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NUMERICAL QUENCHING FOR A SEMILINEAR PARABOLIC EQUATION WITH A POTENTIAL AND GENERAL NONLINEARITIES

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Abstract

This paper concerns the study of the numerical approximation a semilinear parabolic equation subject to Neumann boundary conditions and positive initial data. We find some conditions under which the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. A similar study has been also investigated taking a discrete form of the above problem. Finally, we give some numerical experiments to illustrate our analysis.

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1. Introduction

Consider the following boundary value problem

$$u_t(x,t) - a(x)u_{xx}(x,t) = -b(x)f(u(x,t)), \quad (x,t) \in (0,1) \times (0,T), \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) > 0, \quad x \in [0,1], \tag{1.3}$$

where $f : (0,\infty) \to (0,\infty)$ is a C^1 convex, nonincreasing function, $\lim_{s\to 0^+} f(s) = \infty$, $\int_0^\gamma \frac{d\sigma}{f(\sigma)} < \infty$ for any positive real γ, $a \in C^0([0,1])$, $a(x) > 0$, $x \in [0,1]$. The initial data $u_0 \in C^2([0,1])$, $u_0(x) > 0$, $x \in [0,1]$, $u_0^{'}(0) = 0$ and $u_0^{'}(1) = 0$, $a(x)u_0''(x) - b(x)f(u_0(x)) < 0$, $\quad x \in (0,1)$. The potential $b \in C^1([0,1])$, $b(x) > 0$, $x \in (0,1)$, $b^{'}(0) = 0$, $b^{'}(1) = 0$.

Here, $(0,T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. This means that $u(x,t) > 0$ in $[0,1] \times (0,\infty)$. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T} u_{\min}(t) = 0,$$

where $u_{\min}(t) = \min_{0\le x\le 1} u(x,t)$. In this last case, we say that the solution u quenches in a finite time, and the time T is called the quenching time of the solution u.

The theoretical study of solutions for semilinear parabolic equations which quench in a finite time has been the subject of investigations of many authors (see [2], [4]–[7], [12], [17], and the references cited therein). Local in time existence of a classical solution has been proved and this solution is unique. In addition, it is shown that if the initial data at (1.3) satisfies $a(x)u_0^{''}(x) - b(x)f(u_0(x)) < 0$ in $(0,1)$, then the classical solution u

of (1.1)–(1.3) quenches in a finite time T, and there exist positive constants $c_0,\ c_1,\ C_0,\ C_1$ such that the following estimates hold

$$C_0 \int_0^{u_{0min}} \frac{d\sigma}{f(\sigma)} \leq T \leq C_1 \int_0^{u_{0min}} \frac{d\sigma}{f(\sigma)},$$

$$H(c_0(T-t)) \leq u_{\min}(t) \leq H(c_1(T-t)),$$

where $H(s)$ is the inverse of the function $F(s) = \int_0^s \frac{d\sigma}{f(\sigma)}$ (see [4]-[7]).

In this paper, we are interested in the numerical study of the phenomenon of quenching. Under some assumptions, we show that the solution of a semidiscrete form of (1.1)–(1.3) quenches in a finite time and estimate its semidiscrete quenching time. We also prove that the semidiscrete quenching time goes to the real one when the mesh size goes to zero. Similar results have been also given for a discrete form of (1.1)–(1.3). Recently, an analogous study has been investigated by Nabongo and Boni in [19], where they have considered the problem (1.1)-(1.3) for the case $a(x) = 1, b(x) = 1$ and $f(u) = u^{-p}$ with $p > 0$. Let us notice that, in the present paper, because of the potentials $a(x)$ and $b(x)$, we study the effect of a pertubation of these last on the different approximations of the real quenching time.

In the same way, in [16] and [18], Nabongo and Boni have used semidiscrete schemes to study the phenomenon of quenching for other parabolic problems. Our work was also motived by the papers in [1], [3] and [15]. In [1] and [15], the authors have used semidiscrete and discrete forms for some parabolic equations to study the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). In [3], some schemes have been utilized to study the phenomenon of extinction (we say that a solution extincts in a finite time if it becomes zero after a finite time for equations without singularities). One may also consult the papers in [8]–[10], where the authors have studied theoretically the dependence with respect to the initial data of the blow-up time of nonlinear parabolic problems. Concerning the numerical study, one may find some results in [13], [14], [21], [22], where the authors have proposed some numerical schemes for computing the numerical solutions for parabolic problems which present a solution with one singularity. Let us remark that in these last papers, there is a lack of information about the convergence of the numerical quenching time.

This paper is organized as follows. In the next section, we give some results about the semidiscrete maximum principle. In the third section, under some conditions, we prove that the solution of a semidiscrete form of (1.1)–(1.3) quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we prove the convergence of the semidiscrete quenching time. In the fifth section, we study the results of sections 3 and 4 taking a discrete form of (1.1)–(1.3). Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of a semidiscrete problem

In this section, we give some results about the semidiscrete maximum principle. We start by the construction of a semidiscrete scheme as follows. Let I be a positive integer and let $h = \frac{1}{I}$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution u of the problem (1.1)–(1.3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} - \alpha_i \delta^2 U_i(t) = -\beta_i f(U_i(t)), \quad 0 \leq i \leq I, \quad t \in (0, T_q^h), \tag{2.1}$$

$$U_i(0) = \varphi_i, \quad 0 \leq i \leq I, \tag{2.2}$$

where $\varphi_h > 0$, $\beta_h > 0$, $\alpha_h > 0$,

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

Let us notice that, in our scheme, we pick α_h, β_h and φ_h so that α_i, β_i, φ_i are approximations of $a(x_i)$, $b(x_i)$ and $u_0(x_i)$, respectively. The interest to choose these approximations is that sometimes, it is difficult to have the exact values of the different potentials. It is the case when one of then is, for instance, the solution of a complicated differential equation.

Here $(0, T_q^h)$ is the maximal time interval on which $U_{hmin}(t) > 0$, where

$$U_{hmin}(t) = \min_{0 \le i \le I} U_i(t).$$

When the time T_q^h is finite, then we say that the solution $U_h(t)$ of (2.1)–(2.2) quenches in a finite time, and the time T_q^h is called the quenching time of the solution $U_h(t)$.
The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1. *Let $\gamma_h \in C^0([0,T), \mathbf{R}^{I+1})$ and let $V_h \in C^1([0,T), \mathbf{R}^{I+1})$ be such that*

$$\frac{dV_i(t)}{dt} - \alpha_i \delta^2 V_i(t) + \gamma_i(t) V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in (0,T), \tag{2.3}$$

$$V_i(0) \ge 0, \quad 0 \le i \le I. \tag{2.4}$$

Then, we have $V_i(t) \ge 0$, $0 \le i \le I$, $t \in (0,T)$.

Proof. Let T_0 be any positive quantity satisfying the inequality $T_0 < T$, and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$, where λ is such that

$$\gamma_i(t) - \lambda > 0 \quad \text{for} \quad 0 \le i \le I, \quad t \in [0, T_0].$$

Set $m = \min_{0 \le t \le T_0} Z_{hmin}(t)$. Since $Z_h(t)$ is a continuous vector on the compact $[0, T_0]$, there exist $i_0 \in \{0, ..., I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \tag{2.5}$$

$$\delta^2 Z_{i_0}(t_0) \geq 0. \tag{2.6}$$

According to (2.3), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \alpha_{i_0}\delta^2 Z_{i_0}(t_0) + (\alpha_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0. \tag{2.7}$$

We infer from (2.5)–(2.7) that $(\alpha_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0$, which entails that $Z_{i_0}(t_0) \geq 0$. Therefore, $V_h(t) \geq 0$ for $t \in [0, T_0]$ and the proof is complete. □

Another form of the maximum principle for semidiscrete equations is the following comparison lemma.

