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SUMMABILITY RESULTS FOR MATRICES OF QUASI - HOMOGENEOUS OPERATORS

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Abstract

Some summability results are established for matrices of quasi-homogeneous operators by uniformly vanishing sets.

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1. Introduction

In 2001, Li et al. [4] firstly gave the definition of quasi-homogeneous operators and showed the family of quasi-homogeneous operators included all linear and much more nonlinear operators. The introduction of quasi-homogeneous operators has strongly broadened our research scope of operators. In [4, 5], the authors characterized some matrix families for matrices of quasi-homogeneous operators between topological vector spaces. In [7, 8], Qiu obtained some resonance theorems for families of quasi-homogeneous operators between some special topological vector spaces. In [10], Song and Fang proved some resonance theorems for families of quasi-homogeneous operators between fuzzy normed linear spaces. The fact says that quasi-homogeneous operators are useful and interesting. So it is necessary for us to study it further.

In 2007, Li et al. [6] introduced the definition of uniformly vanishing sets, and especially, obtained the strongest intrinsic meaning of sequential-evaluation convergence by uniformly vanishing sets.

In this paper, we will give new characterizations of some matrix families for matrices of quasi-homogeneous operators by uniformly vanishing sets. Especially, from our results, the results of [4, 5] can easily be obtained.

2. Preliminaries

Let X, Y be topological vector spaces. For sequence families $\lambda(X) \subset X^{\mathbf{N}}$, $\mu(Y) \subset Y^{\mathbf{N}}$ and $f_{ij} : X \to Y$ $(i, j \in \mathbf{N})$ we say that the matrix $(f_{ij})_{i,j\in\mathbf{N}} \in (\lambda(X), \mu(Y))$ if $\{\sum_{j=1}^{\infty} f_{ij}(x_j)\}_{i=1}^{\infty} \in \mu(Y)$ for each $(x_j) \in \lambda(X)$. Let $\mathcal{N}(X)$ be the family of neighborhoods of $0 \in X$, $c_0(X) = \{(x_j) \in X^{\mathbf{N}} : \lim_j x_j = 0\}$, $c(X) = \{(x_j) \in X^{\mathbf{N}} : \lim_j x_j \text{ exists }\}$, $\ell^\infty(X) = \{(x_j) \in X^{\mathbf{N}} : \{x_j : j \in \mathbf{N}\}$ is bounded in $X\}$, and $c_0(X)^{\beta Y} = \{(A_j) \subset Y^X : \sum_{j=1}^{\infty} A_j(x_j)$ converges for each $(x_j) \in c_0(X)\}$.

A topological vector space X is said to be braked if for every $(x_j) \in c_0(X)$ there is a scalar sequence $\lambda_j \to \infty$ such that $\lambda_j x_j \to 0$, i.e., X is braked if and only if for every $(x_j) \in c_0(X)$ there exist $(t_j) \in c_0 = \{(t_j)_1^\infty :$ each t_j is a scalar and $\lim_j t_j = 0\}$ and $(u_j) \in c_0(X)$ such that $x_j = t_j u_j$ for all $j \in \mathbf{N}$ [9, p.43].

Every metrizable locally convex space is braked [2, p.382] and the Schur lemma shows that the nonmetrizable $(\ell^1, weak)$ is also braked. The strict inductive limit X of a sequence $\{X_n\}$ of locally convex Fréchet spaces is called an (LF) space, e.g., the space $\mathcal{D}$ of test functions, the space of rapidly decreasing functions, etc. (LF) spaces are not metrizable but every (LF) space is braked [5, Example 2].

Definition 2.1. *Let X, Y be vector spaces. An operator T from X to Y is said to be quasi-homogeneous if there exists a function $\varphi : \mathbf{C} \to \mathbf{C}$ satisfying $\lim_{t\to 0} \varphi(t) = \varphi(0) = 0$ such that $T(tx) = \varphi(t)T(x)$ for all $t \in \mathbf{C}$ and $x \in X$.*

Obviously, if $\varphi : \mathbf{C} \to \mathbf{C}$ is a function such that $T : X \to Y$ is quasi-homogeneous, then φ satisfies: $\varphi(ts) = \varphi(t)\varphi(s)$, $\forall\, t, s \in \mathbf{C}$.

