt{-\frac{\prod_{i=1}^{n}\left(a_k^2-\sigma_i^2\right)}{a_k^2\prod_{\substack{i=2\\ i\neq k}}^{n}\left(a_i^2-a_k^2\right)}},\ k=2,3,...,n. \tag{1.3}$$

For $b_1$ we take $\lambda=\sigma_1^2$ in (1.2). Then

$$\left(b_1^2+\sum_{i=2}^{n}b_i^2-\sigma_1^2\right)\prod_{i=2}^{n}\left(a_i^2-\sigma_1^2\right)-\sum_{i=2}^{n}\left(a_i b_i\right)^2\prod_{\substack{j=2\\ j\neq i}}^{n}\left(a_j^2-\sigma_1^2\right)=p\left(\sigma_1^2\right)=0$$

and

$$\begin{aligned} b_1 &= \sqrt{\frac{1}{\prod_{i=2}^{n}\left(a_i^2-\sigma_1^2\right)}\left(\sum_{i=2}^{n}a_i^2b_i^2\prod_{\substack{j=2\\ j\neq i}}^{n}\left(a_j^2-\sigma_1^2\right)+\sigma_1^2\prod_{i=2}^{n}\left(a_i^2-\sigma_1^2\right)\right)-\sum_{i=2}^{n}b_i^2}\\ &= \sqrt{\frac{\sum_{i=2}^{n}a_i^2b_i^2\prod_{\substack{j=2\\ j\neq i}}^{n}\left(a_j^2-\sigma_1^2\right)}{\prod_{i=2}^{n}\left(a_i^2-\sigma_1^2\right)}+\sigma_1^2-\sum_{i=2}^{n}b_i^2}\\ &= \sqrt{\sum_{i=2}^{n}\frac{a_i^2b_i^2}{a_i^2-\sigma_1^2}-\sum_{i=2}^{n}b_i^2+\sigma_1^2} \end{aligned}$$

Thus,

$$b_1=\sqrt{\left[\sum_{i=2}^{n}\left(\frac{b_i^2}{a_i^2-\sigma_1^2}\right)\right]\sigma_1^2+\sigma_1^2} \tag{1.4}$$

□

**Lemma 5.** *[4] Let $A$ be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, $r = \min\{m, n\}$. Let $\mathbf{u}_i$ $\mathbf{v}_i$, respectively, the left and right singular vectors corresponding to $\sigma_i$, $i = 1, \ldots, r$. Let $\alpha \in \mathbf{R}$ such that $\alpha + \sigma_i \geq 0$. Then $A + \alpha \mathbf{u}_i \mathbf{v}_i^T$ has singular values*

$$\sigma_1, \ldots, \sigma_{i-1}, \sigma_i + \alpha, \sigma_{i+1}, \ldots, \sigma_r.$$

The paper is organized as follows: In section 2 we construct an $n \times n$ positive matrix with prescribed singular values by using results related with the *symmetric nonnegative inverse eigenvalue problem.* In section 3 we construct a positive matrix, not necessarily symmetric, from Lemmas 4 and 5. In section 4 we consider the rectangular case $m \times n$, $m > n$. Finally, in section 5 we give some examples to illustrate the results.

## 2. Positive matrices with prescribed singular values I

In this section we construct a positive symmetric matrix with prescribed nonnegative eigenvalues, which in this case, are also its singular values.

**Theorem 1.** *Given the real numbers $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, there exists a positive matrix $A$ with singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$.*

**Proof.** Let

$$B_k = \begin{pmatrix} \frac{\sigma_k + \sigma_{n-k+1}}{2} & \frac{\sigma_k - \sigma_{n-k+1}}{2} \\ \frac{\sigma_k - \sigma_{n-k+1}}{2} & \frac{\sigma_k + \sigma_{n-k+1}}{2} \end{pmatrix}, \quad k = 1, 2, \ldots, \frac{n}{2}, \text{ even } n,$$

with $B_{\frac{n+1}{2}} = \left(\sigma_{\frac{n+1}{2}}\right)$ for odd $n$. Thus, $B_k$ is a positive symmetric matrix with singular values $\sigma_k$ and $\sigma_{n-k+1}$ (eigenvalues $\sigma_k$ and $\sigma_{n-k+1}$) . Then

