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Spectral properties of horocycle flows for compact surfaces of constant negative curvature

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Abstract

We consider flows, called $W^{\rm u}$ flows, whose orbits are the unstable manifolds of a codimension one Anosov flow. Under some regularity assumptions, we give a short proof of the strong mixing property of $W^{\rm u}$ flows and we show that $W^{\rm u}$ flows have purely absolutely continuous spectrum in the orthocomplement of the constant functions. As an application, we obtain that time changes of the classical horocycle flows for compact surfaces of constant negative curvature have purely absolutely continuous spectrum in the orthocomplement of the constant functions for time changes in a regularity class slightly less than C^2. This generalises recent results on time changes of horocycle flows.

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1. Introduction

Horocycle flows for compact surfaces of constant negative curvature, and their generalisations, are a classical object of study in dynamical systems. Since the papers of G. A. Hedlund [8, 9] and E. Hopf [10] in the 1930's, many properties of these flows have been put into evidence (far too many to be listed here). As for spectral properties, in particular the structure of the essential spectrum, not that many results are available.

In 1953, O. S. Parasyuk [19] has shown that the classical horocycle flows for compact surfaces of constant negative curvature have countable Lebesgue spectrum. In 1974, A. G. Kushnirenko [13] has shown that some small time changes of the classical horocycle flows for compact surfaces of constant negative curvature are mixing, and thus have purely continuous spectrum in the orthocomplement of the constant functions. In 1977, B. Marcus [15] has shown that a large class of minimal W^{u} flows associated with codimension one Anosov flows (in particular, a large class of reparametrisations of the classical horocycle flows for compact surfaces of negative curvature) are mixing. Finally, more recently, G. Forni and C. Ulcigrai [6], and the author [23, 25] have shown that sufficiently smooth time changes of the classical horocycle flows for compact surfaces of constant negative curvature have purely absolutely continuous spectrum in the orthocomplement of the constant functions (in [6] Lebesgue spectrum is also obtained, and in [23] surfaces of finite volume are also considered).

The purpose of this paper is to extend these last results on the absolutely continuous spectrum of time changes of horocycle flows to a very general class of time changes, namely, time changes in a regularity class slightly less than C^2. Our approach is the following. We consider as B. Marcus a continuous minimal W^{u} flow, that is, a continuous minimal flow whose orbits are the unstable manifolds of a $C^{1+\varepsilon}$ codimension one Anosov flow on a compact connected Riemannian manifold (minimality is required for the W^{u} flow to be uniquely ergodic). Under some regularity assumptions on the W^{u} flow and the Anosov flow, we give a short proof of the strong mixing property of the W^{u} flow. Then, under some additional regularity assumption, we construct a self-adjoint operator, called conjugate operator, satisfying a suitable positive commutator estimate with the self-adjoint generator of the W^{u} flow (a Mourre estimate). Finally, we deduce from this positive commutator estimate and commutator methods that the generator of the W^{u} flow has purely absolutely continuous spectrum in the orthocomplement of the constant functions. As an application, we obtain that time changes of the classical horocycle flows for compact surfaces of constant negative curvature have purely absolutely continuous spectrum in the orthocomplement of the constant functions for time changes in a regularity class slightly less than C^2.

Here is a description of the content of the paper. In Section 2, we recall some definitions and results on codimension one Anosov flows and minimal W^{u} flows, we introduce our regularity assumptions (Assumptions 2.1 and 2.4), and we give a proof of the strong mixing property of the W^{u} flow (Theorem 2.6). In Section 3, we recall the needed facts on commutators of operators and regularity classes associated with them. Then, under some additional regularity assumption (Assumption 3.4), we construct the conjugate operator (Proposition 3.5), we prove the positive commutator estimate (Proposition 3.7), and we show that the generator of the W^{u} flow has purely absolutely continuous spectrum in the orthocomplement of the constant functions (Theorem 3.8). Finally, we present the application of this result to time changes of the classical horocycle flows for compact surfaces of constant negative curvature (Remark 3.9).

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2. Strong mixing

In this section, we recall some definitions and results on codimension one Anosov flows and minimal W^{u} flows, and we present a short proof of the strong mixing property of a class of minimal W^{u} flows. We follow fairly closely the presentation and notations of B. Marcus [15], but we refer to the review papers [2, 17, 20] for additional information.

A $C^{1+\varepsilon}$ Anosov flow on a compact connected Riemannian manifold M with distance $d: M \times M \to [0,\infty)$ is a $C^{1+\varepsilon}$ flow $\{f_t\}_{t\in\mathbf{R}}$ with $\varepsilon > 0$, without fixed points, satisfying the following property: through each point $x \in M$ pass three submanifolds $W^{\mathrm{u}}(x)$, $W^{\mathrm{s}}(x)$, and $\mathrm{Orb}(x)$ whose tangent spaces E^{u}_x, E^{s}_x, and E_x (respectively) vary continuously with $x \in M$ and satisfy

$$T_xM = E^{\mathrm{u}}_x \oplus E^{\mathrm{s}}_x \oplus E_x.$$

The regularity assumption on $\{f_t\}_{t\in\mathbf{R}}$ means that the function $\mathbf{R} \times M \ni (t,x) \mapsto f_t(x) \in M$ is of class $C^{1+\varepsilon}$. So, $\{f_t\}_{t\in\mathbf{R}}$ has a vector field X_f

which is Hölder continuous with exponent ε. The submanifolds $W^{\mathrm{u}}(x)$, $W^{\mathrm{s}}(x)$, and $\mathrm{Orb}(x)$ are called unstable manifolds, stable manifolds, and orbits (respectively), and are characterised by

$$\begin{aligned} W^{\mathrm{u}}(x) &= \Big\{ y \in M \mid \lim_{t\to-\infty} d\Big(f_t(x), f_t(y)\Big) = 0 \Big\}, \\ W^{\mathrm{s}}(x) &= \Big\{ y \in M \mid \lim_{t\to+\infty} d\Big(f_t(x), f_t(y)\Big) = 0 \Big\}, \\ \mathrm{Orb}(x) &= \Big\{ f_t(x) \mid t \in \mathbf{R} \Big\}. \end{aligned}$$

The following two facts are well-known [2]:

(i) The families $\{W^{\mathrm{u}}(x)\}_{x\in M}$ and $\{W^{\mathrm{s}}(x)\}_{x\in M}$ each form a continuous foliation of M (that is, if $y \in W^{\mathrm{u}}(x)$ then $W^{\mathrm{u}}(y) = W^{\mathrm{u}}(x)$, and $W^{\mathrm{u}}(x)$ varies locally continuously with $x \in M$).

(ii) $f_t\Big(W^{\mathrm{u}}(x)\Big) = W^{\mathrm{u}}\Big(f_t(x)\Big)$ and $f_t\Big(W^{\mathrm{s}}(x)\Big) = W^{\mathrm{s}}\Big(f_t(x)\Big)$ for all $t \in \mathbf{R}$ and $x \in M$.

In this paper, we consider a codimension one Anosov flow $\{f_t\}_{t\in\mathbf{R}}$ such that $\{W^{\mathrm{u}}(x)\}_{x\in M}$ is a one-dimensional orientable continuous foliation of M which defines a continuous minimal flow $\{\phi_s\}_{s\in\mathbf{R}}$ whose orbits are the unstable manifolds. Such a flow $\{\phi_s\}_{s\in\mathbf{R}}$ is called a minimal W^{u} flow or a minimal W^{u} parametrisation, and it is uniquely ergodic with respect to a Borel probability mesure μ on M [5, 14]. However, the unique invariant measure μ is in general not absolutely continuous with respect to the volume element [15, Sec. 6] [16].

