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Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT$_m$-preinvex functions

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Abstract

In the present paper, the notion of MT_m-preinvex function is introduced and some new integral inequalities involving MT_m-preinvex functions along with beta function are given. Moreover, some generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT_m-preinvex functions via classical integrals and Riemann-Liouville fractional integrals are established. These results not only extends the results appeared in the literature (see [10], [11], [12]), but also provide new estimates on these types.

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1. Introduction and Preliminaries

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbf{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbf{R}^n$, K° is used to denote the interior of K. $\mathbf{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbf{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Also the following result is known in the literature as the Ostrowski inequality (see [11]) and the references cited therein, which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.2. *Let $f : I \longrightarrow \mathbf{R}$, where $I \subseteq \mathbf{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad \forall x \in [a, b]. \tag{1.2}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [10]) and the references cited therein, also (see [9]) and the references cited therein. For other recent results concerning Ostrowski type inequalities (see [11]) and the references cited therein, also (see [12]) and the references cited therein.

In (see [11]) and the references cited therein, Tunç and Yidirim defined the following so-called MT-convex function:

Definition 1.3. *A function $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}$ is said to belong to the class of $MT(I)$, if it is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ satisfies the following inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.3}$$

In (see [12]), Tunç derived some inequalities of Ostrowski type for MT-convex functions.

Fractional calculus (see [10]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.4. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*
$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$
and
$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$ *where* $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du.$
Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard, Grüss, or Ostrowski type inequalities for functions of different classes (see [10]) and the references cited therein.

Now, let us recall some definitions of various convex functions.

Definition 1.5. *(see [2]) A nonnegative function $f : I \subseteq \mathbf{R} \longrightarrow \mathbf{R}_\circ$ is said to be P-function or P-convex, if*
f(tx+(1-t)y)≤ $f(x) + f(y), \quad \forall x, y \in I,\ t \in [0,1].$

Definition 1.6. *(see [3]) A set $K \subseteq \mathbf{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbf{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please see (see [3],[4]) and the references therein.

Definition 1.7. *(see [5]) The function f defined on the invex set $K \subseteq \mathbf{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that*

$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y)$.

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.
The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{1.4}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [6]).
Recently, Liu (see [7]) obtained several integral inequalities for the left hand side of (1.4) under the Definition 1.5 of P-function.

Also in (see [8]), Özdemir et al. established several integral inequalities concerning the left-hand side of (1.4) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of MT_m-preinvex function is introduced and some new integral inequalities involving MT_m-preinvex functions along with beta function are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via classical integrals are given. In Section 4, some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via fractional integrals are given. In Section 5, some generalizations of Ostrowski type inequalities for MT_m-preinvex functions via classical integrals are given. In Section 6, some generalizations of Ostrowski type inequalities for MT_m-preinvex functions via fractional integrals are given. These results given in sections (3-6) not only extends the results appeared in the literature (see [10], [11], [12]), but also provide new estimates on these types.

2. New integral inequalities for MT_m-preinvex functions

Definition 2.1. *(see [1]) A set $K \subseteq \mathbf{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbf{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

Remark 2.2. *In Definition 2.1, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$.*

We next give new definition, to be referred as MT_m-preinvex function.

Definition 2.3. *Let $K \subseteq \mathbf{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbf{R}^n$. For $f : K \longrightarrow \mathbf{R}$ and $m \in (0,1]$, if*

$$f(my + t\eta(x,y,m)) \le \frac{m\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y), \tag{2.1}$$

is valid for all $x, y \in K$ and $t \in (0,1)$, then we say that $f(x)$ belong to the class of $MT_m(K)$ with respect to η.

Remark 2.4. *In Definition 2.3, it is worthwhile to note that the class $MT_m(K)$ is a generalization of the class $MT(I)$ given in Definition 1.3 on $K = I$ with respect to $\eta(x,y,m) = x - my$ and $m = 1$.*

In this section, in order to prove our main results regarding some new integral inequalities involving MT_m-preinvex functions along with beta function, we need the following Lemma:

Lemma 2.5. *Let $f : K = [ma, ma + \eta(b,a,m)] \longrightarrow \mathbf{R}$ be a continuous function on the interval of real numbers K° with $a < b$ and $ma < ma + \eta(b,a,m)$. Then for some fixed $m \in (0,1]$ and $p, q > 0$, we have*

$$\int_{ma}^{ma+\eta(b,a,m)} (x - ma)^p (ma + \eta(b,a,m) - x)^q f(x)dx$$

$$= \eta(b,a,m)^{p+q+1} \int_0^1 t^p (1-t)^q f(ma + t\eta(b,a,m))dt.$$

Proof. It is easy to observe that
$\int_{ma}^{ma+\eta(b,a,m)}(x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$
$=\eta(b,a,m)\int_0^1(ma+t\eta(b,a,m)-ma)^p(ma+\eta(b,a,m)-ma-t\eta(b,a,m))^q$
$f(ma+t\eta(b,a,m))dt$
$=\eta(b,a,m)^{p+q+1}\int_0^1 t^p(1-t)^q f(ma+t\eta(b,a,m))dt.$ □

The following definition will be used in the sequel.