Lemma 2.2. *Let $f \in C^0(\mathbf{R} \times \mathbf{R}, \mathbf{R})$. If V_h, $W_h \in C^1([0,T), \mathbf{R}^{I+1})$ are such that*

$$\frac{dV_i(t)}{dt} - \alpha_i\delta^2 V_i(t) + f(V_i(t), t) < \frac{dW_i(t)}{dt} - \alpha_i\delta^2 W_i(t) + f(W_i(t), t),$$

$$0 \leq i \leq I, \quad t \in (0, T),$$

$$V_i(0) < W_i(0), \quad 0 \leq i \leq I,$$

then $V_i(t) < W_i(t), \quad 0 \leq i \leq I, \quad t \in (0, T)$.

Proof. Let $Z_h(t) = W_h(t) - V_h(t)$ and let t_0 be the first $t \in (0, T)$ such that $Z_h(t) > 0$ for $t \in [0, t_0)$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, ..., I\}$. We remark that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0,$$

$$\delta^2 Z_{i_0}(t_0) \geq 0.$$

Using these inequalities and the fact that $W_{i_0}(t_0) = V_{i_0}(t_0)$, we derive the following estimate

$$\frac{dZ_{i_0}(t_0)}{dt} - \alpha_{i_0}\delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \leq 0.$$

But, this contradicts the first strict inequality of the lemma and the proof is complete. □

3. Quenching in the semidiscrete problem

In this section, under some assumptions, we show that the solution U_h of (2.1)–(2.2) quenches in a finite time and estimate its semidiscrete quenching time. We need the following result about the operator δ^2.

Lemma 3.1. *Let $U_h \in \mathbf{R}^{I+1}$ be such that $U_h > 0$. Then, we have*

$$\delta^2(f(U))_i \geq f'(U_i)\delta^2 U_i, \quad 0 \leq i \leq I.$$

Proof. Applying Taylor's expansion, we find that

$$\delta^2(f(U))_0 = f'(U_0)\delta^2 U_0 + \frac{(U_1 - U_0)^2}{h^2} f''(\theta_0),$$

$$\delta^2(f(U))_i = f'(U_i)\delta^2 U_i + \frac{(U_{i+1} - U_i)^2}{2h^2} f''(\theta_i) + \frac{(U_{i-1} - U_i)^2}{2h^2} f''(\eta_i),$$
$$0 \leq i \leq I,$$
$$\delta^2(f(U))_I = f'(U_I)\delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{h^2} f''(\eta_I),$$

where θ_i is an intermediate value between U_i and U_{i+1}, η_i the one between U_{i-1} and U_i. Use the fact that $f''(s) \geq 0$ for $s > 0$ and $U_h > 0$ to complete the rest of the proof. □

The statement of the result about solutions which quench in a finite time is the following.

Theorem 3.1. *Let U_h be the solution of (2.1)–(2.2). Assume that there exists a positive constant $A \in (0, 1]$ such that the initial data at (2.2) satisfies*

$$\alpha_i \delta^2 \varphi_i - \beta_i f(\varphi_i) \leq -A f(\varphi_i), \quad 0 \leq i \leq I. \tag{3.1}$$

Then, the solution U_h quenches in a finite time T_q^h, and the following estimate holds

$$T_q^h \leq \frac{1}{A} \int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)}.$$

Proof. Since $(0, T_q^h)$ is the maximal time interval on which $U_{hmin}(t) > 0$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} + Af(U_i(t)), \quad 0 \le i \le I, \quad t \in [0, T_q^h).$$

A straightforward calculation gives

$$\frac{dJ_i}{dt} - \alpha_i \delta^2 J_i = \frac{d}{dt}\left(\frac{dU_i}{dt} - \alpha_i \delta^2 U_i\right)$$

$$+Af'(U_i)\frac{dU_i}{dt} - \alpha_i A\delta^2(f(U))_i, \quad 0 \le i \le I, \quad t \in (0, T_q^h).$$

Taking into account Lemma 3.1, we see that $\delta^2(f(U))_i \ge f'(U_i)\delta^2 U_i$, $0 \le i \le I$, which implies that

$$\frac{dJ_i}{dt} - \alpha_i \delta^2 J_i \le \frac{d}{dt}\left(\frac{dU_i}{dt} - \alpha_i \delta^2 U_i\right) + Af'(U_i)\left(\frac{dU_i}{dt} - \alpha_i \delta^2 U_i\right),$$

$$0 \le i \le I, \quad t \in (0, T_q^h).$$

Using (2.1), we arrive at

$$\frac{dJ_i}{dt} - \alpha_i \delta^2 J_i \le -\beta_i f'(U_i)\frac{dU_i}{dt} - \beta_i Af'(U_i)f(U_i), \ \ 0 \le i \le I, \ \ t \in (0, T_q^h).$$

Making use of the expression of J_h, we discover that

$$\frac{dJ_i}{dt} - \alpha_i \delta^2 J_i \le -\beta_i f'(U_i) J_i, \quad 0 \le i \le I, \quad t \in (0, T_q^h).$$

Exploiting (3.1), we observe that $J_h(0) \le 0$. We infer from Lemma 2.1 that $J_h(t) \le 0$ for $t \in (0, T_q^h)$, which implies that

$$\frac{dU_i(t)}{dt} \le -Af(U_i(t)), \quad 0 \le i \le I, \quad t \in (0, T_q^h). \tag{3.2}$$

These estimates may be rewritten in the following form

$$\frac{dU_i}{f(U_i)} \le -Adt, \quad 0 \le i \le I, \quad t \in (0, T_q^h).$$

Integrating the above inequalities over the interval (t, T_q^h), we get

$$T_q^h - t \leq \frac{1}{A}\int_0^{U_i(t)} \frac{d\sigma}{f(\sigma)}, \quad 0 \leq i \leq I. \tag{3.3}$$

Using the fact that $\varphi_{hmin} = U_{i_0}(0)$ for a certain $i_0 \in \{0, ..., I\}$ and taking $t = 0$ in (3.3), we obtain the desired result. □

Remark 3.1. *The inequalities (3.3) imply that*

$$T_q^h - t_0 \leq \frac{1}{A}\int_0^{U_{hmin}(t_0)} \frac{d\sigma}{f(\sigma)} \quad for \quad t_0 \in (0, T_q^h),$$

and

$$U_{hmin}(t) \geq H(A(T_q^h - t)) \quad for \quad t \in (0, T_q^h),$$

where $H(s)$ is the inverse of the function $F(s) = \int_0^s \frac{d\sigma}{f(\sigma)}$.

Remark 3.2. *Let U_h be the solution of (2.1)–(2.2). Then, we derive the following inequalities*

$$T_q^h \geq \frac{1}{\|\beta_h\|_\infty}\int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)},$$

and

$$U_{hmin}(t) \leq H(\|\beta_h\|_\infty(T_q^h - t)) \quad for \quad t \in (0, T_q^h).$$

To prove these estimates, we proceed as follows. Introduce the function $v(t)$ defined as follows $v(t) = U_{hmin}(t)$ for $t \in [0, T_q^h)$. Let $t_1, t_2 \in [0, T_q^h)$. Then, there exist $i_1, i_2 \in \{0, ..., I\}$ such that $v(t_1) = U_{i_1}(t_1)$ and $v(t_2) = U_{i_2}(t_2)$. We observe that

$$v(t_2) - v(t_1) \geq U_{i_2}(t_2) - U_{i_2}(t_1) = (t_2 - t_1)\frac{dU_{i_2}(t_2)}{dt} + o(t_2 - t_1),$$

$$v(t_2) - v(t_1) \leq U_{i_1}(t_2) - U_{i_1}(t_1) = (t_2 - t_1)\frac{dU_{i_1}(t_1)}{dt} + o(t_2 - t_1),$$

which implies that $v(t)$ is Lipschitz continuous. Further, if $t_2 > t_1$, then

$$\frac{v(t_2) - v(t_1)}{t_2 - t_1} \geq \frac{dU_{i_2}(t_2)}{dt} + o(1) = \alpha_{i_2}\delta^2 U_{i_2}(t_2) - \beta_{i_2} f(U_{i_2}(t_2)) + o(1).$$