Since $f_t\Big(W^{\mathrm{u}}(x)\Big) = W^{\mathrm{u}}\Big(f_t(x)\Big)$ for all $t \in \mathbf{R}$ and $x \in M$, there exists a function $s^* : \mathbf{R} \times \mathbf{R} \times M \to \mathbf{R}$ such that

$$\Big(f_t \circ \phi_s\Big)(x) = \Big(\phi_{s^*(t,s,x)} \circ f_t\Big)(x) \quad \text{for all } s, t \in \mathbf{R} \text{ and } x \in M. \tag{2.1}$$

This commutation relation, which describes how the Anosov flow $\{f_t\}_{t\in\mathbf{R}}$ expands W^{u} orbits, is the starting point of our analysis. It generalises the well-known commutation relation [4, Rem. IV.1.2] between the geodesic flow and the classical horocycle flow on the unit tangent bundle of compact orientable surfaces of constant negative curvature. We recall three facts in relation with (2.1):

(iii) The family $\{W^{\mathrm{u}}(x)\}_{x\in M}$ admits a uniformly expanding parametrisation, that is, a continuous parametrisation $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ such that $f_t \circ \widetilde{\phi}_s = \widetilde{\phi}_{\lambda^t s} \circ f_t$ for some constant $\lambda > 1$ and all $s, t \in \mathbf{R}$ [14, Rem. 1.8 & Prop. 2.1] (the constant λ is equal to $e^{h(f_1)}$, with $h(f_1)$ the topological entropy of f_1).

(iv) Since $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ is a continuous reparametrisation of $\{\phi_s\}_{s\in\mathbf{R}}$, and since $\{\phi_s\}_{s\in\mathbf{R}}$ is uniquely ergodic with respect to the measure μ, the flow $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ is uniquely ergodic with respect to a measure $\widetilde{\mu}$ given in terms of μ (however, the measure $\widetilde{\mu}$ is in general not absolutely continuous with respect to the measure μ [3, § 3 & 4]).

(v) The measure $\widetilde{\mu}$ is invariant under the Anosov flow $\{f_t\}_{t\in\mathbf{R}}$ [15, Rem. 6.4].

In order to be able to define a self-adjoint generator for the flow $\{\phi_s\}_{s\in\mathbf{R}}$ and to have a simple relation between the measures μ and $\widetilde{\mu}$, we assume the following regularity condition:

Assumption 2.1. *The flow $\{\phi_s\}_{s\in\mathbf{R}}$ is of class C^1, and $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ is a C^1 reparametristation of $\{\phi_s\}_{s\in\mathbf{R}}$.*

Under Assumption 2.1, the flow $\{\phi_s\}_{s\in\mathbf{R}}$ has a continuous vector field X_ϕ, and there exists a function $\tau\in C^1(M\times\mathbf{R};\mathbf{R})$ such that $\phi_s(x)=\widetilde{\phi}_{\tau(x,s)}(x)$, $\tau(x,0)=0$, $\tau(x,\ \cdot\ )$ is strictly increasing and $\tau(x,s+t)=\tau(x,s)+\tau\big(\phi_s(x),t\big)$ for all $s,t\in\mathbf{R}$ and $x\in M$ [3, § 1] (the function $\tau(x,\cdot)$ can be chosen strictly increasing for all $x\in M$ because M is connected). These properties imply in particular that $\big(\partial_2\tau\big)(x,0)>0$ for all $x\in M$. Therefore, the function

$$\rho : M\to\mathbf{R},\quad x\mapsto\frac{1}{(\partial_2\tau)(x,0)},$$

is well-defined and belongs to $C\big(M;(0,\infty)\big)$, the flow $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ has continuous vector field ρX_ϕ, and the measure $\widetilde{\mu}$ satisfies [11, Prop. 3]:

$$\widetilde{\mu}=\mu/\widetilde{\rho}\quad\text{with}\quad\widetilde{\rho}:=\rho\int_M d\mu\,\rho^{-1}. \tag{2.2}$$

Also, one verifies that the pullback operators in $\mathcal{H}:=L^2(M,\mu)$ associated with the flow $\{\phi_s\}_{s\in\mathbf{R}}$,

$$U_s^\phi\varphi:=\varphi\circ\phi_s,\quad s\in\mathbf{R},\ \varphi\in\mathcal{H},$$

define a strongly continuous 1-parameter group of unitary operators with $U_s^\phi C^1(M)\subset C^1(M)$ for all $s\in\mathbf{R}$. Thus, Nelson's criterion [21, Thm. VIII.10] implies that the generator of the group $\{U_s^\phi\}_{s\in\mathbf{R}}$,

$$H_\phi\varphi:=s\text{-}\lim_{s\to0} is^{-1}\big(U_s^\phi-1\big)\varphi,$$

$$\varphi \in D(H_\phi) := \left\{ \varphi \in \mathcal{H} \mid \lim_{s \to 0} |s|^{-1} \left\| \left(U_s^\phi - 1 \right) \varphi \right\| < \infty \right\},$$

is essentially self-adjoint on $C^1(M)$ and given by

$$H_\phi \varphi = i X_\phi \varphi, \quad \varphi \in C^1(M).$$

On another hand, the pullback operators associated with the flow $\{f_t\}_{t \in \mathbf{R}}$,

$$U_t^f \varphi := \varphi \circ f_t, \quad t \in \mathbf{R},\ \varphi \in \mathcal{H},$$

are not unitary if $\rho \not\equiv 1$, but they define a strongly continuous 1-parameter group of bounded operators:

Lemma 2.2. *Suppose that Assumption 2.1 is satisfied. Then, $U_t^f \in \mathcal{B}(\mathcal{H})$ for all $t \in \mathbf{R}$, $U_s^f U_t^f = U_{s+t}^f$ for all $s, t \in \mathbf{R}$, $U_0^f = 1$, and $\lim_{\varepsilon \to 0} \left\| \left(U_{t+\varepsilon}^f - U_t^f \right) \varphi \right\| = 0$ for all $t \in \mathbf{R}$ and $\varphi \in \mathcal{H}$.*

Proof. A direct calculation using (2.2) and the fact that $\widetilde{\mu}$ is invariant under $\{f_t\}_{t \in \mathbf{R}}$ implies for $t \in \mathbf{R}$ and $\varphi \in \mathcal{H}$ that

$$\begin{aligned}
\left\| U_t^f \varphi \right\|^2 &= \int_M d\widetilde{\mu}\, \widetilde{\rho} \left| \varphi \circ f_t \right|^2 = \int_M d\widetilde{\mu} \left(\widetilde{\rho} \circ f_{-t} \right) |\varphi|^2 \\
&= \int_M d\mu\, \frac{\rho \circ f_{-t}}{\rho}\, |\varphi|^2 \le \frac{\max(\rho)}{\min(\rho)}\, \|\varphi\|^2.
\end{aligned}$$

Thus, $U_t^f \in \mathcal{B}(\mathcal{H})$ with

$$\left\| U_t^f \right\| \le \sqrt{\frac{\max(\rho)}{\min(\rho)}}. \tag{2.3}$$

The group properties $U_s^f U_t^f = U_{s+t}^f$ for $s, t \in \mathbf{R}$ and $U_0^f = 1$ are evident. To show the last property, take $t \in \mathbf{R}$ and $\varphi \in C(M)$. Then, the continuity of $\left(\varphi \circ f_{t+\varepsilon} - \varphi \circ f_t \right)$ and Lebesgue dominated convergence theorem imply that

$$\lim_{\varepsilon \to 0} \left\| \left(U_{t+\varepsilon}^f - U_t^f \right) \varphi \right\|^2 = \int_M d\mu \lim_{\varepsilon \to 0} \left| \varphi \circ f_{t+\varepsilon} - \varphi \circ f_t \right|^2 = 0.$$

Since $C(M)$ is dense in $\mathcal{H}$ and $\left(U_{t+\varepsilon}^f - U_t^f \right) \in \mathcal{B}(\mathcal{H})$, this implies that $\lim_{\varepsilon \to 0} \left\| \left(U_{t+\varepsilon}^f - U_t^f \right) \varphi \right\| = 0$ for all $t \in \mathbf{R}$ and $\varphi \in \mathcal{H}$. $\square$

For the next lemma, we need the following result of B. Marcus:

Lemma 2.3 (Lemma 3.1 of [15]). *Let $\{\phi_s\}_{s\in\mathbf{R}}$ be a minimal W^{u} flow on M. Then,*

$$\lim_{s\to\infty} s^{-1}s^*(t,s,x) = \lambda^t$$

uniformly in $x \in M$ and t in a given compact interval of $\mathbf{R}$.

We also need to introduce as B. Marcus in [15, Sec. 4] the following regularity condition on the function s^*:

Assumption 2.4. *The derivative*

$$u_{t,s}(x) := \Big(\partial_1\partial_2 s^*\Big)(t,s,x)$$

exists and is continuous in $s, t \in \mathbf{R}$ and $x \in M$.

