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A weakened version of Davis-Choi-Jensen's inequality for normalised positive linear maps

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Abstract

In this paper we show that the celebrated Davis-Choi-Jensen's inequality for normalised positive linear maps can be extended in a weakened form for convex functions. A reverse inequality and applications for important instances of convex (concave) functions are also given.

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1. Introduction

The following result that provides an vector operator version for the Jensen inequality is well known, see for instance [6] or [7, p. 5]:

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\mathrm{Sp}(A) \subseteq [m, M]$ *for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \le \langle f(A)x, x\rangle \tag{1.1}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the *Hölder-McCarthy inequality* [5]: Let A be a selfadjoint positive operator on a Hilbert space H, then
(i) $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;
(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;
(iii) If A is invertible, then $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

In [2] (see also [3, p. 16]) we obtained the following additive reverse of (1.1):

Theorem 2. *Let I be an interval and $f : I \to \mathbf{R}$ be a convex and differentiable function on $\mathring{I}$ $(the interior of I) whose derivative f'$ is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with* $\mathrm{Sp}(A) \subset I$, *then*

$$\begin{aligned}(0 \le)\, &\langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\ &\le \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \cdot \langle f'(A)x, x\rangle \end{aligned}\tag{1.2}$$

for any $x \in H$ with $\|x\| = 1$.

This is a generalization of the scalar discrete inequality obtained in [4]. For other reverse inequalities of this type see [3, p. 16].

The following particular cases are of interest: If A is a selfadjoint operator on H, then we have the inequality:

$$\begin{aligned}(0 \le)\, &\langle \exp(A)x, x\rangle - \exp(\langle Ax, x\rangle) \\ &\le \langle A\exp(A)x, x\rangle - \langle Ax, x\rangle \langle \exp(A)x, x\rangle, \end{aligned}\tag{1.3}$$

for each $x \in H$ with $\|x\| = 1$.

Let A be a positive definite operator on the Hilbert space H. Then we have the following inequality for the logarithm:

$$\begin{aligned}(0 \leq) \ln(\langle Ax, x\rangle) - \langle \ln(A)\, x, x\rangle \\ \leq \langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1,\end{aligned} \tag{1.4}$$

for each $x \in H$ with $\|x\| = 1$.

If $p \geq 1$ and A is a positive operator on H, then

$$(0 \leq) \langle A^p x, x\rangle - \langle Ax, x\rangle^p \leq p\left[\langle A^p x, x\rangle - \langle Ax, x\rangle \left\langle A^{p-1}x, x\right\rangle\right], \tag{1.5}$$

for each $x \in H$ with $\|x\| = 1$. If A is positive definite, then the inequality (1.5) also holds for $p < 0$. If $0 < p < 1$ and A is a positive definite operator then the reverse inequality also holds

$$(0 \leq) \langle Ax, x\rangle^p - {}^p x, x\rangle \leq p\left[\langle Ax, x\rangle \cdot \left\langle A^{p-1}x, x\right\rangle - \langle A^p x, x\rangle\right], \tag{1.6}$$

for each $x \in H$ with $\|x\| = 1$.

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [1] (see also [7, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely $\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$ for any λ, $\mu \in \mathbf{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order,i.e. if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in P[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e. $\Phi(1_H) = 1_K$. We write $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$.*

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1_H \leq A \leq \beta 1_H$, then $\alpha 1_K \leq \Phi(A) \leq \beta 1_K$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we get that $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known:

Theorem 3 (Davis-Choi-Jensen's Inequality). *Let $f : I \to \mathbf{R}$ be an operator convex function on the interval I and $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.7}$$

We observe that if $\Psi \in P[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ in (1.7) we get

$$f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right) \leq \Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps*

$$\begin{aligned} &\Psi^{1/2}(1_H) f\left(\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\right)\Psi^{1/2}(1_H) \\ &\leq \Psi(f(A)). \end{aligned} \tag{1.8}$$