Obviously, $\delta^2 U_{i_2}(t_2) \geq 0$. *Letting* $t_1 \to t_2$, *and using the fact that* $\beta_{i_2} \leq \|\beta_h\|_\infty$, *we obtain* $\frac{dv(t)}{dt} \geq -\|\beta_h\|_\infty f(v)$ *for* $t \in (0, T_q^h)$ *or equivalently* $\frac{dv}{f(v)} \geq -\|\beta_h\|_\infty dt$ *for* $t \in (0, T_q^h)$. *Integrate the above inequality over* (t, T_q^h) *to obtain* $T_q^h - t \geq \frac{1}{\|\beta_h\|_\infty} \int_0^{v(t)} \frac{d\sigma}{f(\sigma)}$. *Since* $v(t) = U_{hmin}(t)$, *we arrive at* $T_q^h - t \geq \frac{1}{\|\beta_h\|_\infty} \int_0^{U_{hmin}(t)} \frac{d\sigma}{f(\sigma)}$ *and the second estimate follows. To obtain the first one, it suffices to replace* t *by* 0 *in the above inequality and use the fact that* $\varphi_{hmin} = U_{hmin}(0)$.

4. Convergence of the semidiscrete quenching time

In this section, under some assumptions, we show that the solution of the semidiscrete problem quenches in a finite time, and its semidiscrete quenching time converges to the real one when the mesh size goes to zero.

We denote $a_h = (a(x_0), \ldots, a(x_I))^T$, $b_h = (b(x_0), \ldots, b(x_I))^T$,

$$u_h(t) = (u(x_0, t), ..., u(x_I, t))^T \quad \text{and} \quad \|U_h(t)\|_\infty = \max_{0 \leq i \leq I} |U_i(t)|.$$

In order to obtain the convergence of the semidiscrete quenching time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

Theorem 4.1. *Assume that the problem (1.1)-(1.3) has a solution* $u \in C^{4,1}([0,1] \times [0, T-\tau])$ *such that* $\min_{t \in [0, T-\tau]} u_{\min}(t) = \varrho > 0$ *with* $\tau \in (0, T)$. *Suppose that* φ_h, β_h *and* α_h *satisfy*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0, \tag{4.1}$$

$$\|\beta_h - b_h\|_\infty = o(1) \quad \text{as} \quad h \to 0, \quad \|\alpha_h - a_h\|_\infty = o(1) \quad as \quad h \to 0. \tag{4.2}$$

Then, for h sufficiently small, the problem (2.1)–(2.2) has a unique solution $U_h \in C^1([0, T_q^h), \mathbf{R}^{I+1})$ *such that the following relation holds*

$$\max_{0 \leq t \leq T-\tau} \|U_h(t) - u_h(t)\|_\infty$$

$$= 0(\|\varphi_h - u_h(0)\|_\infty + \|\beta_h - b_h\|_\infty + \|\alpha_h - a_h\|_\infty + h^2) \quad \text{as} \quad h \to 0.$$

Proof. Let $L > 0$ be such that

$$-(\|b_h\|_\infty + 1)f'(\frac{\rho}{2}) \le L. \tag{4.3}$$

Let us notice that the term on the left hand side of the above inequality is positive because the function $f(s)$ is positive and nonincreasing for positive values of s. The problem (2.1)–(2.2) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbf{R}^{I+1})$. Let $t(h) \le \min\{T - \tau, T_q^h\}$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \frac{\varrho}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{4.4}$$

The relation (4.1) implies that $t(h) > 0$ for h sufficiently small. Invoking the triangle inequality, we have

$$U_i(t) \ge u(x_i, t) - |U_i(t) - u(x_i, t)|, \quad 0 \le i \le I, \quad t \in (0, t(h)).$$

This implies that

$$U_i(t) \ge u(x_i, t) - \|U_h(t) - u_h(t)\|_\infty, \quad 0 \le i \le I, \quad t \in (0, t(h)).$$

Let $j \in \{0, \cdots, I\}$ be such that $U_{hmin}(t) = U_j(t)$. Replacing i by j in the above inequalities, and using the fact that $u(x_j, t) \ge u_{hmin}(t)$, we find that

$$U_{hmin}(t) \ge u_{hmin}(t) - \|U_h(t) - u_h(t)\|_\infty \quad \text{for} \quad t \in (0, t(h)),$$

which implies that

$$U_{hmin}(t) \ge \varrho - \frac{\varrho}{2} = \frac{\varrho}{2} \quad \text{for} \quad t \in (0, t(h)). \tag{4.5}$$

Since $u \in C^{4,1}$, taking the derivative in x on both sides of (1) and due to the fact that u_x, u_{xt}, b_x vanish at $x = 0$ and $x = 1$, we observe that u_{xxx} also vanishes at $x = 0$ and $x = 1$. Applying Taylor's expansion, we discover that

$$u_{xx}(x_i, t) = \delta^2 u(x_i, t) - \frac{h^2}{12} u_{xxxx}(\widetilde{x}_i, t), \quad 0 \le i \le I, \quad t \in (0, t(h)).$$

To establish the above equalities for $i = 0$ and $i = I$, we have used the fact that u_x and u_{xxx} vanish at $x = 0$ and $x = 1$. A direct calculation renders

$$u_t(x_i,t) - \alpha_i\delta^2 u(x_i,t) = -\beta_i f(u(x_i,t)) - a(x_i)\frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t)$$

$$+(a(x_i) - \alpha_i)\delta^2 u(x_i,t) + (\beta_i - b(x_i))f(u(x_i,t)),\ 0 \leq i \leq I,\ t \in (0,t(h)).$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. From the mean value theorem, we have

$$\frac{de_i(t)}{dt} - \alpha_i\delta^2 e_i(t) = -\beta_i f'(\theta_i(t))e_i(t) + a(x_i)\frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t)$$

$$-(a(x_i) - \alpha_i)\delta^2 u(x_i,t) - (\beta_i - b(x_i))f(u(x_i,t)),\ \ 0 \leq i \leq I,\ \ t \in (0,t(h)),$$

where $\theta_i(t)$ is an intermediate value between $U_i(t)$ and $u(x_i,t)$. We observe that for $i \in \{0,\cdots,I\}$ and $t \in (0,t(h))$, $f(u(x_i,t))$ is bounded from above $f(\rho)$. Since $u \in C^{4,1}$, then making use of (4.3) and (4.5), we realize that there exists a positive constant K such that

$$\frac{de_i(t)}{dt} - \alpha_i\delta^2 e_i(t) \leq L|e_i(t)| + K\|\beta_h - b_h\|_\infty + K\|\alpha_h - a_h\|_\infty + Kh^2,$$

$$0 \leq i \leq I, \quad t \in (0,t(h)). \tag{4.6}$$

Introduce the vector $z_h(t)$ defined as follows

$$z_i(t) = e^{(L+1)t}(\|\varphi_h - u_h(0)\|_\infty + K\|\beta_h - b_h\|_\infty + K\|\alpha_h - a_h\|_\infty + Kh^2),$$

$$0 \leq i \leq I, \quad t \in (0,t(h)). \tag{4.7}$$

A straightforward computation reveals that

$$\frac{dz_i}{dt} - \delta^2 z_i > L|z_i| + K\|\beta_h - b_h\|_\infty + K\|\alpha_h - a_h\|_\infty + Kh^2,$$