Lemma 2.5. *Suppose that Assumption 2.4 is satisfied. Then,*

$$\lim_{s\to\infty} s^{-1}\Big(\partial_1 s^*\Big)(t,s,x) = \int_M d\mu\, u_{t,0} = \ln(\lambda)\,\lambda^t \tag{2.4}$$

uniformly in $x \in M$ and t in a given compact interval of $\mathbf{R}$.

Proof. Let $I \subset \mathbf{R}$ be a compact interval, and take $r, s \in \mathbf{R}$, $t \in I$ and $x \in M$. Using successively the relation $\Big(\partial_1 s^*\Big)(t,0,x) = 0$, Assumption 2.4 and the cocycle equation

$$s^*(t,r+s,x) = s^*(t,r,x) + s^*\Big(t,s,\phi_r(x)\Big), \tag{2.5}$$

we obtain

$$\begin{aligned}\Big(\partial_1 s^*\Big)(t,s,x) &= \int_0^s dr\,\frac{d}{dr}\Big(\partial_1 s^*\Big)(t,r,x) = \int_0^s dr\,\frac{d}{dt}\frac{d}{ds}\Big|_{s=0} s^*(t,r+s,x)\\ &= \int_0^s dr\, u_{t,0}\Big(\phi_r(x)\Big).\end{aligned}$$

Therefore, it follows by the unique ergodicity of $\{\phi_s\}_{s\in\mathbf{R}}$ that

$$\lim_{s\to\infty} s^{-1}\Big(\partial_1 s^*\Big)(t,s,x) = \lim_{s\to\infty} s^{-1}\int_0^s dr\, u_{t,0}\Big(\phi_r(x)\Big) = \int_M d\mu\, u_{t,0} \tag{2.6}$$

uniformly in $x \in M$ and $t \in I$ (the uniformity in t follows from the continuity of $u_{t,0}$ in t). To show the second equality in (2.4), we note from Lemma 2.3 that

$$\lim_{s\to\infty} s^{-1}s^*(t,s,x) = \lambda^t$$

uniformly in $x \in M$ and $t \in I$. We also note from (2.6) that $s^{-1}\big(\partial_1 s^*\big)(t,s,x)$ converges to some function of t uniformly in $x \in M$ and $t \in I$. So, it follows by a uniform convergence argument that

$$\lim_{s\to\infty} s^{-1}\big(\partial_1 s^*\big)(t,s,x) = \frac{d}{dt}\lim_{s\to\infty} s^{-1}s^*(t,s,x) = \frac{d}{dt}\lambda^t = \ln(\lambda)\,\lambda^t$$

uniformly in $x \in M$ and $t \in I$. $\square$

In the following theorem, we give a short proof, inspired by [24, Thm. 4.1], of the strong mixing property of the flow $\{\phi_s\}_{s\in\mathbf{R}}$. It can be viewed as a simplified version of the proof of B. Marcus [15, Sec. 4] in the case Assumption 2.1 is satisfied (the proof of B. Marcus works without Assumption 2.1).

Theorem 2.6 (Strong mixing). *Suppose that Assumptions 2.1 and 2.4 are satisfied. Then,* $\lim_{s\to\infty}\big\langle\psi, U_s^\phi\varphi\big\rangle = 0$ *for all* $\psi \in \mathcal{H}$ *and* $\varphi \in \ker(H_\phi)^\perp$. *In particular, the flow* $\{\phi_s\}_{s\in\mathbf{R}}$ *is strongly mixing with respect to the measure* μ.

Proof. Take $\varphi \in C^1(M)$, $s,t \in \mathbf{R}$ and $x \in M$. Then, we know from (2.1) that

$$\big(\varphi\circ f_t\circ\phi_s\big)(x) = \big(\varphi\circ\phi_{s^*(t,s,x)}\circ f_t\big)(x).$$

Applying the derivative $\frac{d}{dt}\big|_{t=0}$ and using the relation $s^*(0,s,x) = s$, we obtain

$$\begin{aligned}
&\big(X_f\varphi\big)\big(\phi_s(x)\big)\\
&= \tfrac{d}{dt}\Big|_{t=0}\big(\varphi\circ\phi_{s^*(t,s,x)}\circ f_t\big)(x)\\
&= \lim_{\varepsilon\to 0}\varepsilon^{-1}\Big\{\big(\varphi\circ\phi_{s^*(\varepsilon,s,x)}\circ f_\varepsilon\big)(x) - \big(\varphi\circ\phi_{s^*(0,s,x)}\circ f_\varepsilon\big)(x)\Big\}\\
&\quad + \lim_{\varepsilon\to 0}\varepsilon^{-1}\Big\{\big(\varphi\circ\phi_s\circ f_\varepsilon\big)(x) - \big(\varphi\circ\phi_s\circ f_0\big)(x)\Big\}\\
&= \big(\partial_1 s^*\big)(0,s,x)\cdot\big(X_\phi\varphi\big)\big(\phi_s(x)\big) + \big(X_f(\varphi\circ\phi_s)\big)(x),
\end{aligned}$$

which is equivalent to

$$\big(\partial_1 s^*\big)(0,s,\,\cdot\,)\,U_s^\phi H_\phi\varphi = iU_s^\phi X_f\varphi - iX_f U_s^\phi\varphi. \tag{2.7}$$

So, for $\psi \in C^1(M)$, we infer from Lemma 2.5 that

$$\lim_{s\to\infty}\left\langle\rho^{-1}\psi,U_s^\phi H_\phi\varphi\right\rangle$$

$$=\ln(\lambda)^{-1}\lim_{s\to\infty}\left\langle\rho^{-1}\psi,\ln(\lambda)U_s^\phi H_\phi\varphi\right\rangle$$

$$=\ln(\lambda)^{-1}\lim_{s\to\infty}\left\langle\rho^{-1}\psi,s^{-1}\Big(\partial_1 s^*\Big)(0,s,\,\cdot\,)\,U_s^\phi H_\phi\varphi\right\rangle$$

$$=\ln(\lambda)^{-1}\lim_{s\to\infty}s^{-1}\Big\langle\rho^{-1}\psi,iU_s^\phi X_f\varphi\Big\rangle-\ln(\lambda)^{-1}\lim_{s\to\infty}s^{-1}\Big\langle\rho^{-1}\psi,iX_fU_s^\phi\varphi\Big\rangle$$

$$=0-\ln(\lambda)^{-1}\lim_{s\to\infty}s^{-1}\Big\langle\psi,i\rho^{-1}X_fU_s^\phi\varphi\Big\rangle.$$

But, the operator $i\rho^{-1}X_f$ is symmetric on $C^1(M)$. So,

$$\lim_{s\to\infty}s^{-1}\Big\langle\psi,i\rho^{-1}X_fU_s^\phi\varphi\Big\rangle=\lim_{s\to\infty}s^{-1}\Big\langle i\rho^{-1}X_f\psi,U_s^\phi\varphi\Big\rangle=0,$$

and thus

$$\lim_{s\to\infty}\left\langle\rho^{-1}\psi,U_s^\phi H_\phi\varphi\right\rangle=0. \tag{2.8}$$

Moreover, the set $\rho^{-1}C^1(M)$ is dense in $\mathcal{H}$ because $C^1(M)$ is dense in $\mathcal{H}$ and $\rho^{-1}:\mathcal{H}\to\mathcal{H}$ is an homeomorphism, and the set $H_\phi C^1(M)$ is dense in $\ker(H_\phi)^\perp$ because $C^1(M)$ is a core for H_ϕ and $H_\phi D(H_\phi)$ is dense in $\ker(H_\phi)^\perp$. Therefore, (2.8) implies that $\lim_{s\to\infty}\left\langle\psi,U_s^\phi\varphi\right\rangle=0$ for all $\psi\in\mathcal{H}$ and $\varphi\in\ker(H_\phi)^\perp$. $\square$

Under Assumptions 2.1 and 2.4, the result of Theorem 2.6 applies in particular to the case of reparametrisations of classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of constant negative curvature (see [15, Cor. 4.2]).