It is obvious that, by (1.7) we have the vector inequality

$$\langle f(\Phi(A))y, y\rangle \leq \langle \Phi(f(A))y, y\rangle \tag{1.9}$$

for any $y \in K$. By (1.1) we also have

$$f(\langle \Phi(A)y, y\rangle) \leq \langle f(\Phi(A))y, y\rangle \tag{1.10}$$

for any $y \in K$, $\|y\| = 1$. Therefore, for an *operator convex function* on I we have

$$f(\langle \Phi(A) y, y \rangle) \leq \langle f(\Phi(A)) y, y \rangle \leq \langle \Phi(f(A)) y, y \rangle \tag{1.11}$$

for any $y \in K$, $\|y\| = 1$.

It is then natural to ask the following question:

Does the inequality between the first and last term in (1.11) remains valid in the more general case of convex functions defined on the interval I ?

A positive answer to this question and some reverse inequalities are provided below. Some applications for important instances of convex (concave) functions are also given.

2. A Jensen's Type Inequality

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbf{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $t, s \in \mathring{I}$ and $t < s$, then $f'_-(t) \leq f'_+(t) \leq f'_-(s) \leq f'_+(s)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbf{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi(\mathring{I}) \subset \mathbf{R}$ and

$$f(t) \geq f(a) + (t - a)\varphi(a) \text{ for any } t, a \in I. \tag{2.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(t) \leq \varphi(t) \leq f'_+(t) \text{ for any } t \in \mathring{I}.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

We have:

Theorem 1. *Let $f : I \to \mathbf{R}$ be a convex function on the interval I and $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ a normalised positive linear map. Then for any selfadjoint operator A whose spectrum $\mathrm{Sp}(A)$ is contained in $\mathring{I}$ we have*

$$f(\langle \Phi(A) y, y \rangle) \leq \langle \Phi(f(A)) y, y \rangle \tag{2.2}$$

for any $y \in K$, $\|y\| = 1$.

Proof. Let m, M with $m < M$ and such that $\operatorname{Sp}(A) \subseteq [m, M] \subset I$. Then $m1_H \leq A \leq M1_H$ and since $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$ we have that $m1_K \leq \Phi(A) \leq M1_K$ showing that $\langle \Phi(A) y, y\rangle \in [m, M]$ for any $y \in K$, $\|y\| = 1$.

By the gradient inequality (2.1) we have for $a = \langle \Phi(A) y, y\rangle \in [m, M]$ that

$$f(t) \geq f(\langle \Phi(A) y, y\rangle) + (t - \langle \Phi(A) y, y\rangle) \varphi(\langle \Phi(A) y, y\rangle)$$

for any $t \in I$.

Using the continuous functional calculus for the operator A we have for a fixed $y \in K$ with $\|y\| = 1$ that

$$f(A) \geq f(\langle \Phi(A) y, y\rangle) 1_H + \varphi(\langle \Phi(A) y, y\rangle)(A - \langle \Phi(A) y, y\rangle 1_H). \tag{2.3}$$

Since $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by taking the functional Φ in the inequality (2.3) we get

$$\Phi(f(A)) \geq f(\langle \Phi(A) y, y\rangle) 1_K + \varphi(\langle \Phi(A) y, y\rangle)(\Phi(A) - \langle \Phi(A) y, y\rangle 1_K) \tag{2.4}$$

for any $y \in K$ with $\|y\| = 1$.

This inequality is of interest in itself.

Taking the inner product in (2.4) we have for any $y \in K$ with $\|y\| = 1$ that

$$\begin{aligned}
&\langle \Phi(f(A)) y, y\rangle \\
&\geq f(\langle \Phi(A) y, y\rangle)\|y\|^2 + \varphi(\langle \Phi(A) y, y\rangle)\left(\langle \Phi(A) y, y\rangle - \langle \Phi(A) y, y\rangle \|y\|^2\right) \\
&= f(\langle \Phi(A) y, y\rangle)
\end{aligned}$$

and the inequality (2.2) is proved. □

Corollary 1. *Let $f : I \to \mathbf{R}$ be a convex function on the interval I and $\Psi \in P[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$. Then for any selfadjoint operator A whose spectrum $\operatorname{Sp}(A)$ is contained in I we have*

$$f\left(\frac{\langle\Psi(A)v,v\rangle}{\langle\Psi(1_H)v,v\rangle}\right)\leq\frac{\langle\Psi(f(A))v,v\rangle}{\langle\Psi(1_H)v,v\rangle} \tag{2.5}$$

for any $v\in K$ with $v\neq 0$.