$$B = \begin{pmatrix} B_1 & 0 & \ddots & 0 \\ 0 & B_2 & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & B_{\frac{n}{2}} \end{pmatrix},$$

with $B_{\frac{n+1}{2}}$ in the last diagonal position if $n$ is odd, is an $n \times n$ nonnegative symmetric matrix with singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$. In order to obtain a

positive matrix with prescribed singular values let us consider $\mu_1 > \mu_2 \geq \cdots \geq \mu_n \geq 0$, where

$$\begin{aligned}
\mu_1 &= \frac{\sigma_1 + \sigma_2}{2}, \\
\mu_i &= \sigma_i + \epsilon_i \leq \mu_{i-1}, \quad i = 2, \ldots, \frac{n}{2} \ (\frac{n+1}{2} \text{ for odd } n), \\
\mu_i &= \sigma_i, \ i = \frac{n+2}{2}, \ldots, n \ (\frac{n+3}{2} \text{ for odd } n)
\end{aligned}$$

with

$$\epsilon = \frac{\sigma_1 - \sigma_2}{2} \text{ and } \epsilon_i = \frac{2\epsilon}{n-2} = \frac{\sigma_1 - \sigma_2}{n-2}$$

Now we apply the Lemma 2, $\frac{n-2}{2}$ times if $n$ is even ($\frac{n-1}{2}$ times if $n$ is odd). In the first step of this process we apply Lemma 2 to the matrices

$$B_1 = \begin{pmatrix} \frac{\mu_1+\sigma_n}{2} & \frac{\mu_1-\sigma_n}{2} \\ \frac{\mu_1-\sigma_n}{2} & \frac{\mu_1+\sigma_n}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \frac{\mu_2+\sigma_{n-1}}{2} & \frac{\mu_2-\sigma_{n-1}}{2} \\ \frac{\mu_2-\sigma_{n-1}}{2} & \frac{\mu_2+\sigma_{n-1}}{2} \end{pmatrix},$$

with singular values $\mu_1, \sigma_n$ and $\mu_2, \sigma_{n-1}$, respectively. From Lemma 1, the matrix

$$\begin{pmatrix} \mu_1 & \rho \\ \rho & \mu_2 \end{pmatrix}, \text{ where } \rho = (\epsilon_2(\mu_1 - \mu_2 + \epsilon_2))^{\frac{1}{2}},$$

has eigenvalues $\mu_1 + \epsilon_2$ and $\mu_2 - \epsilon_2 = \sigma_2$. The unitary vectors

$$\mathbf{u} = \mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$$

are the Perron eigenvectors of $B_1$ and $B_2$, respectively. Then the $4 \times 4$ matrix

$$\mathcal{B}_1 = \begin{pmatrix} B_1 & \rho \mathbf{u}\mathbf{v}^T \\ \rho \mathbf{v}\mathbf{u}^T & B_2 \end{pmatrix}$$

is positive symmetric with singular values (eigenvalues) $\mu_1 + \epsilon_2, \sigma_2, \sigma_{n-1}, \sigma_n$. In the second step of the process we again apply Lemma 2 to the matrices $\mathcal{B}_1$ and

$$B_3 = \begin{pmatrix} \frac{\mu_3+\sigma_{n-2}}{2} & \frac{\mu_3-\sigma_{n-2}}{2} \\ \frac{\mu_3-\sigma_{n-2}}{2} & \frac{\mu_3+\sigma_{n-2}}{2} \end{pmatrix},$$

which has singular values $\mu_3$ and $\sigma_{n-2}$. The matrix

$$\begin{pmatrix} \mu_1+\epsilon_2 & \rho \\ \rho & \mu_3 \end{pmatrix}, \text{ where } \rho = (\epsilon_3(\mu_1+\epsilon_2-\mu_3+\epsilon_3))^{\frac{1}{2}}$$

has eigenvalues $\mu_1+\epsilon_2+\epsilon_3$ and $\mu_3-\epsilon_3=\sigma_3$. Now let $\mathcal{B}_1\mathbf{u}=(\mu_1+\epsilon_2)\mathbf{u}$, $\|\mathbf{u}\|=1$ and $B_3\mathbf{v}=\mu_3\mathbf{v}$, $\mathbf{v}=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})^T$. Then the $6\times 6$ matrix