3. Absolutely continuous spectrum

We know from Theorem 2.6 that, under Assumptions 2.1 and 2.4, the flow $\{\phi_s\}_{s\in\mathbf{R}}$ is strongly mixing with respect to the measure μ. Therefore, its generator H_ϕ has purely continuous spectrum in $\mathbf{R}\setminus\{0\}$. Our goal in this section is to show that the spectrum of H_ϕ is even *purely absolutely continuous* in $\mathbf{R}\setminus\{0\}$ under some additional regularity assumption. For this, we first need to recall some results on commutator methods borrowed from [1, 22] (see also the original paper [18] of É. Mourre).

3.1. Commutators and regularity classes

Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ antilinear in the first argument, denote by $\mathcal{B}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$, and write $\| \cdot \|$ both for the norm on $\mathcal{H}$ and the norm on $\mathcal{B}(\mathcal{H})$. Let A be a self-adjoint operator in $\mathcal{H}$ with domain $D(A)$, and take $S \in \mathcal{B}(\mathcal{H})$. For any $k \in \mathbf{N}$, we say that S belongs to $C^k(A)$, with notation $S \in C^k(A)$, if the map

$$\mathbf{R} \ni t \mapsto e^{-itA}\, S\, e^{itA} \in \mathcal{B}(\mathcal{H}) \tag{3.1}$$

is strongly of class C^k. In the case $k = 1$, one has $S \in C^1(A)$ if and only if the quadratic form

$$D(A) \ni \varphi \mapsto \big\langle \varphi, SA\varphi \big\rangle - \big\langle A\varphi, S\varphi \big\rangle \in \mathbf{C}$$

is continuous for the topology induced by $\mathcal{H}$ on $D(A)$. We denote by $[S, A]$ the bounded operator associated with the continuous extension of this form, or equivalently $-i$ times the strong derivative of the function (3.1) at $t = 0$.

If H is a self-adjoint operator in $\mathcal{H}$ with domain $D(H)$ and spectrum $\sigma(H)$, we say that H is of class $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some $z \in \mathbf{C} \setminus \sigma(H)$. So, H is of class $C^1(A)$ if and only if the quadratic form

$$D(A) \ni \varphi \mapsto \big\langle \varphi, (H - z)^{-1} A\varphi \big\rangle - \big\langle A\varphi, (H - z)^{-1}\varphi \big\rangle \in \mathbf{C}$$

extends continuously to a bounded form defined by the operator $[(H - z)^{-1}, A] \in \mathcal{B}(\mathcal{H})$. In such a case, the set $D(H) \cap D(A)$ is a core for H and the quadratic form

$$D(H) \cap D(A) \ni \varphi \mapsto \big\langle H\varphi, A\varphi \big\rangle - \big\langle A\varphi, H\varphi \big\rangle \in \mathbf{C}$$

is continuous in the topology of $D(H)$ [1, Thm. 6.2.10(b)]. This form then extends uniquely to a continuous quadratic form on $D(H)$ which can be identified with a continuous operator $[H, A]$ from $D(H)$ to the adjoint space $D(H)^*$. In addition, the following relation holds in $\mathcal{B}(\mathcal{H})$:

$$\big[(H - z)^{-1}, A\big] = -(H - z)^{-1}[H, A](H - z)^{-1}. \tag{3.2}$$

Let $E^H(\cdot)$ denote the spectral measure of the self-adjoint operator H, and assume that H is of class $C^1(A)$. If there exist a Borel set $I \subset \mathbf{R}$, a number $a > 0$ and a compact operator $K \in \mathcal{B}(\mathcal{H})$ such that

$$E^H(I)[iH, A]E^H(I) \ge aE^H(I) + K, \tag{3.3}$$

then one says that H satisfies a Mourre estimate on I and that A is a conjugate operator for H on I. Also, one says that H satisfies a strict Mourre estimate on I if (3.3) holds with $K = 0$. One of the consequences of a Mourre estimate is to imply spectral results for H on I. We recall here these spectral results in the case where H is of class $C^2(A)$ (see [1, Sec. 7.1.2] and [22, Thm. 0.1] for more details).

Theorem 3.1. *Let H and A be self-ajoint operators in a Hilbert space $\mathcal{H}$, with H of class $C^2(A)$. Suppose there exist a bounded Borel set $I \subset \mathbf{R}$, a number $a > 0$ and a compact operator $K \in \mathcal{B}(\mathcal{H})$ such that*

$$E^H(I)[iH, A]E^H(I) \geq aE^H(I) + K. \tag{3.4}$$

Then, H has at most finitely many eigenvalues in I, each one of finite multiplicity, and H has no singular continuous spectrum in I. Furthermore, if (3.4) holds with $K = 0$, then H has only purely absolutely continuous spectrum in I (no singular spectrum).

3.2. Absolutely continuous spectrum

We show in this section that the spectrum of H_ϕ is purely absolutely continuous in $\mathbf{R} \setminus \{0\}$ under some additional regularity assumption. We start with two technical lemmas.

Lemma 3.2. *Suppose that Assumptions 2.1 and 2.4 are satisfied.*

(a) If $s \in \mathbf{R}$ and $\varphi \in C^1(M)$, then $U^\phi_s X_f U^\phi_{-s}\varphi = X_f\varphi + \int_0^s dr \left(u_{0,0} \circ \phi_r\right) X_\phi \varphi$.

(b) If $z \in \mathbf{C} \setminus \mathbf{R}$ and $\varphi \in C^1(M)$, then $X_f \left(H_\phi - z\right)^{-1} \varphi \in \mathcal{H}$.

(c) If $X_f(\rho) \in C(M)$, then $u_{0,0} = \ln(\lambda) + \rho^{-1} X_f(\rho)$.

Proof.
(a) Take $s \in \mathbf{R}$ and $\varphi \in C^1(M)$. Then, (2.7), the identity $\left(\partial_1 s^*\right)(0, 0, \cdot\,) \equiv 0$ and Assumption 2.4 imply

$$U^\phi_s X_f U^\phi_{-s}\varphi = X_f\varphi + \left(\partial_1 s^*\right)(0, s, \cdot\,) X_\phi \varphi = X_f\varphi + \int_0^s dr\, u_{0,r} X_\phi \varphi.$$

(3.5)

On another hand, the cocycle equation (2.5) implies for all $r, s, t \in \mathbf{R}$ that

$$\begin{aligned} s^*(t,s,\cdot\,) &= s^*\Big(t,s-r,\phi_r(\,\cdot\,)\Big) - s^*\Big(t,-r,\phi_r(\,\cdot\,)\Big) \\ &= U^\phi_r\Big(s^*(t,s-r,\,\cdot\,) - s^*\Big(t,-r,\,\cdot\,\Big)\Big)U^\phi_{-r}. \end{aligned}$$

Deriving with respect to t and s, we thus obtain

$$u_{t,s} = U^\phi_r u_{t,s-r} U^\phi_{-r} = u_{t,s-r} \circ \phi_r.$$

In particular, we have $u_{0,r} = u_{0,0} \circ \phi_r$, and the claim follows from (3.5).

(b) We give the proof in the case $Im(z) > 0$ since the case $Im(z) < 0$ is analogous. Take $\varphi \in C^1(M)$. Then, the formula $\big(H_\phi - z\big)^{-1}\varphi = i\int_0^\infty dr\; e^{irz}\, U^\phi_r \varphi$ and point (a) imply

$$\begin{aligned} &X_f\Big(H_\phi - z\Big)^{-1}\varphi \\ &= i\lim_{t\to 0}\textstyle\int_0^\infty dr\; e^{irz}\, t^{-1}\Big(U^f_t - 1\Big)U^\phi_r\varphi \\ &= i\lim_{t\to 0}\textstyle\int_0^\infty dr\; e^{irz}\, t^{-1}\int_0^t ds\, U^f_s X_f U^\phi_r\varphi \\ &= i\lim_{t\to 0}\textstyle\int_0^\infty dr\; e^{irz}\, t^{-1}\int_0^t ds\, U^f_s U^\phi_r\left(X_f + \int_0^{-r} dq\,\Big(u_{0,0}\circ\phi_q\Big)X_\phi\right)\varphi. \end{aligned} \tag{3.6}$$