Proof. If we write the inequality (2.2) for $\Phi=\Psi^{-1/2}(1_H)\Psi\Psi^{-1/2}(1_H)$ we have

$$f\left(\left\langle\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)y,y\right\rangle\right)$$

$$\leq\left\langle\Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H)y,y\right\rangle$$

for any $y\in K$, $\|y\|=1$.

Now, let $v\in K$ with $v\neq 0$ and take $y=\frac{1}{\|\Psi^{1/2}(1_H)v\|}\Psi^{1/2}(1_H)v$ in (2) to get

$$f\left(\left\langle\Psi^{-1/2}(1_H)\Psi(A)\Psi^{-1/2}(1_H)\frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|},\frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|}\right\rangle\right)$$

$$\leq\left\langle\Psi^{-1/2}(1_H)\Psi(f(A))\Psi^{-1/2}(1_H)\frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|},\frac{\Psi^{1/2}(1_H)v}{\|\Psi^{1/2}(1_H)v\|}\right\rangle$$

that is equivalent to

$$f\left(\left\langle\frac{\Psi(A)v}{\|\Psi^{1/2}(1_H)v\|},\frac{v}{\|\Psi^{1/2}(1_H)v\|}\right\rangle\right)\leq\left\langle\frac{\Psi(f(A))v}{\|\Psi^{1/2}(1_H)v\|},\frac{v}{\|\Psi^{1/2}(1_H)v\|}\right\rangle$$

and since

$$\left\|\Psi^{1/2}(1_H)v\right\|^2=\langle\Psi(1_H)v,v\rangle$$

for $v\in K$ with $v\neq 0$, then we obtain the desired inequality (2.5). $\square$

By taking some example of fundamental convex (concave) functions, we can state the following results:

Let $\Phi:\mathcal{B}(H)\to\mathcal{B}(K)$ be a normalised positive linear map.
(i) If A is a selfadjoint operator on H and $r\geq 1$, then we have

$$|\langle\Phi(A)y,y\rangle|^r\leq\langle\Phi(|A|^r)y,y\rangle \tag{2.6}$$

and in particular

$$|\langle \Phi(A) y, y\rangle| \leq \langle \Phi(|A|) y, y\rangle \tag{2.7}$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$\|\Phi(A)\|^r \leq \|\Phi(|A|^r)\|. \tag{2.8}$$

(ii) If A is a positive operator on a Hilbert space H, then for any $p \geq 1$ ($p \in (0,1)$) we have

$$\langle \Phi(A) y, y\rangle^p \leq (\geq) \langle \Phi(A^p) y, y\rangle \tag{2.9}$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$\|\Phi(A)\|^p \leq (\geq) \|\Phi(A^p)\|. \tag{2.10}$$

If A is a positive definite operator on a Hilbert space H, then for any $p < 0$ we have

$$\langle \Phi(A) y, y\rangle^p \leq \langle \Phi(A^p) y, y\rangle \tag{2.11}$$

for all $y \in K$, $\|y\| = 1$.
(iii) If A is a selfadjoint operator on H then we have

$$\exp(\langle \Phi(A) y, y\rangle) \leq \langle \Phi(\exp(A)) y, y\rangle \tag{2.12}$$

for all $y \in K$, $\|y\| = 1$. We have the norm inequality

$$\exp(\|\Phi(A)\|) \leq \|\Phi(\exp(A))\|. \tag{2.13}$$

Let $P_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with

$$\sum_{j=1}^{k} P_j^* P_j = 1_H. \tag{2.14}$$

The map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by [7]

$$\Phi(A) := \sum_{j=1}^{k} P_j^* A P_j$$

is a normalized positive linear map on $\mathcal{B}(H)$. Therefore, if $f : I \to \mathbf{R}$ be a convex function on the interval I and A is selfadjoint operator whose spectrum $\mathrm{Sp}(A)$ is contained in I, we have by (2.2) that

$$f\left(\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right) \leq \left\langle \sum_{j=1}^{k} P_j^* f(A) P_j y, y\right\rangle \tag{2.15}$$

for all $y \in K$, $\|y\| = 1$.