Let $C(0) = \{\varphi \in \mathbf{C}^{\mathbf{C}} : \lim_{t\to 0} \varphi(t) = \varphi(0) = 0,\ \varphi(ts) = \varphi(t)\varphi(s),\ \forall\, t, s \in \mathbf{C}\}$ and for each $\varphi \in C(0)$, $QH_\varphi(X, Y) = \{T \in Y^X : T(tx) = \varphi(t)T(x),$ $\forall\, t \in \mathbf{C},\ x \in X\}$.

The following Uniform Convergence Principle can be found in [1, p.25; 3; 11].

Lemma 2.1 (Uniform Convergence Principle). *Let G be an abelian topological group and Ω a sequentially compact space and $F_j : \Omega \to G$ a sequentially continuous function for all $j \in \mathbf{N}$. If every sequence $j_1 < j_2 < \cdots$ in $\mathbf{N}$ has a subsequence $j_{k_1} < j_{k_2} < \cdots$ such that $\sum_{v=1}^{\infty} F_{j_{k_v}}(\omega)$ converges at each $\omega \in \Omega$ and $\sum_{v=1}^{\infty} F_{j_{k_v}}(\cdot) : \Omega \to G$ is sequentially continuous, then $\lim_j F_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$.*

Definition 2.2. *Let X be a Banach space. $M \subset c_0(X)$ is said to be uniformly vanishing if $\lim_j x_j = 0$ uniformly for $(x_j) \in M$.*

For uniformly vanishing sets, [6] has obtained the following very good results.

Lemma 2.2. *For $M \subset c_0(X)$ the following* (1) *and* (2) *are equivalent:*

(1) *M is uniformly vanishing.*

(2) *For every Fréchet space E and $(A_j) \in c_0(X)^{\beta E}$, $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly for $(x_j) \in M$.*

Now we give the similar definition and result in topological vector spaces as Definition 2.2 and Lemma 2.2.

Definition 2.3. *Let X be a topological vector space. $M \subset c_0(X)$ is said to be uniformly vanishing if $\lim_j x_j = 0$ uniformly for $(x_j) \in M$.*

Lemma 2.3. *Let X be a topological vector space. If $M \subset c_0(X)$ is uniformly vanishing, then for every topological vector space E and $(A_j) \in c_0(X)^{\beta E}$, $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly for $(x_j) \in M$.*

Proof. Assume that $\sum_{j=1}^{\infty} A_j(x_j)$ is not uniform for $(x_j) \in M$. Then we have a $V \in \mathcal{N}(X)$ and integers $m_1 < n_1 < m_2 < n_2 < \cdots$ and $\{(x_{kj})_{j=1}^{\infty} : k \in \mathbf{N}\} \subset M$ such that $\sum_{j=m_k}^{n_k} A_j(x_{kj}) \notin V$, $k = 1, 2, 3, \cdots$.

Let

$$x_j = \begin{cases} x_{kj}, & m_k \le j \le n_k, \quad k = 1, 2, 3, ..., \\ 0, & \text{otherwise.} \end{cases}$$

Since M is uniformly vanishing, $(x_j) \in c_0(X)$ but $\sum_{j=m_k}^{n_k} A_j(x_j) = \sum_{j=m_k}^{n_k} A_j(x_{kj}) \notin V$, $k = 1, 2, 3, \cdots$. This contradicts $(A_j) \in c_0(X)^{\beta E}$ and so $\sum_{j=1}^{\infty} A_j(x_j)$ converges uniformly with respect to $(x_j) \in M$ for each topological vector space E and $(A_j) \in c_0(X)^{\beta E}$.

3. Main Results

Theorem 3.1. *Let X, Y be topological vector spaces and X be braked. If $(T_{ij})_{i,j \in \mathbf{N}} \subset QH_{\varphi}(X, Y)$, then the following* (3) *and* (4) *are equivalent:*

(3) $(T_{ij})_{i,j \in \mathbf{N}} \in (c_0(X), \ell^{\infty}(Y))$.