$$0 \leq i \leq I, \quad t \in (0,t(h)),$$

$$z_i(0) > e_i(0), \quad 0 \le i \le I.$$

It follows from Lemma 2.2 that

$$z_i(t) > e_i(t) \quad \text{for} \quad t \in (0, t(h)), \quad 0 \le i \le I.$$

In the same way, we also prove that

$$z_i(t) > -e_i(t) \quad \text{for} \quad t \in (0, t(h)), \quad 0 \le i \le I,$$

which implies that

$$\|U_h(t) - u_h(t)\|_\infty \le e^{(L+1)t}(\|\varphi_h - u_h(0)\|_\infty + K\|\beta_h - b_h\|_\infty$$

$$+K\|\alpha_h - a_h\|_\infty + Kh^2) \quad \text{for} \quad t \in (0, t(h)).$$

Let us show that $t(h) = \min\{T-\tau, T_q^h\}$. Suppose that $t(h) < \min\{T-\tau, T_q^h\}$. Due to the fact that $t(h)$ is the greatest value of $t > 0$ such that (4.4) holds, we infer that $\|U_h(t(h)) - u_h(t(h))\|_\infty \ge \frac{\varrho}{2}$, which implies that

$$\frac{\varrho}{2} \le \|U_h(t(h)) - u_h(t(h))\|_\infty$$

$$\le e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + K\|\beta_h - b_h\|_\infty + K\|\alpha_h - a_h\|_\infty + Kh^2).$$

Let us notice that both last formulas for $t(h)$ are valid for sufficiently small h. Since the term on the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $\frac{\varrho}{2} \le 0$, which is impossible. Consequently $t(h) = \min\{T-\tau, T_q^h\}$.
Now, let us show that $t(h) = T - \tau$. Suppose that $t(h) = T_q^h < T - \tau$. Reasoning as above, we prove that we have a contradiction and the proof is complete. □
Now, we are in a position to prove the main theorem of this section.

Theorem 4.2. *Suppose that the problem (1.1)–(1.3) has a solution u which quenches in a finite time T such that $u \in C^{4,1}([0,1] \times [0,T))$. Assume that*

φ_h, β_h and α_h satisfy the conditions (4.1) and (4.2). Under the hypothesis of Theorem 3.1, the problem (2.1)–(2.3) admits a unique solution U_h which quenches in a finite time T_q^h, and the following relation holds

$$\lim_{h\to 0} T_q^h = T.$$

Proof. Let $0 < \varepsilon < T/2$. There exists $\varrho \in (0,1)$ such that

$$\frac{1}{A}\int_0^{\varrho} \frac{d\sigma}{f(\sigma)} \le \frac{\varepsilon}{2}. \tag{4.8}$$

Since u quenches in a finite time T, there exist $h_0(\varepsilon) > 0$ and a time $T_0 \in (T - \frac{\varepsilon}{2}, T)$ such that $0 < u_{\min}(t) < \frac{\varrho}{2}$ for $t \in [T_0, T)$, $h \le h_0(\varepsilon)$. It is not hard to see that $u_{\min}(t) > 0$ for $t \in [0, T_0]$, $h \le h_0(\varepsilon)$. According to Theorem 4.1, the problem (2.1)–(2.3) admits a unique solution $U_h(t)$, and the following estimate holds $\|U_h(t) - u_h(t)\|_\infty \le \frac{\varrho}{2}$ for $t \in [0, T_0]$, $h \le h_0(\varepsilon)$, which implies that $\|U_h(T_0) - u_h(T_0)\|_\infty \le \frac{\varrho}{2}$ for $h \le h_0(\varepsilon)$. Applying the triangle inequality, we find that

$$U_{hmin}(T_0) \le \|U_h(T_0) - u_h(T_0)\|_\infty + u_{hmin}(T_0) \le \frac{\varrho}{2} + \frac{\varrho}{2} = \varrho \quad \text{for} \quad h \le h_0(\varepsilon).$$

Invoking Theorem 3.1, we note that $U_h(t)$ quenches at the time T_q^h. We deduce from Remark 3.1 and (4.8) that for $h \le h_0(\varepsilon)$,

$$|T_q^h - T| \le |T_q^h - T_0| + |T_0 - T| \le \frac{1}{A}\int_0^{U_{hmin}(T_0)} \frac{d\sigma}{f(\sigma)} + \frac{\varepsilon}{2} \le \varepsilon.$$

This completes the proof. □

5. Full discretizations

In this section, we pursue our study concerning the phenomenon of quenching using a full discrete explicit scheme of (1.1)–(1.3). Approximate the solution $u(x,t)$ of the problem (1.1)–(1.3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following explicit scheme

$$\delta_t U_i^{(n)} = \alpha_i \delta^2 U_i^{(n)} - \beta_i f(U_i^{(n)}), \quad 0 \le i \le I, \tag{5.1}$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \le i \le I, \tag{5.2}$$

where $n \ge 0$,

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}.$$

We observe that $\frac{f(s)}{s}$ is nonincreasing by the following

$$\left(\frac{f(s)}{s}\right)' = \frac{f'(s)s - f(s)}{s^2} \le 0 \quad \text{for} \quad s > 0.$$

Hence, if $U_h^{(n)} > 0$, then $-\frac{f(U_i^{(n)})}{U_i^{(n)}} \ge -\frac{f(U_{hmin}^{(n)})}{U_{hmin}^{(n)}}$, $0 \le i \le I$, and a straightforward computation reveals that

$$U_0^{(n+1)} \ge \frac{2\alpha_0 \Delta t_n}{h^2} U_1^{(n)} + \left(1 - 2\|\alpha_h\|_\infty \frac{\Delta t_n}{h^2} - \|\beta_h\|_\infty \Delta t_n \frac{f(U_{hmin}^{(n)})}{U_{hmin}^{(n)}}\right) U_0^{(n)},$$

$$U_i^{(n+1)} \ge \frac{\alpha_i \Delta t_n}{h^2} U_{i+1}^{(n)} + \left(1 - 2\|\alpha_h\|_\infty \frac{\Delta t_n}{h^2} - \|\beta_h\|_\infty \Delta t_n \frac{f(U_{hmin}^{(n)})}{U_{hmin}^{(n)}}\right) U_i^{(n)}$$

$$+ \frac{\alpha_i \Delta t_n}{h^2} U_{i-1}^{(n)}, \quad 1 \le i \le I-1,$$

$$U_I^{(n+1)} \ge \frac{2\alpha_I \Delta t_n}{h^2} U_{I-1}^{(n)} + \left(1 - 2\|\alpha_h\|_\infty \frac{\Delta t_n}{h^2} - \|\beta_h\|_\infty \Delta t_n \frac{f(U_{hmin}^{(n)})}{U_{hmin}^{(n)}}\right) U_I^{(n)}.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T, we need to adapt the size of the time step so that we pick

$$\Delta t_n = \min\left\{\frac{(1-\tau)h^2}{2\|\alpha_h\|_\infty}, \tau \frac{U_{hmin}^{(n)}}{\|\beta_h\|_\infty f(U_{hmin}^{(n)})}\right\}$$

with $0 < \tau < 1$. We observe that $1 - 2\|\alpha_h\|_\infty \frac{\Delta t_n}{h^2} - \|\beta_h\|_\infty \Delta t_n \frac{f(U_{hmin}^{(n)})}{U_{hmin}^{(n)}} \ge 0$, which implies that $U_h^{(n+1)} > 0$. Thus, since by hypothesis $U_h^{(0)} = \varphi_h > 0$, if we take Δt_n as defined above, then using a recursion argument, we see that

the positivity of the discrete solution is guaranteed. Here, τ is a parameter which will be chosen later to allow the discrete solution $U_h^{(n)}$ to satisfy certain properties useful to get the convergence of the numerical quenching time defined below.
If necessary, we may take $\Delta t_n = \min\{\frac{(1-\tau)h^2}{K\|\alpha_h\|_\infty}, \tau\frac{U_{hmin}^{(n)}}{\|\beta_h\|_\infty f(U_{hmin}^{(n)})}\}$ with $K > 2$ because in this case, the positivity of the discrete solution is also guaranteed.
The following lemma is a discrete form of the maximum principle.