$$\mathcal{B}_2 = \begin{pmatrix} \mathcal{B}_1 & \rho\mathbf{u}\mathbf{v}^T \\ \rho\mathbf{v}\mathbf{u}^T & B_3 \end{pmatrix},$$

is positive symmetric with singular values $\mu_1+\epsilon_2+\epsilon_3, \sigma_2, \sigma_3, \sigma_{n-2}, \sigma_{n-1}, \sigma_n$. The process is continued until in the last step we apply Lemma 2 to the matrices $\mathcal{B}_{\frac{n-4}{2}}$ ($\mathcal{B}_{\frac{n-3}{2}}$ for odd $n$), of order $n-2$ ($n-1$ for odd $n$) and $B_{\frac{n}{2}}$ ($B_{\frac{n+1}{2}}$ for odd $n$), which has the singular values $\mu_{\frac{n}{2}}$ and $\mu_{\frac{n+2}{2}}$ ($\mu_{\frac{n+1}{2}}$ for odd $n$). The matrix

$$\begin{pmatrix} \mu_1+(\epsilon-\epsilon_{\frac{n}{2}}) & \rho \\ \rho & \mu_{\frac{n}{2}} \end{pmatrix}, \text{ where } \rho = (\epsilon_{\frac{n}{2}}(\mu_1+(\epsilon-\epsilon_{\frac{n}{2}})-\mu_{\frac{n}{2}}+\epsilon_{\frac{n}{2}}))^{\frac{1}{2}},$$

has eigenvalues $\mu_1+\epsilon$ and $\mu_{\frac{n}{2}}-\epsilon_{\frac{n}{2}}=\sigma_{\frac{n}{2}}$. Then the $n\times n$ matrix

$$A = \mathcal{B}_{\frac{n-2}{2}} = \begin{pmatrix} \mathcal{B}_{\frac{n-4}{2}} & \rho\mathbf{u}\mathbf{v}^T \\ \rho\mathbf{v}\mathbf{u}^T & B_{\frac{n}{2}} \end{pmatrix} \quad \text{for even } n$$

or

$$A = \mathcal{B}_{\frac{n-1}{2}} = \begin{pmatrix} \mathcal{B}_{\frac{n-4}{2}} & \rho\mathbf{u}\mathbf{v}^T \\ \rho\mathbf{v}\mathbf{u}^T & B_{\frac{n+1}{2}} \end{pmatrix} \quad \text{for odd } n,$$

where $\mathbf{u}$ and $\mathbf{v}$ are the unitary eigenvectors of $\mathcal{B}_{\frac{n-2}{2}}$ ($\mathcal{B}_{\frac{n-1}{2}}$ for odd $n$) and $B_{\frac{n}{2}}$ ($B_{\frac{n+1}{2}}$ for odd $n$), respectively, is positive and since

$$\mu_1+\epsilon_2+\cdots\epsilon_{\frac{n}{2}} = \mu_1+\epsilon = \sigma_1,$$

$A$ has the prescribed singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$. $\square$

## 3. Positive matrices with prescribed singular values II

In this section we construct an $n\times n$ positive matrix $A$, no necessarily symmetric, with prescribed singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$. First, by applying Lemma 4 we construct a nonnegative semibordered diagonal matrix $C$ of the form

$$C = \begin{pmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ & a_2 & & & \\ & & a_3 & & \\ & & & \ddots & \\ & & & & a_n \end{pmatrix}, \quad b_j > 0, \ j = 1, \ldots, n. \tag{3.1}$$

and then, by applying Lemma 5, we add to the matrix $C$ the positive matrix $\alpha \mathbf{u}\mathbf{v}^T$, where $\alpha > 0$ and $\mathbf{u}$ and $\mathbf{v}$ are the left and right singular vectors of $C$ corresponding to its maximal singular value.

**Theorem 2.** Let the nonnegative real numbers $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_n > 0$ be given. Then there exists an $n \times n$ positive matrix $A$ with singular values $\sigma_1, \sigma_2, ..., \sigma_n$.