(4) *$\{T_{ij}(x)\}_{i=1}^{\infty}$ is bounded for each $x \in X$ and $j \in \mathbf{N}$, and for every uniformly vanishing $M \subset c_0(X)$, $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbf{N}$ and $(x_j) \in M$.*

Proof. (3)$\Longrightarrow$(4): Since $(0, \cdots, \overset{(j)}{x}, 0, 0, \cdots) \in c_0(X)$ for every $j \in \mathbf{N}$ and $x \in X$, it follows from (3) and $T_{ij}(0) = 0$ for all $i, j \in \mathbf{N}$ that $\{T_{ij}(x)\}_{i=1}^{\infty}$ is bounded, $\forall\, j \in \mathbf{N}$, $x \in X$.

Assume that $M \subset c_0(X)$ is uniformly vanishing but the convergence of $\sum_{j=1}^{\infty} T_{ij}(x_j)$ is not uniform with respect to both $i \in \mathbf{N}$ and $(x_j) \in M$. Then there exists $V \in \mathcal{N}(X)$ such that for every $m_0 \in \mathbf{N}$ we have integers $m > m_0$, $i \in \mathbf{N}$ and $(x_j) \in M$ for which $\sum_{j=m}^{\infty} T_{ij}(x_j) \notin V$. Also there exists $W \in \mathcal{N}(X)$ such that $W + W \subset V$.

There exist integers $m_1 > 1$, $i_1 \in \mathbf{N}$ and $(x_{1j}) \in M$ such that $\sum_{j=m_1}^{\infty} T_{i_1 j}(x_{1j}) \notin V$ and $\sum_{j=n_1+1}^{\infty} T_{i_1 j}(x_{1j}) \in W$ for some $n_1 > m_1$. Hence, $\sum_{j=m_1}^{n_1} T_{i_1 j}(x_{1j}) \notin W$.

By Lemma 2.3, there is an integer $n_0 > n_1$ such that $\sum_{j=m}^{\infty} T_{ij}(x_j) \in V$ for all $m > n_0$, $1 \le i \le i_1$ and all $(x_j) \in M$. Then there exist integers $n_2 > m_2 > n_0$, $i_2 > i_1$ and $(x_{2j}) \in M$ such that $\sum_{j=m_2}^{n_2} T_{i_2 j}(x_{2j}) \notin W$.

Continuing this construction produces integer sequences $m_1 < n_1 < m_2 < n_2 < \cdots$, $i_1 < i_2 < \cdots$ and $\{(x_{kj})_{j=1}^{\infty} : k \in \mathbf{N}\} \subset M$ such that $\sum_{j=m_k}^{n_k} T_{i_k j}(x_{kj}) \notin W$, $k = 1, 2, 3, \cdots$.

Let

$$x_j = \begin{cases} x_{kj}, & m_k \le j \le n_k, \quad k = 1, 2, 3, ..., \\ 0, & \text{otherwise,} \end{cases}$$

then $(x_j) \in c_0(X)$ since M is uniformly vanishing. So

$$\sum_{j=m_k}^{n_k} T_{i_k j}(x_{kj}) \notin W, \ k = 1, 2, 3, \cdots. \tag{3.1}$$

Since X is braked, there exist $(t_j) \in c_0$ and $(z_j) \in c_0(X)$ such that $x_j = t_j z_j$ for all $j \in \mathbf{N}$. Let $\delta_k = \max_{m_k \le j \le n_k} |t_j|$. Then $\delta_k \to 0$ as $k \to \infty$ and, by (3.1), $\delta_k > 0$ for all $k \in \mathbf{N}$.