Now, we know from (2.3) that $\Big\|U^f_s\Big\| \le \sqrt{\frac{\max(\rho)}{\min(\rho)}}$ for all $s \in \mathbf{R}$. Thus, we have

$$\begin{aligned} &\left\|e^{irz}\, t^{-1}\textstyle\int_0^t ds\, U^f_s U^\phi_r\left(X_f + \int_0^{-r} dq\,\Big(u_{0,0}\circ\phi_q\Big)X_\phi\right)\varphi\right\| \\ &\le e^{-r\,Im(z)}\sqrt{\tfrac{\max(\rho)}{\min(\rho)}}\left(\Big\|X_f\varphi\Big\| + r\,\|u_{0,0}\|_{L^\infty(X,\mu)}\Big\|X_\phi\varphi\Big\|\right) \\ &\in L^1\Big([0,\infty), dr\Big), \end{aligned}$$

and we can apply Lebesgue dominated convergence theorem to (3.6) to obtain

$$\begin{aligned} &X_f\Big(H_\phi - z\Big)^{-1}\varphi \\ &= i\textstyle\int_0^\infty dr\; e^{irz}\lim_{t\to 0} t^{-1}\int_0^t ds\, U^f_s U^\phi_r\left(X_f + \int_0^{-r} dq\,\Big(u_{0,0}\circ\phi_q\Big)X_\phi\right)\varphi \\ &= i\textstyle\int_0^\infty dr\; e^{irz}\, U^\phi_r\left(X_f + \int_0^{-r} dq\,\Big(u_{0,0}\circ\phi_q\Big)X_\phi\right)\varphi \\ &\in \mathcal{H}. \end{aligned}$$

(c) The proof is inspired by a result of L. W. Green in the case of the classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of negative curvature (see [7, Eq. (3.3) & Lemma 3.3]). Take $\psi,\varphi\in C^1(M)$. Then, the facts that $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ has vector field ρX_ϕ, that $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ and $\{f_t\}_{t\in\mathbf{R}}$ preserve the measure $\widetilde{\mu}=\mu/\widetilde{\rho}$ and that $f_t\circ\widetilde{\phi}_{-s}=\widetilde{\phi}_{-\lambda^t s}\circ f_t$ imply

$$\begin{aligned}
\left\langle X_\phi\psi, X_f\varphi\right\rangle &= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\left\langle \rho^{-1}\left(\psi\circ\widetilde{\phi}_s\right),\varphi\circ f_t\right\rangle\\
&= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\left\langle \rho^{-1}\psi,\varphi\circ f_t\circ\widetilde{\phi}_{-s}\right\rangle\\
&= \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\left\langle \rho^{-1}\psi,\varphi\circ\widetilde{\phi}_{-\lambda^t s}\circ f_t\right\rangle\\
&= -\frac{d}{dt}\Big|_{t=0}\left\langle \rho^{-1}\psi,\lambda^t\left(\rho X_\phi\varphi\right)\circ f_t\right\rangle\\
&= -\frac{d}{dt}\Big|_{t=0}\left\langle \psi\circ f_{-t},\lambda^t X_\phi\varphi\right\rangle\\
&= \left\langle X_f\psi, X_\phi\varphi\right\rangle-\left\langle \psi,\ln(\lambda)X_\phi\varphi\right\rangle.
\end{aligned}$$

On another hand, point (a) and the symmetricity of $i\rho^{-1}X_f$ on $C^1(M)$ imply

$$\begin{aligned}
&\left\langle \psi, u_{0,0}X_\phi\varphi\right\rangle\\
&= \frac{d}{ds}\Big|_{s=0}\left\langle \rho U^\phi_{-s}\psi,\rho^{-1}X_f U^\phi_{-s}\varphi\right\rangle\\
&= -\left\langle X_\phi\psi, X_f\varphi\right\rangle+\left\langle \rho^{-1}X_f\rho\psi, X_\phi\varphi\right\rangle\\
&= -\left\langle X_\phi\psi, X_f\varphi\right\rangle+\left\langle X_f\psi, X_\phi\varphi\right\rangle+\left\langle \psi,\rho^{-1}X_f(\rho)X_\phi\varphi\right\rangle.
\end{aligned}$$

Combining the two relations, we thus obtain

$$\left\langle \psi,\left(u_{0,0}-\ln(\lambda)-\rho^{-1}X_f(\rho)\right)X_\phi\varphi\right\rangle=0,$$

and we infer from the density of $C^1(M)$ in $\mathcal{H}$ and the density of $X_\phi C^1(M)$ in $\ker(H_\phi)^\perp$ that

$$\left\langle \psi,\left(u_{0,0}-\ln(\lambda)-\rho^{-1}X_f(\rho)\right)\varphi\right\rangle=0 \tag{3.7}$$

for all $\psi\in\mathcal{H}$ and $\varphi\in\ker(H_\phi)^\perp$. Now, Lemma 2.5, the fact that $\{f_t\}_{t\in\mathbf{R}}$ preserves $\widetilde{\mu}=\mu/\widetilde{\rho}$, and the ergodicity of $\{\phi_s\}_{s\in\mathbf{R}}$ imply that

$$u_{0,0}-\ln(\lambda)-\rho^{-1}X_f(\rho)\in\left\{\varphi\in\mathcal{H}\mid\int_M d\mu\,\varphi=0\right\}=\ker(H_\phi)^\perp.$$

So, we can set $\psi=1$ and $\varphi=u_{0,0}-\ln(\lambda)-\rho^{-1}X_f(\rho)$ in (3.7) to get

$$\int_M d\mu\left(u_{0,0}-\ln(\lambda)-\rho^{-1}X_f(\rho)\right)^2=0,$$

and then infer that $u_{0,0} - \ln(\lambda) - \rho^{-1}X_f(\rho) \equiv 0$ by the continuity of $u_{0,0}$ and $\rho^{-1}X_f(\rho)$. $\square$

Lemma 3.3. *Let X be a C^1 vector field on M and $g \in C^1(M;\mathbf{R})$. Assume that the operator*

$$A\varphi := i(X+g)\varphi, \quad \varphi \in C^1(M),$$

is symmetric in $\mathcal{H}$. Then, A is essentially self-adjoint in $\mathcal{H}$.

Proof. Since X is of class C^1, X admits a C^1 flow $\{\zeta_s\}_{s\in\mathbf{R}}$ [12, Thm. 3.43]. Thus, the operators

$$V_s\varphi := e^{\int_0^s dr\,(g\circ\zeta_r)}\,\varphi\circ\zeta_s, \quad s\in\mathbf{R},\ \varphi\in C^1(M),$$

are well-defined operators in $\mathcal{H}$. Simple calculations show that $V_sV_t\varphi = V_{s+t}\varphi$ and $V_0\varphi = \varphi$ for $s,t \in \mathbf{R}$ and $\varphi \in C^1(M)$, that $\lim_{\varepsilon\to 0}\|(V_{s+\varepsilon} - V_s)\varphi\| = 0$ for $s \in \mathbf{R}$ and $\varphi \in C^1(M)$, that $V_sC^1(M) \subset C^1(M)$ for $s \in \mathbf{R}$, and that $\frac{d}{ds}V_s\varphi = -iAV_s\varphi$ for $s \in \mathbf{R}$ and $\varphi \in C^1(M)$. Furthermore, we have for $s \in \mathbf{R}$ and $\psi,\varphi \in C^1(M)$ the equalities

$$\Big\langle V_s\psi, V_s\varphi\Big\rangle - \langle\psi,\varphi\rangle = \int_0^s dr\,\frac{d}{dr}\Big\langle V_r\psi, V_r\varphi\Big\rangle$$

$$= i\int_0^s dr\,\Big(\Big\langle AV_r\psi, V_r\varphi\Big\rangle - \Big\langle V_r\psi, AV_r\varphi\Big\rangle\Big) = 0,$$

due to the symmetricity of A. Therefore, the family $\{V_s\}_{s\in\mathbf{R}}$ satisfies on $C^1(M)$ the properties of a strongly continuous 1-parameter group of isometric operators in $\mathcal{H}$ with $V_sC^1(M) \subset C^1(M)$ for all $s \in \mathbf{R}$, and with generator equal to A on $C^1(M)$. Since $C^1(M)$ is dense in $\mathcal{H}$, it follows that $\{V_s\}_{s\in\mathbf{R}}$ extends to a strongly continuous 1-parameter group of isometric (and thus unitary) operators in $\mathcal{H}$ with $V_sC^1(M) \subset C^1(M)$ for all $s \in \mathbf{R}$, and with generator equal to A on $C^1(M)$. Thus, Nelson's criterion [21, Thm. VIII.10] implies that A is essentially self-adjoint in $\mathcal{H}$. $\square$

In the rest of the paper, in addition to Assumptions 2.1 and 2.4, we assume the following:

Assumption 3.4. *The vector fields X_f and X_ϕ are of class C^1, $X_f(\rho) \in C(M)$ and $\rho^{-1}X_f(\rho) \in C^1(M)$.*

The conditions $X_f(\rho) \in C(M)$ and $\rho^{-1}X_f(\rho) \in C^1(M)$ are equivalent to the condition $X_f\big(\ln(\rho)\big) \in C^1(M)$. So, if one prefers, one can replace the two conditions $X_f(\rho) \in C(M)$ and $\rho^{-1}X_f(\rho) \in C^1(M)$ in Assumption 3.4 by the single condition $X_f\big(\ln(\rho)\big) \in C^1(M)$.