Lemma 5.1. *Let $\gamma_h^{(n)}$ and $V_h^{(n)}$ be two sequences such that $\gamma_h^{(n)}$ is bounded and*

$$\delta_t V_i^{(n)} - \alpha_i \delta^2 V_i^{(n)} + \gamma_i^{(n)} V_i^{(n)} \geq 0, \quad 0 \leq i \leq I, \quad n \geq 0, \tag{5.3}$$

$$V_i^{(0)} \geq 0, \quad 0 \leq i \leq I. \tag{5.4}$$

Then, we have $V_i^{(n)} \geq 0$ for $n \geq 0$, $0 \leq i \leq I$ if $\Delta t_n \leq \frac{h^2}{2\|\alpha_h\|_\infty + \|\gamma_h^{(n)}\|_\infty h^2}$.

Proof. If $V_h^{(n)} \geq 0$, then a routine computation yields

$$V_0^{(n+1)} \geq \frac{2\alpha_0 \Delta t_n}{h^2} V_1^{(n)} + \left(1 - 2\|\alpha_h\|_\infty \frac{\Delta t_n}{h^2} - \Delta t_n \|\gamma_h^{(n)}\|_\infty\right) V_0^{(n)},$$

$$V_i^{(n+1)} \geq \frac{\alpha_i \Delta t_n}{h^2} V_{i+1}^{(n)} + \left(1 - 2\|\alpha_h\|_\infty \frac{\Delta t_n}{h^2} - \Delta t_n \|\gamma_h^{(n)}\|_\infty\right) V_i^{(n)}$$

$$+\frac{\alpha_i \Delta t_n}{h^2} V_{i-1}^{(n)}, \quad 1 \leq i \leq I-1,$$

$$V_I^{(n+1)} \geq \frac{2\alpha_I \Delta t_n}{h^2} V_{I-1}^{(n)} + \left(1 - 2\|\alpha_h\|_\infty \frac{\Delta t_n}{h^2} - \Delta t_n \|\gamma_h^{(n)}\|_\infty\right) V_I^{(n)}.$$

Since $\Delta t_n \leq \frac{h^2}{2\|\alpha_h\|_\infty + \|\gamma_h^{(n)}\|_\infty h^2}$, we see that $1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|\gamma_h^{(n)}\|_\infty$ is nonnegative. Making use of (5.4), we deduce by induction that $V_h^{(n)} \geq 0$, which ends the proof. □

A direct consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 5.2. *Let $V_h^{(n)}$, $W_h^{(n)}$ and $\gamma_h^{(n)}$ be three sequences such that $\gamma_h^{(n)}$ is bounded and*

$$\delta_t V_i^{(n)} - \alpha_i \delta^2 V_i^{(n)} + \gamma_i^{(n)} V_i^{(n)} \leq \delta_t W_i^{(n)} - \alpha_i \delta^2 W_i^{(n)} + \gamma_i^{(n)} W_i^{(n)},$$

$$0 \leq i \leq I, \quad n \geq 0,$$

$$V_i^{(0)} \leq W_i^{(0)}, \quad 0 \leq i \leq I.$$

Then, we have $V_i^{(n)} \leq W_i^{(n)}$ for $n \geq 0$, $0 \leq i \leq I$ if $\Delta t_n \leq \frac{h^2}{2\|\alpha_h\|_\infty + \|\gamma_h^{(n)}\|_\infty h^2}$.

Now, let us give a property of the operator δ_t stated in the following lemma. Its proof is quite similar to that of Lemma 3.1, so we omit it here.

Lemma 5.3. *Let $U^{(n)} \in \mathbf{R}$ be such that $U^{(n)} > 0$ for $n \geq 0$. Then, we have*

$$\delta_t f(U^{(n)}) \geq f'(U^{(n)}) \delta_t U^{(n)}, \quad n \geq 0.$$

We need the result below.

Lemma 5.4. *Let a, b be two positive numbers such that $b \in (0,1)$. Then the following estimate holds*

$$\sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)} \leq \frac{a}{f(a)} - \frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)}.$$

Proof. We have $\int_0^\infty \frac{ab^x dx}{f(ab^x)} = \sum_{n=0}^\infty \int_n^{n+1} \frac{ab^x dx}{f(ab^x)}$. We observe that $ab^x \geq ab^{n+1}$ for $n \leq x \leq n+1$, which implies that $\int_n^{n+1} \frac{ab^x dx}{f(ab^x)} \geq \frac{ab^{n+1}}{f(ab^{n+1})}$. Consequently, we get

$$\int_0^\infty \frac{ab^x dx}{f(ab^x)} \geq \sum_{n=0}^{\infty} \frac{ab^{n+1}}{f(ab^{n+1})} = -\frac{a}{f(a)} + \sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)}.$$

By a change of variables, we see that $\int_0^\infty \frac{ab^x dx}{f(ab^x)} = -\frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)}$, which implies that

$$\sum_{n=0}^{\infty} \frac{ab^n}{f(ab^n)} \leq \frac{a}{f(a)} - \frac{1}{\ln(b)} \int_0^a \frac{d\sigma}{f(\sigma)}.$$

This completes the proof. □

The theorem below is the discrete version of Theorem 4.1.

Theorem 5.1. *Suppose that the problem (1.1)–(1.3) has a solution* $u \in C^{4,2}([0,1] \times [0, T-\tau])$ *such that* $\min_{t\in[0,T]} u_{\min}(t) = \rho > 0$ *with* $\tau \in (0,T)$. *Assume that* φ_h, β_h *and* α_h *satisfy the conditions (4.1) and (4.2). Then, the problem (5.1)–(5.2) has a solution* $U_h^{(n)}$ *for h sufficiently small,* $0 \le n \le J$ *and the following relation holds*

$$\max_{0\le n\le J} \|U_h^{(n)} - u_h(t_n)\|_\infty$$

$$= O(\|\varphi_h - u_h(0)\|_\infty + \|b_h - \beta_h\|_\infty + \|\alpha_h - a_h\|_\infty + h^2) \quad as \quad h \to 0,$$

where J *is any quantity satisfying the inequality*

$$\sum_{j=0}^{J-1} \Delta t_j \le T - \tau \quad \text{and} \quad t_n = \sum_{j=0}^{n-1} \Delta t_j.$$

Proof. For each h, the problem (5.1)–(5.2) has a solution $U_h^{(n)}$. Let $N \le J$ be the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{\rho}{2} \quad \text{for} \quad n < N. \tag{5.5}$$

We know that $N \ge 1$ because of (4.1). Applying the triangle inequality, we have

$$U_i^{(n)} \ge u(x_i, t_n) - |U_i^{(n)} - u(x_i, t_n)|, \quad 0 \le i \le I, \quad n < N.$$

This implies that

$$U_i^{(n)} \ge u(x_i, t_n) - \|U_h^{(n)} - u_h(t_n)\|_\infty, \quad 0 \le i \le I, \quad n < N.$$

Let $i_0 \in \{0, \cdots, I\}$ be such that $U_{i_0}^{(n)} = U_{hmin}^{(n)}$. Replacing i by i_0 in the above inequalities and using the fact that $u(x_{i_0}, t_n) \ge u_{hmin}(t_n)$, we arrive at

$$U_{hmin}^{(n)} \ge u_{hmin}(t_n) - \|U_h^{(n)} - u_h(t_n)\|_\infty \ge \frac{\rho}{2} \quad \text{for} \quad n < N. \tag{5.6}$$