Observing $T_{ij} \in QH_{\varphi}(X, Y)$, $\forall\, i, j \in \mathbf{N}$, for every $m_k \le j \le n_k$ we have $T_{i_k j}(x_j) = T_{i_k j}(t_j z_j) = T_{i_k j}(\delta_k \frac{t_j}{\delta_k} z_j) = \varphi(\delta_k) T_{i_k j}(\frac{t_j}{\delta_k} z_j)$ so, by (3.1), $\sum_{j=m_k}^{n_k} \varphi(\delta_k)\, T_{i_k j}(\frac{t_j}{\delta_k} z_j) \notin W$, $k = 1, 2, 3, \cdots$, i.e.,

$$\varphi(\delta_k) \sum_{j=m_k}^{n_k} T_{i_k j}(\frac{t_j}{\delta_k} z_j) \notin W, \ k = 1, 2, 3, \cdots. \tag{3.2}$$

Now $\Omega = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is a sequentially compact subset of $\mathbf{R}$. For each $k \in \mathbf{N}$, define $F_k : \Omega \to Y$ by

$$F_k(0) = 0, \quad F_k(\frac{1}{n}) = \varphi(\delta_n) \sum_{j=m_k}^{n_k} T_{i_n j}(\frac{t_j}{\delta_k} z_j), \quad n = 1, 2, 3, \cdots.$$

Since $\{\sum_{j=m_k}^{n_k} T_{ij}(\frac{t_j}{\delta_k} z_j)\}_{i=1}^{\infty}$ is bounded for each $k \in \mathbf{N}$ by (3) and $\varphi(\delta_n) \to 0$ as $n \to \infty$, $\lim_n F_k(\frac{1}{n}) = \lim_n \varphi(\delta_n) \sum_{j=m_k}^{n_k} T_{i_n j}(\frac{t_j}{\delta_k} z_j) = 0 = F_k(0)$ so each $F_k : \Omega \to Y$ is sequentially continuous. If $k_1 < k_2 < \cdots$ in $\mathbf{N}$ and

$$u_j = \begin{cases} \frac{t_j}{\delta_{k_v}} z_j, & m_{k_v} \le j \le n_{k_v}, \quad v = 1, 2, 3, ..., \\ 0, & \text{otherwise,} \end{cases}$$

then $u_j \to 0$ so $(u_j) \in c_0(X)$. Thus, $\{\sum_{j=1}^{\infty} T_{ij}(u_j)\}_{i=1}^{\infty}$ is bounded by (3) and

$$\begin{aligned} \lim_n \sum_{v=1}^{\infty} F_{k_v}(\tfrac{1}{n}) &= \lim_n \sum_{v=1}^{\infty} \varphi(\delta_n) \sum_{j=m_{k_v}}^{n_{k_v}} T_{i_n j}(\tfrac{t_j}{\delta_{k_v}} z_j) \\ &= \lim_n \varphi(\delta_n) \sum_{j=1}^{\infty} T_{i_n j}(u_j) \\ &= 0 = \sum_{v=1}^{\infty} F_{k_v}(0). \end{aligned}$$

This shows that $\sum_{v=1}^{\infty} F_{k_v}(\cdot) : \Omega \to Y$ is sequentially continuous and, by Lemma 2.1, $\lim_k F_k(\frac{1}{n}) = 0$ is uniform with respect to $n \in \mathbf{N}$. Therefore,

$$\lim_k \varphi(\delta_k) \sum_{j=m_k}^{n_k} T_{i_k j}(\frac{t_j}{\delta_k} z_j) = \lim_k f_k(\frac{1}{k}) = 0.$$

This contradicts (3.2) and so (3) implies (4).

(4)$\Longrightarrow$(3): Let $(x_j) \in c_0(X)$, and $\{\sum_{j=1}^{\infty} T_{i_k j}(x_j)\}_{k=1}^{\infty} \subset \{\sum_{j=1}^{\infty} T_{ij}(x_j) : i \in \mathbf{N}\}$. Since $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbf{N}$, the series $\sum_{j=1}^{\infty} \frac{1}{k} T_{ij}(x_j)$ also converges uniformly with respect to $i \in \mathbf{N}$ for each $k \in \mathbf{N}$. And $\{T_{ij}(x_j)\}_{i=1}^{\infty}$ is bounded for each $j \in \mathbf{N}$ so

$$\begin{aligned} \lim_k \tfrac{1}{k} \sum_{j=1}^{\infty} T_{i_k j}(x_j) &= \lim_k \tfrac{1}{k} \lim_n \sum_{j=1}^{n} T_{i_k j}(x_j) \\ &= \lim_k \lim_n \sum_{j=1}^{n} \tfrac{1}{k} T_{i_k j}(x_j) \\ &= \lim_n \lim_k \sum_{j=1}^{n} \tfrac{1}{k} T_{i_k j}(x_j) \\ &= \lim_n \sum_{j=1}^{n} \lim_k \tfrac{1}{k} T_{i_k j}(x_j) \\ &= \lim_n \sum_{j=1}^{n} 0 = 0. \end{aligned}$$