**Proof.** Let $\alpha > 0$ such that $\widetilde{\sigma}_1 = \sigma_1 - \alpha > \sigma_2$ and consider $\widetilde{\sigma}_1 > \sigma_2 \geq \cdots \geq \sigma_n$. Then, from Lemma 4 there exists an $n \times n$ nonnegative matrix $C$ of the form (3.1), with singular values $\widetilde{\sigma}_1, \sigma_2, ..., \sigma_n$. The arrow matrix $CC^T$ is nonnegative, irreducible, with Perron eigenvector equal to $\mathbf{u}_1$ corresponding to the Perron root $\widetilde{\sigma}_1^2$ (left singular vector of $C$ corresponding to the singular value $\widetilde{\sigma}_1$). Then $\mathbf{u}_1$ is an entrywise positive vector and so is the right singular vector $\mathbf{v}_1$. Therefore the matrix $\alpha \mathbf{u}_1 \mathbf{v}_1^T$ is positive and from Lemma 5,
$A = C + \alpha \mathbf{u}_1 \mathbf{v}_1^T$ is positive with the prescribed singular values $\sigma_1, \sigma_2, ..., \sigma_n$. $\square$

Lemma 4 and the form of the matrix in (3.1) allow us to construct a nonnegative matrix , say $C$, whose exponential matrix $e^C$ has prescribed singular values: To show this, we consider the matrix $C$ in (3.1). Then a straight forward calculation shows that

$$C^m = \begin{pmatrix} b_1^m & b_2 \sum_{k=0}^{m-1} b_1^{m-1-k} a_2^k & \cdots & \cdots & b_n \sum_{k=0}^{m-1} b_1^{m-1-k} a_n^k \\ & a_2^m & & & \\ & & a_3^m & & \\ & & & \ddots & \\ & & & & a_n^m \end{pmatrix}$$

and

$$e^C = \sum_{j=0}^{\infty} \frac{1}{j!} \begin{pmatrix} b_1^j & b_2 \sum_{k=0}^{j-1} b_1^{j-1-k} a_2^k & \cdots & \cdots & b_n \sum_{k=0}^{j-1} b_1^{j-1-k} a_n^k \\ & a_2^j & & & \\ & & a_3^j & & \\ & & & \ddots & \\ & & & & a_n^j \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=0}^{\infty} \frac{1}{j!} b_1^j & b_2 \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1-k} a_2^k & \cdots & \cdots & b_n \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1-k} a_n^k \\ & \sum_{j=0}^{\infty} \frac{1}{j!} a_2^j & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sum_{j=0}^{\infty} \frac{1}{j!} a_n^j \end{pmatrix}$$

is also a semibordered diagonal matrix. Now, for $i = 2, 3, \ldots, n$, we have

$$\begin{aligned} \left(e^C\right)_{1i} &= b_i \sum_{j=0}^{\infty} \left( \frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1-k} a_i^k \right) \\ &= b_i \sum_{j=0}^{\infty} \left( \frac{1}{j!} \sum_{k=0}^{j-1} b_1^{j-1} \left(\frac{a_i}{b_1}\right)^k \right) \\ &= b_i \sum_{j=0}^{\infty} \left( \frac{b_1^{j-1}}{j!} \sum_{k=0}^{j-1} \left(\frac{a_i}{b_1}\right)^k \right). \end{aligned}$$

Since

$$\sum_{k=0}^{j-1} \left(\frac{a_i}{b_1}\right)^k = \frac{1 - \left(\frac{a_i}{b_1}\right)^j}{1 - \left(\frac{a_i}{b_1}\right)} = \frac{b_1^j - a_i^j}{b_1^{j-1}(b_1 - a_i)}$$

then

$$\begin{aligned}\left(e^C\right)_{1i} &= \frac{b_i}{(b_1-a_i)}\sum_{j=0}^{\infty}\left(\frac{b_1^j-a_i^j}{j!}\right)=\frac{b_i}{(b_1-a_i)}\sum_{j=0}^{\infty}\left(\frac{b_1^j}{j!}-\frac{a_i^j}{j!}\right)\\ &= \frac{b_i\left(e^{b_1}-e^{a_i}\right)}{b_1-a_i},\ i=2,..,n.\end{aligned}$$

Moreover,

$$\left(e^C\right)_{11}=e^{b_1}\ \text{ and }\ \left(e^C\right)_{ii}=e^{a_i},\ i=2,...,n.$$