In the next proposition we define and prove the self-adjointness of the conjugate operator. Intuitively, the conjugate operator is constructed as follows. First, we take the sum of the vector field $2iX_f$ and its "divergence" $i\rho^{-1}X_f(\rho)$ to get a symmetric operator on $C^1(M)$. Then, we take the Birkhoff average of the resulting operator along the flow $\{\phi_s\}_{s\in\mathbf{R}}$ to take into account the unique ergodicity of $\{\phi_s\}_{s\in\mathbf{R}}$.

Proposition 3.5 (Conjugate operator). *Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied. Then, the operator*

$$A_t\varphi := \frac{1}{t}\int_0^t ds\, U_s^\phi\Big(2iX_f + i\rho^{-1}X_f(\rho)\Big)U_{-s}^\phi\varphi, \quad t>0,\ \varphi\in C^1(M),$$

is essentially self-adjoint in $\mathcal{H}$ (and its closure is denoted by the same symbol).

Proof. Since $\rho^{-1}X_f(\rho) = \widetilde{\rho}^{-1}X_f(\widetilde{\rho})$, the operator $\Big(2iX_f + i\rho^{-1}X_f(\rho)\Big)$ is symmetric on $C^1(M)$. Therefore, the operator A_t is also symmetric on $C^1(M)$ because $U_s^\phi C^1(M) \subset C^1(M)$ for all $s \in \mathbf{R}$. To show that A_t is essentially self-adjoint on $C^1(M)$, we take $\psi, \varphi \in C^1(M)$. Then, Lemma 3.2(a) implies that

$$\begin{aligned}
&\Big\langle \psi, A_t\varphi\Big\rangle \\
&= \Big\langle \psi, \tfrac{1}{t}\textstyle\int_0^t ds\, \Big(2iX_f + 2i\int_0^s dr\,\Big(u_{0,0}\circ\phi_r\Big)X_\phi + i\Big(\rho^{-1}X_f(\rho)\Big)\circ\phi_s\Big)\,\varphi\Big\rangle \\
&= \Big\langle \psi, i\Big(2X_f + a_tX_\phi + b_t\Big)\varphi\Big\rangle
\end{aligned} \tag{3.8}$$

with

$$a_t := \frac{2}{t}\int_0^t ds\int_0^s dr\,\Big(u_{0,0}\circ\phi_r\Big) \quad\text{and}\quad b_t := \frac{1}{t}\int_0^t ds\,\Big(\rho^{-1}X_f(\rho)\Big)\circ\phi_s.$$

Furthermore, Assumption 3.4, Lemma 3.2(c) and Leibniz integral rule imply that X_f, X_ϕ, a_t and b_t are of class C^1. Therefore, we can apply Lemma 3.3 with $X := 2X_f + a_tX_\phi$ and $g := b_t$ to conclude that A_t is essentially self-adjoint in $\mathcal{H}$. $\square$

Lemma 3.6. *Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied. Then, we have for $t>0$*

(a) $\big(H_\phi - i\big)^{-1} \in C^1(A_t)$ *with*

$$\Big[i\big(H_\phi - i\big)^{-1}, A_t\Big] = 2\big(H_\phi - i\big)^{-1} c_t H_\phi \big(H_\phi - i\big)^{-1} - \Big[\big(H_\phi - i\big)^{-1}, c_t\Big] \quad \text{and}$$

$$c_t := \frac{1}{t}\int_0^t ds\,\big(u_{0,0}\circ \phi_s\big),$$

(b) $\big(H_\phi - i\big)^{-1} \in C^2(A_t)$.

Proof. (a) Set $A := \big(2iX_f + i\rho^{-1}X_f(\rho)\big)$ on $C^1(M)$ and take $\varphi \in C^1(M)$. Then, we know from Lemma 3.2(b)-(c) that

$$\begin{aligned}
&\Big\langle \big(H_\phi + i\big)^{-1}\varphi, A\varphi\Big\rangle - \Big\langle A\varphi, \big(H_\phi - i\big)^{-1}\varphi\Big\rangle \\
&= \Big\langle \big(2iX_f + i\rho^{-1}X_f(\rho)\big)\big(H_\phi + i\big)^{-1}\varphi, \big(H_\phi - i\big)^{-1}\big(H_\phi - i\big)\varphi\Big\rangle \\
&\quad - \Big\langle \big(H_\phi + i\big)^{-1}\big(H_\phi + i\big)\varphi, \big(2iX_f + i\rho^{-1}X_f(\rho)\big)\big(H_\phi - i\big)^{-1}\varphi\Big\rangle \\
&= i\,\tfrac{d}{ds}\Big|_{s=0}\Big\{\Big\langle \big(2iX_f + i\rho^{-1}X_f(\rho)\big)\big(H_\phi + i\big)^{-1}\varphi, \big(H_\phi - i\big)^{-1}\big(U^\phi_s - s\big)\varphi\Big\rangle \\
&\qquad\qquad - \Big\langle \big(H_\phi + i\big)^{-1}\big(U^\phi_{-s} - s\big)\varphi, \big(2iX_f + i\rho^{-1}X_f(\rho)\big)\big(H_\phi - i\big)^{-1}\varphi\Big\rangle\Big\} \\
&= -\tfrac{d}{ds}\Big|_{s=0}\Big\{\Big\langle \big(H_\phi + i\big)^{-1}\varphi, \big(2X_f + u_{0,0}\big)\big(H_\phi - i\big)^{-1}U^\phi_s\varphi\Big\rangle \\
&\qquad\qquad - \Big\langle \big(H_\phi + i\big)^{-1}\varphi, U^\phi_s\big(2X_f + u_{0,0}\big)\big(H_\phi - i\big)^{-1}\varphi\Big\rangle\Big\} \\
&= -2\Big\langle \varphi, \tfrac{d}{ds}\Big|_{s=0}\big(H_\phi - i\big)^{-1}\Big\{X_f\big(H_\phi - i\big)^{-1}U^\phi_s - U^\phi_s X_f\big(H_\phi - i\big)^{-1}\Big\}\varphi\Big\rangle \\
&\quad - \Big\langle \varphi, \tfrac{d}{ds}\Big|_{s=0}\big(H_\phi - i\big)^{-1}\Big\{u_{0,0}\big(H_\phi - i\big)^{-1}U^\phi_s - U^\phi_s u_{0,0}\big(H_\phi - i\big)^{-1}\Big\}\varphi\Big\rangle.
\end{aligned}$$

For the first term, the equation $\big(H_\phi - i\big)^{-1} = i\int_0^\infty dr\ e^{-r}\,U^\phi_r$ (valid in the strong sense) and Lemma 3.2(a) imply

$$\begin{aligned}
&\tfrac{d}{ds}\Big|_{s=0}\big(H_\phi - i\big)^{-1}\Big\{X_f\big(H_\phi - i\big)^{-1}U^\phi_s - U^\phi_s X_f\big(H_\phi - i\big)^{-1}\Big\}\varphi \\
&= i\,\tfrac{d}{ds}\Big|_{s=0}\int_0^\infty dr\ e^{-r}\,U^\phi_s\big(H_\phi - i\big)^{-1}\big(U^\phi_{-s}X_fU^\phi_s - X_f\big)U^\phi_r\varphi \\
&= i\,\tfrac{d}{ds}\Big|_{s=0}\int_0^\infty dr\ e^{-r}\,U^\phi_s\big(H_\phi - i\big)^{-1}\int_0^{-s} dt\,\big(u_{0,0}\circ\phi_t\big)U^\phi_r X_\phi\varphi \\
&= -i\int_0^\infty dr\ e^{-r}\,\big(H_\phi - i\big)^{-1}u_{0,0}U^\phi_r X_\phi\varphi \\
&= i\big(H_\phi - i\big)^{-1}u_{0,0}H_\phi\big(H_\phi - i\big)^{-1}\varphi.
\end{aligned}$$