This shows that $\{\sum_{j=1}^{\infty} T_{ij}(x_j) : i \in \mathbf{N}\}$ is bounded. Thus, (4)$\Longrightarrow$(3) holds.

Corollary 3.1. *Let X, Y be topological vector spaces and X be braked. If $(T_{ij})_{i,j \in \mathbf{N}} \subset QH_{\varphi}(X, Y)$, then the following (5) and (6) are equivalent:*

(5) $(T_{ij})_{i,j \in \mathbf{N}} \in (c_0(X), c_0(Y))$.

(6) $\lim_i T_{ij}(x) = 0$ *for each $x \in X$ and $j \in \mathbf{N}$, and for each uniformly vanishing $M \subset c_0(X)$, $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbf{N}$ and $(x_j) \in M$.*

Proof. By Theorem 3.1, we just need to prove (6)$\Longrightarrow$(5). By (6), $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges for each $(x_j) \in c_0(X)$ and $i \in \mathbf{N}$, and for each $(x_j) \in c_0(X)$, $\sum_{j=1}^{\infty} T_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbf{N}$.

For $(x_j) \in c_0(X)$, we have

$$\lim_i \sum_{j=1}^{\infty} T_{ij}(x_j) = \lim_i \lim_n \sum_{j=1}^{n} T_{ij}(x_j) = \lim_n \lim_i \sum_{j=1}^{n} T_{ij}(x_j) = \lim_n \sum_{j=1}^{n} \lim_i T_{ij}(x_j) = 0.$$

Corollary 3.2. *Let X, Y be topological vector spaces, X braked and Y sequentially complete. If $(T_{ij})_{i,j\in\mathbf{N}} \subset QH_\varphi(X,Y)$, then the following* (7) *and* (8) *are equivalent:*

(7) $(T_{ij})_{i,j\in\mathbf{N}} \in (c_0(X), c(Y))$.

(8) $\lim_i T_{ij}(x)$ *exists for each $x \in X$ and $j \in \mathbf{N}$, and for each uniformly vanishing $M \subset c_0(X)$, $\sum_{j=1}^\infty T_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbf{N}$ and $(x_j) \in M$.*

Proof. By Theorem 3.1, we just need to prove (8)$\Longrightarrow$(7). By (8), $\sum_{j=1}^\infty T_{ij}(x_j)$ converges for each $(x_j) \in c_0(X)$ and $i \in \mathbf{N}$, and for each $(x_j) \in c_0(X)$ $\sum_{j=1}^\infty T_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbf{N}$.

Let $(x_j) \in c_0(X)$ and $V \in \mathcal{N}(X)$. Then there is balanced $W \in \mathcal{N}(X)$ such that $W + W + W \subset V$. There exists $n_0 \in \mathbf{N}$ such that $\sum_{j=n_0+1}^\infty T_{ij}(x_j) \in W$ for all $i \in \mathbf{N}$. Since $\lim_i \sum_{j=1}^{n_0} T_{ij}(x_j) = \sum_{j=1}^{n_0} \lim_i T_{ij}(x_j)$ exists, there is $i_0 \in \mathbf{N}$ such that $\sum_{j=1}^{n_0} T_{kj}(x_j) - \sum_{j=1}^{n_0} T_{ij}(x_j) \in W$ for all k, $j > i_0$. So

$$\begin{aligned}\sum_{j=1}^\infty T_{kj}(x_j) - \sum_{j=1}^\infty T_{ij}(x_j) &= \sum_{j=1}^{n_0}(T_{kj}(x_j) - T_{ij}(x_j)) \\ &\quad + \sum_{j=n_0+1}^\infty T_{kj}(x_j) - \sum_{j=n_0+1}^\infty T_{ij}(x_j) \\ &\quad \in W + W + W \subset V, \quad \forall\, k,\, i > i_0.\end{aligned}$$

This shows that $\{\sum_{j=1}^\infty T_{ij}(x_j)\}_{i=1}^\infty$ is Cauchy in sequentially complete space Y and so $\lim_i \sum_{j=1}^\infty T_{ij}(x_j)$ exists.