If we take $k = 1$ and $P_1 = 1_H$ in (2.15), then we recapture Jensen's inequality (1.1).

We then have for any selfadjoint operator A and $r \geq 1$ that

$$\left|\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right|^r \leq \left\langle \sum_{j=1}^{k} P_j^* |A|^r P_j y, y\right\rangle \tag{2.16}$$

and

$$\exp\left(\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right) \leq \left\langle \sum_{j=1}^{k} P_j^* (\exp A) P_j y, y\right\rangle \tag{2.17}$$

for all $y \in K$, $\|y\| = 1$. In the case $r = 1$ we have

$$\left|\sum_{j=1}^{k}\left\langle P_j^* A P_j y, y\right\rangle\right| \leq \left\langle \sum_{j=1}^{k} P_j^* |A| P_j y, y\right\rangle. \tag{2.18}$$

By taking the supremum over $y \in K$, $\|y\| = 1$ we also have the norm inequalities

$$\left\|\sum_{j=1}^{k} P_j^* A P_j\right\|^r \leq \left\|\sum_{j=1}^{k} P_j^* |A|^r P_j\right\|, \quad r \geq 1 \tag{2.19}$$

and

$$\exp\left(\left\|\sum_{j=1}^{k} P_j^* A P_j\right\|\right) \leq \left\|\sum_{j=1}^{k} P_j^* (\exp A) P_j\right\|. \tag{2.20}$$

In the case $r = 1$ we have

$$\left\|\sum_{j=1}^{k} P_j^* A P_j\right\| \leq \left\|\sum_{j=1}^{k} P_j^* |A|^r P_j\right\|. \tag{2.21}$$

If A is a positive operator on a Hilbert space H, then for any $p \in (-\infty, 0) \cup [1, \infty)$ ($p \in (0, 1)$) we have by (2.15) for power function that

$$\left\langle \sum_{j=1}^{k} P_j^* A P_j y, y \right\rangle^p \leq (\geq) \left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y \right\rangle \tag{2.22}$$

for all $y \in K$, $\|y\| = 1$.

If we take $k = 1$ and $P_1 = 1_H$ in (2.22), then we recapture Hölder-McCarthy's inequality.

By taking the supremum over $y \in K$, $\|y\| = 1$ we also have the norm inequality

$$\left\| \sum_{j=1}^{k} P_j^* A P_j \right\|^p \leq (\geq) \left\| \sum_{j=1}^{k} P_j^* A^p P_j \right\|, \tag{2.23}$$

where $p \geq 1$ $(p \in (0,1))$.

3. A Reverse Inequality

We have:

Theorem 1. *Let I be an interval and $f : I \to \mathbf{R}$ be a convex and differentiable function on I whose derivative f' is continuous on I. If $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is a normalised positive linear map and A is a selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A) \subset I$, then*

$$\begin{aligned} 0 &\leq \langle \Phi(f(A)) y, y \rangle - f(\langle \Phi(A) y, y \rangle) \\ &\leq \langle \Phi(A f'(A)) y, y \rangle - \langle \Phi(A) y, y \rangle \langle \Phi(f'(A)) y, y \rangle \end{aligned} \tag{3.1}$$

for any $y \in K$, $\|y\| = 1$.

Proof. From the gradient inequality (2.1) we have

$$f(t) \geq f(s) + (t - s) f'(s) \tag{3.2}$$

for any $t, s \in I$.