For the second term, a direct calculation gives

$$\frac{d}{ds}\bigg|_{s=0}\Big(H_\phi-i\Big)^{-1}\Big\{u_{0,0}\Big(H_\phi-i\Big)^{-1}U_s^\phi-U_s^\phi u_{0,0}\Big(H_\phi-i\Big)^{-1}\Big\}\varphi$$

$$=\Big[\Big(H_\phi-i\Big)^{-1},u_{0,0}\Big]\varphi.$$

So, putting together the last equations, we get

$$\begin{aligned}&\Big\langle\Big(H_\phi+i\Big)^{-1}\varphi,A\varphi\Big\rangle-\Big\langle A\varphi,\Big(H_\phi-i\Big)^{-1}\varphi\Big\rangle\\&=\Big\langle\varphi,\Big\{-2i\Big(H_\phi-i\Big)^{-1}u_{0,0}H_\phi\Big(H_\phi-i\Big)^{-1}+i\Big[\Big(H_\phi-i\Big)^{-1},u_{0,0}\Big]\Big\}\varphi\Big\rangle.\end{aligned}$$

Therefore, for each $t>0$ we obtain

$$\begin{aligned}&\Big\langle\Big(H_\phi+i\Big)^{-1}\varphi,A_t\varphi\Big\rangle-\Big\langle A_t\varphi,\Big(H_\phi-i\Big)^{-1}\varphi\Big\rangle\\&=\tfrac{1}{t}\textstyle\int_0^t ds\Big\{\Big\langle\Big(H_\phi+i\Big)^{-1}U_{-s}^\phi\varphi,AU_{-s}^\phi\varphi\Big\rangle-\Big\langle AU_{-s}^\phi\varphi,\Big(H_\phi-i\Big)^{-1}U_{-s}^\phi\varphi\Big\rangle\Big\}\\&=\tfrac{1}{t}\textstyle\int_0^t ds\,\Big\langle U_{-s}^\phi\varphi,\Big\{-2i\Big(H_\phi-i\Big)^{-1}u_{0,0}H_\phi\Big(H_\phi-i\Big)^{-1}\\&+\Big[\Big(H_\phi-i\Big)^{-1},u_{0,0}\Big]\Big\}U_{-s}^\phi\varphi\Big\rangle\\&=\Big\langle\varphi,\Big\{-2i\Big(H_\phi-i\Big)^{-1}c_tH_\phi\Big(H_\phi-i\Big)^{-1}+i\Big[\Big(H_\phi-i\Big)^{-1},c_t\Big]\Big\}\varphi\Big\rangle\end{aligned}$$

with $c_t=\frac{1}{t}\int_0^t ds\,(u_{0,0}\circ\phi_s)$, and the claim follows by the density of $C^1(M)$ in $D(A_t)$.

(b) We know from point (a) that $\Big(H_\phi-i\Big)^{-1}\in C^1(A_t)$ with

$$\begin{aligned}&\Big[i\Big(H_\phi-i\Big)^{-1},A_t\Big]\\&=2\Big(H_\phi-i\Big)^{-1}c_tH_\phi\Big(H_\phi-i\Big)^{-1}-\Big[\Big(H_\phi-i\Big)^{-1},c_t\Big]\\&=2\Big(H_\phi-i\Big)^{-1}c_t+2i\Big(H_\phi-i\Big)^{-1}c_t\Big(H_\phi-i\Big)^{-1}-\Big[\Big(H_\phi-i\Big)^{-1},c_t\Big].\end{aligned}$$

So, it is sufficient to show that $c_t\in C^1(A_t)$. For this, we note that $c_t\in C^1(M)$ due to the assumption $\rho^{-1}X_f(\rho)\in C^1(M)$, Lemma 3.2(c) and Leibniz integral rule. Then, we use (3.8) to get for $\varphi\in C^1(M)$

$$\begin{aligned}&\Big\langle c_t\varphi,A_t\varphi\Big\rangle-\Big\langle A_t\varphi,c_t\varphi\Big\rangle\\&=\Big\langle\varphi,ic_t\Big(2X_f+a_tX_\phi+b_t\Big)\varphi\Big\rangle-\Big\langle\varphi,i\Big(2X_f+a_tX_\phi+b_t\Big)c_t\varphi\Big\rangle\\&=\Big\langle\varphi,-i\Big(2X_f(c_t)+a_tX_\phi(c_t)\Big)\varphi\Big\rangle\end{aligned}$$

with $\Big(2X_f(c_t)+a_tX_\phi(c_t)\Big)$ a bounded multiplication operator, and we note that this implies the claim because $C^1(M)$ is dense in $D(A_t)$. $\square$

In the next proposition, we show that H_ϕ satisfies a strict Mourre estimate on compact subsets of $(0,\infty)$ and $(-\infty,0)$. We use the operator $-\ln(\lambda)A_t$ as conjugate operator on $(0,\infty)$ and the operator $\ln(\lambda)A_t$ as conjugate operator on $(-\infty,0)$.

Proposition 3.7 (Strict Mourre estimate). *Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied.*

(a) If $I\subset(0,\infty)$ is a compact set with $I\cap\sigma(H_\phi)\neq\emptyset$, then there exist $t>0$ and $a>0$ such that

$$E^{H_\phi}(I)\Big[iH_\phi,-\ln(\lambda)A_t\Big]E^{H_\phi}(I)\geq a\,E^{H_\phi}(I).$$

(b) If $J\subset(-\infty,0)$ is a compact set with $J\cap\sigma(H_\phi)\neq\emptyset$, then there exist $t>0$ and $a>0$ such that

$$E^{H_\phi}(J)\Big[iH_\phi,\ln(\lambda)A_t\Big]E^{H_\phi}(J)\geq a\,E^{H_\phi}(J).$$

Before the proof, we recall that the flow $\{\phi_s\}_{s\in\mathbf{R}}$ is ergodic. Therefore, the spectrum of the operator H_ϕ in $\mathbf{R}\setminus\{0\}$ is not empty, and thus there exist compact sets $I\subset(0,\infty)$ such that $I\cap\sigma(H_\phi)\neq\emptyset$ and/or compacts sets $J\subset(-\infty,0)$ such that $J\cap\sigma(H_\phi)\neq\emptyset$.

Proof.

(a) Let $t>0$. Then, we know from Lemma 3.6(a) that $\big(H_\phi-i\big)^{-1}\in C^1\big(-\ln(\lambda)A_t\big)$ with

$$\Big[i\big(H_\phi-i\big)^{-1},-\ln(\lambda)A_t\Big]=-2\ln(\lambda)\big(H_\phi-i\big)^{-1}c_tH_\phi\big(H_\phi-i\big)^{-1}$$
$$+\ln(\lambda)\Big[\big(H_\phi-i\big)^{-1},c_t\Big].$$