As in the proof of Theorem 4.1, using Taylor's expansion, we find that for $n < N$, $0 \le i \le I$,

$$\delta_t u(x_i, t_n) - \alpha_i \delta^2 u(x_i, t_n) + \beta_i f(u(x_i, t_n)) + (b(x_i) - \beta_i) f(u(x_i, t_n))$$

$$+(\alpha_i - a(x_i))\delta^2 u(x_i,t_n) = -a(x_i)\frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t_n) + \frac{\Delta t_n}{2}u_{tt}(x_i,\widetilde{t}_n).$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem, we get for $n < N$, $0 \le i \le I$,

$$\delta_t e_i^{(n)} - \alpha_i \delta^2 e_i^{(n)} = -\beta_i f'(\xi_i^{(n)})e_i^{(n)} + a(x_i)\frac{h^2}{12}u_{xxxx}(\widetilde{x}_i,t_n)$$

$$+(\alpha_i - a(x_i))\delta^2 u(x_i,t_n) - \frac{\Delta t_n}{2}u_{tt}(x_i,\widetilde{t}_n) + (b(x_i)-\beta_i)f(u(x_i,t_n)),$$

where $\xi_i^{(n)}$ is an intermediate value between $u(x_i,t_n)$ and $U_i^{(n)}$. Since $u_{xxxx}(x,t)$, $\delta^2 u(x_i,t_n)$, $0 \le i \le I$, $u_{tt}(x,t)$ are bounded, $u(x,t) \ge \rho$ and $\Delta t_n = O(h^2)$, then there exists a positive constant M such that

$$\delta_t e_i^{(n)} - \alpha_i \delta^2 e_i^{(n)} \le -\beta_i f'(\xi_i^{(n)})e_i^{(n)} + M\|b_h - \beta_h\|_\infty$$

$$+M\|\alpha_h - a_h\|_\infty + Mh^2, \quad 0 \le i \le I, \quad n < N. \tag{5.7}$$

Set $L = -(\|b_h\|_\infty + 1)f'(\frac{\rho}{2})$ and introduce the vector $V_h^{(n)}$ defined as follows

$$V_i^{(n)} = e^{(L+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + M\|b_h - \beta_h\|_\infty + M\|\alpha_h - a_h\|_\infty + Mh^2),$$

$0 \le i \le I$, $\quad n < N$. It is clear that L is positive because the function $f(s)$ is positive and nonincreasing for positive values of s. A straightforward computation gives

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} > -\beta_i f'(\xi_i^{(n)})V_i^{(n)} + M\|b_h - \beta_h\|_\infty$$

$$+M\|\alpha_h - a_h\|_\infty + Mh^2, \quad 0 \le i \le I, \quad n < N, \tag{5.8}$$

$$V_i^{(0)} > e_i^{(0)}, \quad 0 \le i \le I. \tag{5.9}$$

For $i \in \{0,\cdots,I\}$, according to the fact that $\xi_i^{(n)}$ is between $U_i^{(n)}$ and $u(x_i,t_n)$, we infer from (5.6) that $\xi_i^{(n)} \ge \frac{\rho}{2}$ because $u(x_i,t_n) \ge \rho$. This implies that $-\beta_i f'(\xi_i^{(n)})$ is bounded from above by the quantity $-\|\beta_h\|_\infty f'(\frac{\rho}{2})$

which is positive because $f(s)$ is positive and nonincreasing for positive values of s. It follows from Lemma 5.2 that $V_h^{(n)} \geq e_h^{(n)}$. In the same way, we also prove that $V_h^{(n)} \geq -e_h^{(n)}$, which implies that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty$$

$$\leq e^{(L+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + M\|b_h - \beta_h\|_\infty + M\|\alpha_h - a_h\|_\infty + Mh^2), \ n < N. \tag{5.10}$$

Let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (5.10) and use (5.5), we find that

$$\frac{\rho}{2} \leq \|U_h^{(N)} - u_h(t_N)\|_\infty$$

$$\leq e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + M\|b_h - \beta_h\|_\infty + M\|\alpha_h - a_h\|_\infty + Mh^2).$$

Since the term on the right hand side of the second inequality goes to zero as h goes to zero, we deduce that $\frac{\rho}{2} \leq 0$, which is a contradiction and the proof is complete. $\square$

To handle the phenomenon of quenching for discrete equations, we need the following definition.

Definition 5.1. *We say that the solution $U_h^{(n)}$ of (5.1)-(5.2) quenches in a finite time if $U_{hmin}^{(n)} > 0$ for $n \geq 0$, but*

$$\lim_{n\to\infty} U_{hmin}^{(n)} = 0 \quad and \quad T_h^{\Delta t} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta t_i < \infty.$$

The number $T_h^{\Delta t}$ is called the numerical quenching time of $U_h^{(n)}$.

The following theorem reveals that the discrete solution $U_h^{(n)}$ of (5.1)-(5.2) quenches in a finite time under some hypotheses.

Theorem 5.2. *Let $U_h^{(n)}$ be the solution of (5.1)-(5.2). Suppose that there exists a constant $A \in (0, 1]$ such that the initial data at (5.2) satisfies*

$$\alpha_i \delta^2 \varphi_i - \beta_i f(\varphi_i) \leq -Af(\varphi_i), \quad 0 \leq i \leq I. \tag{5.11}$$

Then $U_h^{(n)}$ is nonincreasing and quenches in a finite time $T_h^{\Delta t}$ which satisfies the following estimate

$$T_h^{\Delta t} \leq \frac{\tau\varphi_{hmin}}{\|\beta_h\|_\infty f(\varphi_{hmin})} - \frac{\tau}{\|\beta_h\|_\infty \ln(1-\tau')} \int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)},$$

where $\tau' = A\min\{\frac{(1-\tau)h^2 f(\varphi_{hmin})}{2\|\alpha_h\|_\infty \varphi_{hmin}}, \tau\}$.

Proof. Introduce the vector $J_h^{(n)}$ defined as follows

$$J_i^{(n)} = \delta_t U_i^{(n)} + Af(U_i^{(n)}), \quad 0 \leq i \leq I, \quad n \geq 0.$$

A straightforward computation yields for $0 \leq i \leq I$, $n \geq 0$,

$$\delta_t J_i^{(n)} - \alpha_i \delta^2 J_i^{(n)} = \delta_t \left(\delta_t U_i^{(n)} - \alpha_i \delta^2 U_i^{(n)}\right) + A\delta_t f(U_i^{(n)}) - A\alpha_i \delta^2 f(U_i^{(n)}).$$

Using (5.1), we arrive at

$$\delta_t J_i^{(n)} - \alpha_i \delta^2 J_i^{(n)} = -(\beta_i - A)\delta_t f(U_i^{(n)})$$

$$-A\alpha_i \delta^2 f(U_i^{(n)}), \quad 0 \leq i \leq I, \quad n \geq 0.$$

It follows from Lemmas 5.3 and 3.1 that for $0 \leq i \leq I$, $n \geq 0$,

$$\delta_t J_i^{(n)} - \alpha_i \delta^2 J_i^{(n)} \quad \leq \quad -(\beta_i - A) f'(U_i^{(n)})\delta_t U_i^{(n)} - A\alpha_i f'(U_i^{(n)})\delta^2 U_i^{(n)}.$$

After a little transformation, the above estimates become

$$\delta_t J_i^{(n)} - \alpha_i \delta^2 J_i^{(n)} \quad \leq \quad -\beta_i f'(U_i^{(n)})\delta_t U_i^{(n)}$$

$$+Af'(U_i^{(n)})(\delta_t U_i^{(n)} - \alpha_i \delta^2 U_i^{(n)}), \quad 0 \leq i \leq I.$$