Let $y \in K$, $\|y\| = 1$. If we take in (3.2) $t = \langle \Phi(A) y, y \rangle \in I$, then we get

$$f(\langle \Phi(A) y, y \rangle) \geq f(s) + (\langle \Phi(A) y, y \rangle - s) f'(s)$$

for any $s \in I$ that can be written as

$$(s - \langle \Phi(A) y, y \rangle) f'(s) \geq f(s) - f(\langle \Phi(A) y, y \rangle)$$

for any $s \in I$.

Let $y \in K$, $\|y\| = 1$. Using the continuous functional calculus for the operator A we have

$$Af'(A) - \langle \Phi(A) y, y \rangle f'(A) \geq f(A) - f(\langle \Phi(A) y, y \rangle) 1_H. \tag{3.3}$$

Since $\Phi \in P_N[\mathcal{B}(H), \mathcal{B}(K)]$, then by taking the functional Φ in the inequality (3.3) we have

$$\Phi(Af'(A)) - \langle \Phi(A) y, y \rangle \Phi(f'(A))$$

$$\geq \Phi(f(A)) - f(\langle \Phi(A) y, y \rangle) 1_K, \tag{3.4}$$

for any $y \in K$, $\|y\| = 1$.

This is an inequality of interest in itself.

Taking the inner product in (3.4) we obtain the desired result (3.1). □

Corollary 2. *Let I be an interval and $f : I \to \mathbf{R}$ be a convex and differentiable function on I whose derivative f' is continuous on I. If $\Psi \in P[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1_H) \in \mathcal{B}^{++}(K)$ and A is a selfadjoint operators on the Hilbert space H with* $\operatorname{Sp}(A) \subset I$, *then*

$$\begin{aligned} 0 &\leq \frac{\langle \Psi(f(A))v,v\rangle}{\langle \Psi(1_H)v,v\rangle} - f\left(\frac{\langle \Psi(A)v,v\rangle}{\langle \Psi(1_H)v,v\rangle}\right) \\ &\leq \frac{\langle \Psi(Af'(A))v,v\rangle}{\langle \Psi(1_H)v,v\rangle} - \frac{\langle \Psi(A)v,v\rangle}{\langle \Psi(1_H)v,v\rangle}\frac{\langle \Psi(f'(A))v,v\rangle}{\langle \Psi(1_H)v,v\rangle} \end{aligned} \tag{3.5}$$

for any $v \in K$ with $v \neq 0$.

The proof follows from the inequality (3.1) by a similar argument to the one from the proof of Corollary 1 and the details are omitted.

Let $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a normalised positive linear map.

(i) If A is a selfadjoint operator on H, then we have

$$\begin{aligned} 0 &\leq \langle \Phi(\exp(A)) y, y \rangle - \exp(\langle \Phi(A) y, y \rangle) \\ &\leq \langle \Phi(A \exp(A)) y, y \rangle - \langle \Phi(A) y, y \rangle \langle \Phi(\exp(A)) y, y \rangle \end{aligned} \tag{3.6}$$

for all $y \in K$, $\|y\| = 1$.

(ii) If A is a positive (positive definite) operator on a Hilbert space H, then for any $p \geq 1$ ($p \in (-\infty, 0)$) we have

$$\begin{aligned} 0 &\leq \langle \Phi(A^p) y, y \rangle - \langle \Phi(A) y, y \rangle^p \\ &\leq p\left[\langle \Phi(A^p) y, y \rangle - \langle \Phi(A) y, y \rangle \langle \Phi(A^{p-1}) y, y \rangle\right] \end{aligned} \tag{3.7}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive operator on a Hilbert space H, then for any $p \in (0,1)$ we have

$$\begin{aligned} 0 &\leq \langle \Phi(A) y, y\rangle^p - \langle \Phi(A^p) y, y\rangle \\ &\leq p\left[\langle \Phi(A) y, y\rangle \langle \Phi(A^{p-1}) y, y\rangle - \langle \Phi(A^p) y, y\rangle\right] \end{aligned} \tag{3.8}$$

for all $y \in K$, $\|y\| = 1$.