Hence, given the real positive numbers $\sigma_1>\sigma_2\geq\cdots\geq\sigma_n>0$, from Lemma 4, we may construct a matrix $\widetilde{C}$ of the form (3.1), with singular values $\sigma_1,\ldots,\sigma_n$, where the $\widetilde{b}_i$ are given by (1.3) and (1.4) and can be chosen as positive. Clearly, the $\widetilde{a}_i$ can also be chosen positive.
Then, if $e^C=\widetilde{C}$, we have

$$\left(e^C\right)_{1i}=\frac{b_i\left(e^{b_1}-e^{a_i}\right)}{b_1-a_i}=\widetilde{b}_i,\ i=2,..,n$$

and

$$b_i=\frac{(b_1-a_i)}{(e^{b_1}-e^{a_i})}\widetilde{b}_i\quad i=2,...,n. \tag{3.2}$$

In the same way,

$$\left(e^C\right)_{11}=e^{b_1}=\widetilde{b}_1$$

and

$$b_1=\frac{1}{2}\ln\widetilde{b}_1 \tag{3.3}$$

From (1.3) we see that

$$\sum_{i=2}^{n}\left(\frac{\widetilde{b}_i^2}{\widetilde{a}_i^2-\sigma_1^2}\right)>0.$$

Finally, from

$$\left(e^C\right)_{ii}=e^{a_i}=\widetilde{a}_i=\sqrt{\frac{\sigma_{i-1}+\sigma_i}{2}},\ i=2,...,n,$$

we have

$$a_i = \frac{1}{2}\ln\left(\frac{\sigma_{i-1}+\sigma_i}{2}\right),\ i = 2, ..., n. \tag{3.4}$$

Observe that if $\sigma_n > 1$, then $\frac{\sigma_i+\sigma_{i-1}}{2} > 1$ and the $a_i$ are all positive. Thus, if $\sigma_1 > \sigma_2 \geq \cdots \geq \sigma_n > 1$, we can construct a nonnegative matrix $C$ of the form (3.1), in such a way that $e^C$ has prescribed singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$. To illustrate this construction, let us consider the numbers $\sigma_i : 7, 5, 3, 2$. Then, from ( 3.2), (3.3) and (3.4) we compute the matrix $C$ such that

$$e^C = \begin{bmatrix} 3.2845 & 2.428 & 2.7309 & 1.5365 \\ 0 & 6.083 & 0 & 0 \\ 0 & 0 & 4.1231 & 0 \\ 0 & 0 & 0 & 2.5495 \end{bmatrix}$$

has the prescribed singular values $7, 5, 3, 2$.

## 4. Positive matrices with prescribed singular values III

In this section we consider the construction of a rectangular positive matrix with prescribed singular values. First we construct an $m \times n$ nonnegative matrix $B$, $m > n$, of the form

$$B = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \\ a_{n+1} & & & \\ \vdots & & & \\ a_m & & & \end{pmatrix},\ m > n \tag{4.1}$$

and then we apply the perturbation result given by Lemma 5 to fill out $B$ with positive entries.

**Theorem 3.** Given the numbers $\sigma_1 > \sigma_2 > \cdots > \sigma_n > 0$, there exists an $m \times n$ nonnegative matrix, $m > n$, with singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$.

**Proof.** We define

$$a_i = \frac{\sigma_{i-1} + \sigma_i}{2}, \quad i = 2, \ldots, n,$$

and we choose $a_{n+1}, a_{n+2} \ldots, a_m$ in such a way that

$$a_1^2 = \sum_{n+1}^{m} a_j^2 < \sigma_n.$$

Since we want to construct the matrix $B$ in (4.1), then we observe that

$$\mathrm{B}^T B = \begin{pmatrix} b_1^2 + \sum\limits_{j=n+1}^{m} a_j^2 & b_1 b_2 & b_1 b_3 & \cdots & b_1 b_n \\ b_1 b_2 & b_2^2 + a_2^2 & b_2 b_3 & \ddots & b_2 b_n \\ b_1 b_3 & b_2 b_3 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & b_n^2 + a_n^2 \end{pmatrix} = D^2 + \mathbf{b}\mathbf{b}^T,$$

where $D = diag\{a_1, \ldots, a_n\}$ and $\mathbf{b} = (b_1, \ldots, b_n)^T$. Then

$$\begin{aligned} \prod_{k=1}^{n} (\sigma_k^2 - \sigma) &= \det(B^T B - \sigma I) = \det(D^2 + \mathbf{b}\mathbf{b}^T - \sigma I) \\ &= \det(D^2 - \sigma I) \det(I + (D^2 - \sigma I)^{-1} \mathbf{b}\mathbf{b}^T) \end{aligned}$$