We deduce from (5.1) that

$$\delta_t J_i^{(n)} - \alpha_i \delta^2 J_i^{(n)} \leq -\beta_i f'(U_i^{(n)}) J_i^{(n)}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

Obviously, the inequalities (5.11) ensure that $J_h^{(0)} \leq 0$. Applying Lemma 5.1, we get $J_h^{(n)} \leq 0$ for $n \geq 0$, which implies that

$$U_i^{(n+1)} \leq U_i^{(n)} \left(1 - A\Delta t_n \frac{f(U_i^{(n)})}{U_i^{(n)}}\right), \qquad 0 \leq i \leq I, \quad n \geq 0. \tag{5.12}$$

These estimates reveal that the sequence $U_h^{(n)}$ is nonincreasing. By induction, we obtain $U_h^{(n)} \leq U_h^{(0)} = \varphi_h$. Thus, the following holds

$$(5.13) A\Delta t_n \frac{f(U_{hmin}^{(n)})}{U_{hmin}^{(n)}} \geq A \min\left\{ \frac{(1-\tau)h^2 f(\varphi_{hmin})}{2\|\alpha_h\|_\infty \varphi_{hmin}}, \frac{\tau}{\|\beta_h\|} \right\} = \tau'.$$

Let i_0 be the index such that $U_{hmin}^{(n)} = U_{i_0}^{(n)}$. Replacing i by i_0 in (5.2), we obtain

$$(5.14) \qquad U_{hmin}^{(n+1)} \leq U_{hmin}^{(n)}(1-\tau'), \quad n \geq 0,$$

and by iteration, we arrive at

$$(5.15) \qquad U_{hmin}^{(n)} \leq U_{hmin}^{(0)}(1-\tau')^n = \varphi_{hmin}(1-\tau')^n, \quad n \geq 0.$$

Since the term on the right hand side of the above equality goes to zero as n approaches infinity, we conclude that $U_{hmin}^{(n)}$ tends to zero as n approaches infinity. Now, let us estimate the numerical quenching time. Due to (5.15) and the restriction $\Delta t_n \leq \frac{\tau U_{hmin}^{(n)}}{\|\beta_h\|_\infty f(U_{hmin}^{(n)})}$, it is not hard to see that

$$\sum_{n=0}^{\infty} \Delta t_n \leq \frac{\tau}{\|\beta_h\|_\infty} \sum_{n=0}^{\infty} \frac{\varphi_{hmin}(1-\tau')^n}{f(\varphi_{hmin}(1-\tau')^n)},$$

because $\frac{s}{f(s)}$ is nondecreasing for $s > 0$. It follows from Lemma 5.4 that

$$\sum_{n=0}^{\infty} \Delta t_n \leq \frac{\tau \varphi_{hmin}}{\|\beta_h\|_\infty f(\varphi_{hmin})} - \frac{\tau}{\|\beta_h\|_\infty \ln(1-\tau')} \int_0^{\varphi_{hmin}} \frac{d\sigma}{f(\sigma)}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof. □

Remark 5.1. *From (5.14), we deduce by induction that*

$$U_{hmin}^{(n)} \leq U_{hmin}^{(q)}(1-\tau')^{n-q} \quad \textit{for} \quad n \geq q,$$

and we see that

$$T_h^{\Delta t} - t_q = \sum_{n=q}^{\infty} \Delta t_n \leq \frac{\tau}{\|\beta_h\|_\infty} \sum_{n=q}^{\infty} \frac{U_{hmin}^{(q)}(1-\tau')^{n-q}}{f(U_{hmin}^{(q)}(1-\tau')^{n-q})},$$

because $\frac{s}{f(s)}$ *is nondecreasing for* $s > 0$. *It follows from Lemma 5.4 that*

$$T_h^{\Delta t} - t_q \leq \frac{\tau U_{hmin}^{(q)}}{\|\beta_h\|_\infty f(U_{hmin}^{(q)})} - \frac{\tau}{\|\beta_h\|_\infty \ln(1-\tau')} \int_0^{U_{hmin}^{(q)}} \frac{d\sigma}{f(\sigma)}.$$

Since $\tau' = A \min\{\frac{(1-\tau)h^2 f(\varphi_{hmin})}{2\|\alpha_h\|_\infty \varphi_{hmin}}, \frac{\tau}{\|\beta_h\|_\infty}\}$, *if we take* $\tau = h^2$, *then we get*

$$\frac{\tau'}{\tau} = A \min\left\{\frac{(1-h^2)f(\varphi_{hmin})}{2\|\alpha_h\|_\infty \varphi_{hmin}}, \frac{1}{\|\beta_h\|_\infty}\right\} \geq A \min\left\{\frac{f(\varphi_{hmin})}{4\|\alpha_h\|_\infty \varphi_{hmin}}, \frac{1}{\|\beta_h\|_\infty}\right\}.$$

Therefore, there exist constants c_0, c_1 *such that* $0 < c_0 \leq \tau/\tau' \leq c_1$ *and* $\frac{-\tau}{\ln(1-\tau')} = O(1)$, *for the choice* $\tau = h^2$.

In the sequel, we take $\tau = h^2$.
Now, we are in a position to state the main theorem of this section.

Theorem 5.3. *Suppose that the problem (1.1)–(1.3) has a solution* u *which quenches in a finite time* T *and* $u \in C^{4,2}([0,1] \times [0,T))$. *Assume that* φ_h, β_h *and* α_h *satisfy the conditions (4.1) and (4.2). Under the assumption of Theorem 5.2, the problem (5.1)–(5.2) has a solution* $U_h^{(n)}$ *which quenches in a finite time* $T_h^{\Delta t}$ *and the following relation holds*

$$\lim_{h\to 0} T_h^{\Delta t} = T.$$

Proof. We know from Remark 5.1 that $\frac{-\tau}{\ln(1-\tau')}$ is bounded. Letting $0 < \varepsilon < T/2$, there exists a constant $R \in (0,1)$ such that

$$\frac{\tau R}{\|\beta_h\|_\infty f(R)} - \frac{\tau}{\|\beta_h\|_\infty \ln(1-\tau')} \int_0^R \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}. \tag{5.16}$$

Since u quenches at the time T, there exist $T_1 \in (T - \frac{\varepsilon}{2}, T)$ and $h_0(\varepsilon) > 0$ such that $0 < u_{\min}(t) < \frac{R}{2}$ for $t \in [T_1, T)$, $h \leq h_0(\varepsilon)$. Let q be a positive integer such that $t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_1, T)$ for $h \leq h_0(\varepsilon)$. It follows from Theorem 5.1 that the problem (5.1)–(5.2) has a solution $U_h^{(n)}$ which obeys $\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{R}{2}$ for $n \leq q$, $h \leq h_0(\varepsilon)$, which implies that

$$U_{hmin}^{(q)} \leq \|U_h^{(q)} - u_h(t_q)\|_\infty + u_{hmin}(t_q) < \frac{R}{2} + \frac{R}{2} = R, \quad h \leq h_0(\varepsilon).$$

From Theorem 5.2, $U_h^{(n)}$ quenches at the time $T_h^{\Delta t}$. It follows from Remark 5.1 and (5.16) that

$$|T_h^{\Delta t} - t_q| \leq \frac{\tau U_{hmin}^{(q)}}{\|\beta_h\|_\infty f(U_{hmin}^{(q)})} - \frac{\tau}{\|\beta_h\|_\infty \ln(1-\tau')} \int_0^{U_{hmin}^{(q)}} \frac{d\sigma}{f(\sigma)} < \frac{\varepsilon}{2}$$

because

$U_{hmin}^{(q)} < R$ for $h \leq h_0(\varepsilon)$. We deduce that for $h \leq h_0(\varepsilon)$,

$$|T - T_h^{\Delta t}| \leq |T - t_q| + |t_q - T_h^{\Delta t}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon,$$

which leads us to the result. □

6. Numerical results

In this section, we present some numerical approximations to the quenching time for the solution of the problem (1.1)–(1.3) in the case where $f(u) = u^{-p}$ with $p = \text{const} > 0$, $u_0(x) = \frac{2+\varepsilon\cos(\pi x)}{4}$, $a(x) = 2 - \varepsilon\sin(\pi h)$, and $b(x) = 3 - \varepsilon(x^3+1)$ with $0 < \varepsilon \leq 1$. Firstly, we take the explicit scheme in (5.1)–(5.2). Secondly, we use the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \alpha_i \delta^2 U_i^{(n+1)} - \beta_i (U_i^{(n)})^{-p-1} U_i^{(n+1)}, \quad 0 \leq i \leq I,$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n = h^2 (U_{hmin}^{(n)})^{(p+1)}$.