(iii) If A is a positive definite operator on a Hilbert space H, then

$$\begin{aligned} 0 &\leq \ln(\langle \Phi(A) y, y\rangle) - \langle \Phi(\ln A) y, y\rangle \\ &\leq \langle \Phi(A) y, y\rangle \left\langle \Phi\left(A^{-1}\right) y, y\right\rangle - 1 \end{aligned} \tag{3.9}$$

for all $y \in K$, $\|y\| = 1$.

Let $P_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with the property (2.14). If $f : I \to \mathbf{R}$ is a convex function on the interval I and A is selfadjoint operator whose spectrum $\operatorname{Sp}(A)$ is contained in I, then we have by (3.1) that

$$\begin{aligned} 0 &\leq \left\langle \sum_{j=1}^k P_j^* f(A) P_j y, y\right\rangle - f\left(\sum_{j=1}^k \left\langle P_j^* A P_j y, y\right\rangle\right) \\ &\leq \left\langle \sum_{j=1}^k P_j^* A f'(A) P_j y, y\right\rangle - \left\langle \sum_{j=1}^k P_j^* A P_j y, y\right\rangle \left\langle \sum_{j=1}^k P_j^* f'(A) P_j y, y\right\rangle \end{aligned} \tag{3.10}$$

for all $y \in K$, $\|y\| = 1$. This is a generalization of (1.2).

In particular, if A is a selfadjoint operator on H, then we have

$$\begin{aligned} 0 &\leq \left\langle \sum_{j=1}^k P_j^* \exp(A) P_j y, y\right\rangle - \exp\left(\sum_{j=1}^k \left\langle P_j^* A P_j y, y\right\rangle\right) \\ &\leq \left\langle \sum_{j=1}^k P_j^* A \exp(A) P_j y, y\right\rangle - \left\langle \sum_{j=1}^k P_j^* A P_j y, y\right\rangle \left\langle \sum_{j=1}^k P_j^* \exp P_j y, y\right\rangle \end{aligned} \tag{3.11}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive (positive definite) operator on a Hilbert space H, then for any $p \geq 1$ ($p \in (-\infty, 0)$) we have

$$\begin{aligned} 0 &\leq \left\langle \sum_{j=1}^k P_j^* A^p P_j y, y\right\rangle - \left(\sum_{j=1}^k \left\langle P_j^* A P_j y, y\right\rangle\right)^p \\ &\leq p\left[\left\langle \sum_{j=1}^k P_j^* A^p P_j y, y\right\rangle - \left\langle \sum_{j=1}^k P_j^* A P_j y, y\right\rangle \left\langle \sum_{j=1}^k P_j^* A^{p-1} P_j y, y\right\rangle\right], \end{aligned}$$

(3.12)

for all $y \in K$, $\|y\| = 1$. However, when $p \in (0,1)$ and A is a positive, then

$$\begin{aligned} 0 &\le \left(\sum_{j=1}^{k} \left\langle P_j^* A P_j y, y \right\rangle\right)^p - \left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y \right\rangle \\ &\le p \left[\left\langle \sum_{j=1}^{k} P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^{k} P_j^* A^{p-1} P_j y, y \right\rangle - \left\langle \sum_{j=1}^{k} P_j^* A^p P_j y, y \right\rangle \right], \end{aligned} \tag{3.13}$$

for all $y \in K$, $\|y\| = 1$.

If A is a positive definite operator on H, then

$$\begin{aligned} 0 &\le \ln \left(\sum_{j=1}^{k} \left\langle P_j^* A P_j y, y \right\rangle\right) - \left\langle \sum_{j=1}^{k} P_j^* (\ln A) P_j y, y \right\rangle \\ &\le \left\langle \sum_{j=1}^{k} P_j^* A P_j y, y \right\rangle \left\langle \sum_{j=1}^{k} P_j^* A^{-1} P_j y, y \right\rangle - 1 \end{aligned} \tag{3.14}$$

for all $y \in K$, $\|y\| = 1$.

These inequalities generalize the corresponding results from (1.4)-(1.6).

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