and from the well known formula $\det(I + \mathbf{x}\mathbf{y}^T) = 1 + \mathbf{y}^T \mathbf{x}$, we have

$$\begin{aligned} \prod_{k=1}^{n} (\sigma_k^2 - \sigma) &= \left( \sum_{j=n+1}^{m} a_j^2 - \sigma \right) \prod_{k=2}^{n} (a_k^2 - \sigma) \left( 1 + \frac{b_1^2}{\sum\limits_{j=n+1}^{m} a_j^2 - \sigma} + \sum_{k=2}^{n} \frac{b_k^2}{a_k^2 - \sigma} \right) \\ &= (a_1^2 - \sigma) \prod_{k=2}^{n} (a_k^2 - \sigma) + b_1^2 \prod_{k=2}^{n} (a_k^2 - \sigma) + \\ &\quad + (a_1^2 - \sigma) \prod_{k=2}^{n} (a_k^2 - \sigma) \sum_{k=2}^{n} \frac{b_k^2}{a_k^2 - \sigma} \\ &= (a_1^2 - \sigma) \prod_{k=2}^{n} (a_k^2 - \sigma) + b_1^2 \prod_{k=2}^{n} (a_k^2 - \sigma) + \\ &\quad + (a_1^2 - \sigma) \prod_{k=2}^{n} (a_k^2 - \sigma) \sum_{\substack{k=2 \\ k \neq i}}^{n} \frac{b_k^2}{a_k^2 - \sigma} + b_i^2 \prod_{\substack{k=1 \\ k \neq i}}^{n} (a_k^2 - \sigma). \end{aligned}$$

Thus, for $\sigma = a_i^2$, $i = 1, \ldots, n$, we have

$$b_i^2 = \frac{\prod_{k=1}^{n} (\sigma_k^2 - a_i^2)}{\prod_{\substack{k=1 \\ k \neq i}}^{n} (a_k^2 - a_i^2)}, \ i = 1, \cdots n. \tag{4.2}$$

Since $\sigma_1 > a_2 > \sigma_2 > a_3 > \cdots > \sigma_n > a_1$, then the right side in (4.2) is positive and we may choose $b_i$, $i = 1, ..., n$, as positive. Hence, we may construct the nonnegative matrix $B$ in (4.1), with the prescribed singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$. $\square$

**Corollary 1.** *Given the positive numbers $\sigma_1 > \sigma_2 > \cdots > \sigma_n > 0$, there exists an $m \times n$ positive matrix $A$, $m > n$, with singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$.*

**Proof.** Let $\alpha > 0$ such that $\widetilde{\sigma}_1 = \sigma_1 - \alpha > \sigma_2$. Then from Theorem **??** there exists an $m \times n$ nonnegative matrix $B$ of the form (4.1) with singular values $\widetilde{\sigma}_1, \sigma_2, .., \sigma_n$. The matrix $BB^T$ is irreducible nonnegative and therefore its Perron eigenvector $\mathbf{u}_1$ (left singular vector of $B$ corresponding to $\widetilde{\sigma}_1$ ) is positive and so is the right singular vector $\mathbf{v}_1$. Hence, from Lemma 5, $A = B + \alpha u_1 v_1^T$ is an $m \times n$ positive matrix with singular values $\sigma_1, \sigma_2, ..., \sigma_n$. $\square$

## 5. Examples

**Example 1.** *This example illustrates the construction of the matrix $A$ of Theorem 1: Let $S = \{24, 16, 16, 16, 16, 2\}$ be given. Since $\epsilon = \frac{\sigma_1 - \sigma_2}{2} = 4$, then $\epsilon_i = 2$, $i = 1, 2$. Thus, instead $S$ we consider $\Gamma = \{20, 18, 18, 16, 16, 2\}$ as the prescribed set of singular values. Then*

$$B_1 = \begin{pmatrix} 11 & 9 \\ 9 & 11 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 17 & 1 \\ 1 & 17 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 17 & 1 \\ 1 & 17 \end{pmatrix}.$$