This, together with (3.2), implies that

$$\begin{aligned}
&E^{H_\phi}(I)\Big[iH_\phi,-\ln(\lambda)A_t\Big]E^{H_\phi}(I)\\
&=-\big(H_\phi-i\big)E^{H_\phi}(I)\Big[i\big(H_\phi-i\big)^{-1},-\ln(\lambda)A_t\Big]\big(H_\phi-i\big)E^{H_\phi}(I)\\
&=2\ln(\lambda)E^{H_\phi}(I)c_tH_\phi E^{H_\phi}(I)-\ln(\lambda)\big(H_\phi-i\big)E^{H_\phi}(I)\Big[\big(H_\phi-i\big)^{-1},c_t\Big]\\
&\big(H_\phi-i\big)E^{H_\phi}(I)\\
&=2\big(\ln(\lambda)\big)^2H_\phi E^{H_\phi}(I)+2\ln(\lambda)E^{H_\phi}(J)\big(c_t-\ln(\lambda)\big)H_\phi E^{H_\phi}(I)\\
&\quad-\ln(\lambda)\big(H_\phi-i\big)E^{H_\phi}(I)\Big[\big(H_\phi-i\big)^{-1},c_t-\ln(\lambda)\Big]\big(H_\phi-i\big)E^{H_\phi}(I)\\
&\geq a_IE^{H_\phi}(I)+2\ln(\lambda)E^{H_\phi}(I)\big(c_t-\ln(\lambda)\big)H_\phi E^{H_\phi}(I)\\
&\quad-\ln(\lambda)\big(H_\phi-i\big)E^{H_\phi}(I)\Big[\big(H_\phi-i\big)^{-1},c_t-\ln(\lambda)\Big]\big(H_\phi-i\big)E^{H_\phi}(I)
\end{aligned}$$

with $a_I := 2\big(\ln(\lambda)\big)^2 \inf(I) > 0$. Furthermore, since $\{\phi_s\}_{s\in\mathbf{R}}$ is uniquely ergodic, we obtain from Lemma 2.5 that

$$\lim_{t\to\infty}\Big(c_t - \ln(\lambda)\Big) = \int_M d\mu\, u_{0,0} - \ln(\lambda) = 0$$

uniformly on M. Therefore, if $t > 0$ is large enough, there exists $a \in (0, a_I)$ such that

$$E^{H_\phi}(I)\Big[iH_\phi, -\ln(\lambda)A_t\Big]E^{H_\phi}(I) \geq a\, E^{H_\phi}(I).$$

(b) The proof is similar to that of point (a). □

The next theorem is the main result of the paper.

Theorem 3.8 (Absolutely continuous spectrum). *Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied. Then, H_ϕ has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\mathbf{C}\cdot 1$.*

Proof. We know from Lemma 3.6(b) that H_ϕ is of class $C^2\Big(-\ln(\lambda)A_t\Big)$ for all $t > 0$. Moreover, we know from Proposition 3.7(a) that for each compact set $I \subset (0,\infty)$ with $I \cap \sigma(H_\phi) \neq \emptyset$ there exist $t > 0$ and $a > 0$ such that

$$E^{H_\phi}(I)\Big[iH_\phi, -\ln(\lambda)A_t\Big]E^{H_\phi}(I) \geq a\, E^{H_\phi}(I).$$

Therefore, it follows from Theorem 3.1 that H_ϕ has purely absolutely continuous spectrum in $(0,\infty)$. Since the same holds for $(-\infty, 0)$, H_ϕ has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace $\mathbf{C}\cdot 1$ due to the ergodicity of the flow $\{\phi_s\}_{s\in\mathbf{R}}$.
□

We conclude with some remarks on Theorem 3.8.

Remark 3.9. *(a) Under Assumptions 2.1, 2.4, and 3.4, the result of Theorem 3.8 applies, as in the case of Theorem 2.6, to reparametrisations of classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of constant negative curvature.*

Our regularity assumptions on the function ρ are $X_f(\rho) \in C(M)$ and $\rho^{-1}X_f(\rho) \in C^1(M)$. Therefore, Theorem 3.8 extends the results of [6, 23, 25] on the absolutely continuous spectrum of time changes of the classical horocycle flows on the unit tangent bundle of compact orientable surfaces of constant negative curvature since the function corresponding to ρ in

[6, 23, 25] is at least of class C^3 (in [23], the function ρ is of class C^2, but it satisfies another additional assumption). Also, the result of Theorem 3.8 answers the question on the regularity of the function ρ raised in [25, Rem. 3.4]. The result of [6] on Lebesgue spectrum and the result of [23] for surfaces of finite volume are of a different nature and are not covered by Theorem 3.8.

(b) In the case of the classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of negative curvature, L. W. Green has shown in [7, Thm. B] that the curvature of the surface is necessarily constant if the flow $\{\phi_s\}_{s\in\mathbf{R}}$ admits a C^2 uniformly expanding reparametrisation $\{\widetilde{\phi}_s\}_{s\in\mathbf{R}}$ (and thus a function ρ of class C^1). Since it is possible that a similar argument also applies under our regularity assumptions (an anonymous referee signaled this to us), we prefer not to mention the case of compact surfaces of variable negative curvature. However, we hope that in some future the technics and the regularity assumptions presented in this manuscript could be improved in order to cover explicit examples of compact surfaces of variable negative curvature.

(c) In Lemma 3.6, Proposition 3.7 and Theorem 3.8, we stated and proved the results using only the operator H_ϕ under study, and not its square $(H_\phi)^2$ as we did in [23, 25]. This allowed us to simplify the exposition in comparison to [23, 25].

References

[1] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, volume 135 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, (1996).

[2] D. V. Anosov. *Geodesic flows on closed Riemann manifolds with negative curvature.* Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder. American Mathematical Society, Providence, R.I., 1969.

[3] M. Beboutoff and W. Stepanoff. Sur la mesure invariante dans les systèmes dynamiques qui ne diffèrent que par le temps. *Rec. Math. [Mat. Sbornik] N.S.*, 7 (49), pp. 143–166, (1940).

[4] M. B. Bekka and M. Mayer. *Ergodic theory and topological dynamics of group actions on homogeneous spaces*, volume 269 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, (2000).

[5] R. Bowen and B. Marcus. Unique ergodicity for horocycle foliations. *Israel J. Math.*, 26(1), pp. 43–67, (1977).

[6] G. Forni and C. Ulcigrai. Time-changes of horocycle flows. *J. Mod. Dyn.*, 6(2), pp. 251–273, (2012).

[7] L. W. Green. Remarks on uniformly expanding horocycle parameterizations. *J. Differential Geom.*, 13(2), pp. 263–271, (1978).

[8] G. A. Hedlund. Fuchsian groups and transitive horocycles. *Duke Math. J.*, 2(3), pp. 530–542, (1936).

[9] G. A. Hedlund. Fuchsian groups and mixtures. *Ann. of Math. (2)*, 40(2), pp. 370–383, (1939).

[10] E. Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig*, 91, pp. 261–304, (1939).

[11] P. D. Humphries. Change of velocity in dynamical systems. *J. London Math. Soc. (2)*, 7, pp. 747–757, (1974).

[12] M. C. Irwin. *Smooth dynamical systems*, volume 17 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, (2001). Reprint of the 1980 original, With a foreword by R. S. MacKay.

[13] A. G. Kushnirenko. Spectral properties of certain dynamical systems with polynomial dispersal. *Moscow Univ. Math. Bull.*, 29(1), pp. 82–87, (1974).

[14] B. Marcus. Unique ergodicity of the horocycle flow: variable negative curvature case. *Israel J. Math.*, 21(2-3), pp. 133–144, (1975). Conference on Ergodic Theory and Topological Dynamics Kibbutz Lavi, (1974).

[15] B. Marcus. Ergodic properties of horocycle flows for surfaces of negative curvature. *Ann. of Math. (2)*, 105 (1), pp. 81–105, (1977).

[16] G. A. Margulis. Certain measures that are connected with U-flows on compact manifolds. *Funkcional. Anal. i Priložen.*, 4 (1), pp. 62–76, (1970).

[17] S. Matsumoto. *Codimension one Anosov flows*, volume 27 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, (1995).

[18] É. Mourre. Absence of singular continuous spectrum for certain self-adjoint operators. *Comm. Math. Phys.*, 78 (3), pp. 391–408, 1980/81.

[19] O. S. Parasyuk. Flows of horocycles on surfaces of constant negative curvature. *Uspehi Matem. Nauk (N.S.)*, 8 (3(55)):125–126, 1953.

[20] Y. B. Pesin. Geodesic flows with hyperbolic behavior of trajectories and objects connected with them. *Uspekhi Mat. Nauk*, 36 (4(220)), pp. 3–51, 247, (1981).

[21] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, (1980). Functional analysis.

[22] J. Sahbani. The conjugate operator method for locally regular Hamiltonians. *J. Operator Theory*, 38(2), pp. 297–322, (1997).

[23] R. Tiedra de Aldecoa. Spectral analysis of time changes of horocycle flows. *J. Mod. Dyn.*, 6(2), pp. 275–285, (2012).

[24] R. Tiedra de Aldecoa. The absolute continuous spectrum of skew products of compact lie groups. *Israel J. Math.*, 208(1), pp. 323–350, (2015).

[25] R. Tiedra de Aldecoa. Commutator methods for the spectral analysis of uniquely ergodic dynamical systems. *Ergodic Theory Dynam. Systems*, 35(3), pp. 944–967, (2015).

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