From Theorem 3.1, Corollary 3.1 and Corollary 3.2, we can easily obtain the following results of [4, 5].

Corollary 3.3. *Let X, Y be topological vector spaces and X be braked. If $(T_{ij})_{i,j\in\mathbf{N}} \subset QH_\varphi(X,Y)$, then the following* (3) *and* (4′) *are equivalent:*

(3) $(T_{ij})_{i,j\in\mathbf{N}} \in (c_0(X), \ell^\infty(Y))$.

(4′) $\{T_{ij}(x)\}_{i=1}^\infty$ *is bounded for each $x \in X$ and $j \in \mathbf{N}$, and for each $(x_j) \in c_0(X)$, $\sum_{j=1}^\infty T_{ij}(x_j)$ converges uniformly with respect to $i \in \mathbf{N}$.*

Proof. Suppose that (4′) holds and $M \subset c_0(X)$ is uniformly vanishing but $\sum_{j=1}^\infty T_{ij}(x_j)$ is not uniform with respect to $(x_j) \in M$ and $i \in \mathbf{N}$. As the proof of Theorem 3.1, we have $V \in \mathcal{N}(X)$, integer sequences $m_1 < n_1 < m_2 < n_2 < \cdots$, $i_1 < i_2 < \cdots$ and $\{(x_{kj})_{j=1}^\infty : k = 1, 2, 3, \cdots\} \subset M$ such that $\sum_{j=m_k}^{n_k} T_{i_k j}(x_j) \notin V$, $k = 1, 2, 3, \cdots$.

Let

$$x_j = \begin{cases} x_{kj}, & m_k \le j \le n_k, \quad k = 1, 2, 3, ..., \\ 0, & \text{otherwise,} \end{cases}$$

then $(x_j) \in c_0(X)$ since M is uniformly vanishing. So

$$\sum_{j=m_k}^{n_k} T_{i_k j}(x_j) \notin V, \; k = 1, 2, 3, \cdots .$$

This contradicts $(4')$. Thus $(4')\Longrightarrow(4)$ and $(3)\Longleftrightarrow(4')$ hold, respectively.

Corollary 3.4. *Let X, Y be topological vector spaces and X be braked. If $(T_{ij})_{i,j\in\mathbf{N}} \subset QH_\varphi(X, Y)$, then the following (5) and $(6')$ are equivalent:*

(5) $(T_{ij})_{i,j\in\mathbf{N}} \in (c_0(X), c_0(Y))$.

$(6')$ $\lim_i T_{ij}(x) = 0$ *for each* $x \in X$ *and* $j \in \mathbf{N}$, *and for each* $(x_j) \in c_0(X)$, $\sum_{j=1}^{\infty} T_{ij}(x_j)$ *converges uniformly with respect to* $i \in \mathbf{N}$.

Corollary 3.5. *Let X, Y be topological vector spaces, X braked and Y sequentially complete. If $(T_{ij})_{i,j\in\mathbf{N}} \subset QH_\varphi(X, Y)$, then the following (7) and $(8')$ are equivalent:*

(7) $(T_{ij})_{i,j\in\mathbf{N}} \in (c_0(X), c(Y))$.

$(8')$ $\lim_i T_{ij}(x)$ *exists for each* $x \in X$ *and* $j \in \mathbf{N}$, *and for each* $(x_j) \in c_0(X)$, $\sum_{j=1}^{\infty} T_{ij}(x_j)$ *converges uniformly with respect to* $i \in \mathbf{N}$.

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