*We apply Lemma 2 to the matrices $B_1$ and $B_2$, with $\rho = 2\sqrt{2}$, $\mathbf{u} = \mathbf{v} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$ to obtain*

$$\mathcal{B}_1 = \begin{pmatrix} B_1 & \rho\mathbf{u}\mathbf{v}^T \\ \rho\mathbf{v}\mathbf{u}^T & B_2 \end{pmatrix} = \begin{pmatrix} 11 & 9 & \sqrt{2} & \sqrt{2} \\ 9 & 11 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 17 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 17 \end{pmatrix},$$

*with eigenvalues* $22, 16, 16, 2$. *Next we again apply Lemma 2 to the matrices* $\mathcal{B}_1$ *and* $B_3$, *with*

$$\rho = 2\sqrt{3}, \quad \widehat{\mathbf{u}} = \left(\tfrac{1}{3}\sqrt{3}, \tfrac{1}{3}\sqrt{3}, \tfrac{1}{6}\sqrt{6}, \tfrac{1}{6}\sqrt{6}\right)^T, \quad \mathbf{v} = (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}})^T,$$

*to obtain the positive matrix*

$$A = \mathcal{B}_2 = \begin{pmatrix} \mathcal{B}_1 & \rho\widehat{\mathbf{u}}\mathbf{v}^T \\ \rho\mathbf{v}\widehat{\mathbf{u}}^T & B_3 \end{pmatrix} = \begin{pmatrix} 11 & 9 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 9 & 11 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 17 & 1 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 17 & 1 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 1 & 17 & 1 \\ \sqrt{2} & \sqrt{2} & 1 & 1 & 1 & 17 \end{pmatrix}$$

*with the prescribed singular values:* $24, 16, 16, 16, 16, 2$.

**Example 2.** *This example illustrates the construction of a* $4 \times 4$ *positive nonsymmetric matrix with the prescribed singular values* $3, \sqrt{5}, \sqrt{3}, 1$. *We consider the list* $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$. *Then from Lemma 4 we construct the nonnegative matrix*

$$C = \begin{pmatrix} \frac{\sqrt{35}}{4} & \frac{\sqrt{5}}{4} & \frac{3}{4} & \frac{\sqrt{15}}{4} \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

*with singular values* $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$. *Next, we apply Lemma 5 to obtain the positive matrix*

$$\begin{aligned} A &= C + (3 - \sqrt{7})\mathbf{u}_1\mathbf{v}_1^T \\ &= \begin{pmatrix} 1.5409 & 0.72275 & 0.82322 & 1.0250 \\ 8.4740 \times 10^{-2} & 2.6737 & 0.10026 & 7.7665 \times 10^{-2} \\ 3.0943 \times 10^{-2} & 8.1867 \times 10^{-2} & 2.0366 & 2.8359 \times 10^{-2} \\ 1.6948 \times 10^{-2} & 4.4837 \times 10^{-2} & 2.0052 \times 10^{-2} & 1.4297 \end{pmatrix} \end{aligned}$$

*with the prescribed singular values* $3, \sqrt{5}, \sqrt{3}, 1$.

**Example 3.** *This example illustrates the construction of a* $7 \times 4$ *positive matrix with the prescribed singular values* $3, \sqrt{5}, \sqrt{3}, 1$. *Let* $\alpha = 3 - \sqrt{7}$ *and let us consider the numbers* $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$, *as in the previous example. From Theorem* **??**, *we compute the nonnegative matrix*

$$B = \begin{pmatrix} 0.546\,23 & 0.592\,52 & 0.841\,92 & 1.401\,6 \\ 0 & 2.440\,9 & 0 & 0 \\ 0 & 0 & 1.984\,1 & 0 \\ 0 & 0 & 0 & 1.366 \\ 0.707\,11 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \\ 0.408\,25 & 0 & 0 & 0 \end{pmatrix},$$

*with singular values* $\sqrt{7}, \sqrt{5}, \sqrt{3}, 1$. *Since* $B^T B$ *is irreducible nonnegative, its Perron eigenvector,* $\mathbf{u}_1$, *corresponding to the Perron root* 7 *(left singular vector of* $B$ *corresponding to the singular value* $\sqrt{7}$*), is entrywise positive and so is the right singular vector* $\mathbf{v}_1$. *Therefore the matrix* $\alpha \mathbf{u}_1 \mathbf{v}_1^T$ *is positive and from Lemma 5,* $A = B + \alpha \mathbf{u}_1 \mathbf{v}_1^T$ *is positive with the prescribed singular values* $3, \sqrt{5}, \sqrt{3}, 1$.

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