In both cases, $\varphi_i = \frac{2+\varepsilon\cos(\pi i h)}{4}$, $0 \leq i \leq I$, $\alpha_i = 2 - \varepsilon\sin(i\pi h)$, $\beta_i = 3 - \varepsilon(i^2h^2+1)$. For the above implicit scheme, the existence and positivity of the discrete solution $U_h^{(n)}$ are guaranteed using standard methods (see [3]). In the tables 1–6, in rows, we present the numerical quenching times, the numbers of iterations and the CPU times corresponding to meshes of 16, 32, 64, 128. We take for the numerical quenching time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \leq 10^{-16}.$$

First case: $p = 1; \varepsilon = 1;$

Table 1: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method

I	t_n	n	CPU_t
16	0.055633	3806	2.8
32	0.055445	14556	12.4
64	0.055398	55460	152
128	0.0553867	210677	661

Table 2: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method

I	t_n	n	CPU_t
16	0.055558	3806	5.4
32	0.055426	14555	25
64	0.055393	55459	367
128	0.055863	210676	987

Second case: $p = 1; \varepsilon = 1/100;$

Table 3: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method

I	t_n	n	CPU_t
16	0.0419564	1278	1.2
32	0.0417734	4897	5.4
64	0.0417279	18656	63.5
128	0.0417116	70841	620

Table 4: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method

I	t_n	n	CPU_t
16	0.0419552	1278	1.6
32	0.0417731	4897	8.5
64	0.0417278	18656	130
128	0.0417165	70841	2461

Third case: $p = 1; \varepsilon = 1/10000;$

Table 5: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the explicit Euler method

I	t_n	n	CPU_t
16	0.0419126	1269	1.1
32	0.0417282	4861	4.3
64	0.0416823	18517	63
128	0.0416709	70300	942

Table 6: Numerical quenching times, numbers of iterations and CPU times (seconds) obtained with the implicit Euler method

I	t_n	n	CPU_t
16	0.0418126	1269	1.6
32	0.0417282	4861	8.4
64	0.0416823	18517	124
128	0.0416709	70300	2400

Remark 6.1. *When $\varepsilon = 0$ and $p = 1$, we know that the quenching time of the continuous solution of (1.1)–(1.3) is the same as the one of the solution $\alpha(t)$ of the following differential equation $\alpha'(t) = -3(\alpha(t))^{-p}$, $t > 0$, $\alpha(0) = 0.5$. It is clear that the quenching time of the solution $\alpha(t)$ is 0.0416666. We observe from Tables 1–8 that when ε decays to zero, then the numerical quenching time of the discrete solution goes to 0.0416666.*

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References

[1] **L. M. Abia, J. C. López-Marcos and J. Martinez,** On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, Appl. Numer. Math., **26**, pp. 399-414, (1998).

[2] **A. Acker and B. Kawohl,** Remarks on quenching, Nonl. Anal. TMA, **13**, pp. 53-61, (1989).

[3] **T. K. Boni,** Extinction for discretizations of some semilinear parabolic equations, C. R. Acad. Sci. Paris, Sér. I, **333**, pp. 795-800, (2001).

[4] **T. K. Boni,** On quenching of solutions for some semilinear parabolic equations of second order, Bull. Belg. Math. Soc., **7**, pp. 73-95, (2000).

[5] **M. Fila, B. Kawohl and H. A. Levine,** Quenching for quasilinear equations, Comm. Part. Diff. Equat., **17**, pp. 593-614, (1992).

[6] **J. S. Guo and B. Hu,** The profile near quenching time for the solution of a singular semilinear heat equation, Proc. Edin. Math. Soc., **40**, pp. 437-456, (1997).

[7] **J. Guo,** On a quenching problem with Robin boundary condition, Nonl. Anal. TMA, **17**, pp. 803-809, (1991).

[8] **V. A. Galaktionov and J. L. Vázquez,** Continuation of blow-up solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math., **50**, pp. 1-67, (1997).

[9] **V. A. Galaktionov and J. L. Vázquez,** The problem of blow-up in nonlinear parabolic equations, Discrete Contin. Systems A, **8**, pp. 399-433, (2002).

[10] **M. A. Herrero and J. J. L. Velazquez,** Generic behaviour of one dimensional blow up patterns, Ann. Scuola Norm. Sup. di Pisa, **XIX**, pp. 381-950, (1992).

[11] **C. M. Kirk and C. A. Roberts,** A review of quenching results in the context of nonlinear volterra equations, Dyn. contin. Discrete Impuls. Syst. Ser. A, Math. Anal., **10**, pp. 343-356, (2003).

[12] **H. A. Levine,** Quenching, nonquenching and beyond quenching for solutions of some parabolic equations, Annali Math. Pura Appl., **155**), pp. 243-260, (1990).

[13] **K. W. Liang, P. Lin and R. C. E. Tan,** Numerical solution of quenching problems using mesh-dependent variable temporal steps, Appl. Numer. Math., **57**, pp. 791-800, (2007).

[14] **K. W. Liang, P. Lin, M. T. Ong and R. C. E. Tan,** A splitting moving mesh method for reaction-diffusion equations of quenching type, J. Comput. Phys., **215**, pp. 757-777, (2006).

[15] **T. Nakagawa,** Blowing up on the finite difference solution to $u_t = u_{xx} + u^2$, Appl. Math. Optim., **2**, pp. 337-350, (1976).

[16] **D. Nabongo and T. K. Boni**, Quenching for semidiscretization of a heat equation with a singular boundary condition, *Asympt. Anal.,* **59**, pp. 27-38, (2008).

[17] **D. Nabongo and T. K. Boni**, Quenching time of solutions for some nonlinear parabolic equations, *An. St. Univ. Ovidius Constanta*, **16**, pp. 87-102, (2008).

[18] **D. Nabongo and T. K. Boni,** Quenching for semidiscretization of a semilinear heat equation with Dirichlet and Neumann boundary condition, Comment. Math. Univ. Carolinae, **49**, pp. 463-475, (2008).

[19] **D. Nabongo and T. K. Boni,** Numerical quenching for a semilinear parabolic equation, Math. Modelling and Anal., To appear.

[20] **M. H. Protter and H. F. Weinberger,** Maximum principles in differential equations, Prentice Hall, Englewood Cliffs, NJ, (1967).

[21] **Q. Sheng and A. Q. M. Khaliq,** Adaptive algorithms for convection-diffusion-reaction equations of quenching type, Dyn. Contin. Discrete Impuls. Syst. Ser. A, Math. Anal., **8**, pp. 129-148, (2001).

[22] **Q. Sheng and A. Q. M. Khaliq,** A compound adaptive approach to degenerate nonlinear quenching problems, Numer. Methods PDE, **15**, pp. 29-47, (1999).

[23] **W. Walter,** Differential-und Integral-Ungleichungen, Springer, Berlin, (1964).

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