Definition 2.6. *The Euler Beta function is defined for $x,y>0$ as*
$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$

Theorem 2.7. *Let $f:K=[ma,ma+\eta(b,a,m)]\longrightarrow \mathbf{R}$ be a continuous function on the interval of real numbers K°, $a<b$ with $ma<ma+\eta(b,a,m)$. If $|f|$ is a MT_m-preinvex function on K for some fixed $m\in(0,1]$, then for some fixed $p,q>0$,*
$\int_{ma}^{ma+\eta(b,a,m)}(x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$
$\le \frac{m}{2}\eta(b,a,m)^{p+q+1}\left[|f(a)|\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(b)|\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)\right].$

Proof. Since $|f|$ is a MT_m-preinvex function on K, we have

$$\left|f(ma+t\eta(b,a,m))\right|\le\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)|+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|$$

for all $t\in(0,1)$ and for some fixed $m\in(0,1]$. By Lemma 2.5, Definition 2.6 and the fact that $|f|$ is a MT_m-preinvex function on K, we get
$\int_{ma}^{ma+\eta(b,a,m)}(x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$
$\le \eta(b,a,m)^{p+q+1}\int_0^1 t^p(1-t)^q\left|f(ma+t\eta(b,a,m))\right|dt$

$$\le \eta(b,a,m)^{p+q+1}\int_0^1 t^p(1-t)^q\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)|+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|\right]dt$$

$$=\frac{m}{2}\eta(b,a,m)^{p+q+1}\left[|f(a)|\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(b)|\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)\right].$$

□

Theorem 2.8. *Let $f : K = [ma, ma + \eta(b, a, m)] \longrightarrow \mathbf{R}$ be a continuous function on the interval of real numbers K°, $a < b$ with $ma < ma + \eta(b, a, m)$. Let $k > 1$. If $|f|^{\frac{k}{k-1}}$ is a MT_m-preinvex function on K for some fixed $m \in (0, 1]$, then for some fixed $p, q > 0$,*

$$\int_{ma}^{ma+\eta(b,a,m)}(x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$
$$\leq \left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} \eta(b,a,m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a MT_m-preinvex function on K, combining with Lemma 2.5, Definition 2.6 and Hölder inequality for all $t \in (0, 1)$ and for some fixed $m \in (0, 1]$, we get

$$\int_{ma}^{ma+\eta(b,a,m)}(x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$
$$\leq \eta(b,a,m)^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}\left[\int_0^1 \Big|f(ma+t\eta(b,a,m))\Big|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}}$$
$$\leq \eta(b,a,m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}$$
$$\times\left[\int_0^1\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)|^{\frac{k}{k-1}}+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}}$$

□

$$=\left(\frac{m\pi}{4}\right)^{\frac{k-1}{k}} \eta(b,a,m)^{p+q+1}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}}\left(|f(a)|^{\frac{k}{k-1}}+|f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.$$

Theorem 2.9. *Let $f : K = [ma, ma + \eta(b, a, m)] \longrightarrow \mathbf{R}$ be a continuous function on the interval of real numbers K°, $a < b$ with $ma < ma + \eta(b, a, m)$. Let $l \geq 1$. If $|f|^l$ is a MT_m-preinvex function on K for some fixed $m \in (0, 1]$, then for some fixed $p, q > 0$,*

$$\int_{ma}^{ma+\eta(b,a,m)}(x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$
$$\leq \left(\frac{m}{2}\right)^{\frac{1}{l}} \eta(b,a,m)^{p+q+1}\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}$$
$$\left[|f(a)|^l\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(b)|^l\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)\right]^{\frac{1}{l}}.$$

Proof. Since $|f|^l$ is a MT_m-preinvex function on K, combining with Lemma 2.5, Definition 2.6 and Hölder inequality for all $t \in (0,1)$ and for some fixed $m \in (0,1]$, we get

$$\int_{ma}^{ma+\eta(b,a,m)} (x-ma)^p(ma+\eta(b,a,m)-x)^q f(x)dx$$

$$=\eta(b,a,m)^{p+q+1}\int_0^1 \left[t^p(1-t)^q\right]^{\frac{l-1}{l}}\left[t^p(1-t)^q\right]^{\frac{1}{l}} f(ma+t\eta(b,a,m))dt$$

$$\leq \eta(b,a,m)^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\left[\int_0^1 t^p(1-t)^q\left|f(ma+t\eta(b,a,m))\right|^l dt\right]^{\frac{1}{l}}$$

$$\leq \eta(b,a,m)^{p+q+1}\left[\beta(p+1,q+1)\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 t^p(1-t)^q\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f(b)|^l+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f(a)|^l\right)dt\right]^{\frac{1}{l}}$$

$$=\left(\frac{m}{2}\right)^{\frac{1}{l}}\eta(b,a,m)^{p+q+1}\left[\beta(p+1,q+1)\right]^{\frac{l-1}{l}}$$

$$\left[|f(a)|^l\beta\left(p+\frac{1}{2},q+\frac{3}{2}\right)+|f(b)|^l\beta\left(p+\frac{3}{2},q+\frac{1}{2}\right)\right]^{\frac{1}{l}}.$$

□

Remark 2.10. *In Theorem 2.9, if we choose $l=1$, we get Theorem 2.7.*

3. Hermite-Hadamard type classical integral inequalities for MT_m-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via classical integrals, we need the following Lemma:

Lemma 3.1. *Let $K \subseteq \mathbf{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0,1]$ and let $a,b \in K$, $a<b$ with $ma < ma+\eta(b,a,m)$. Assume that $f : K \longrightarrow \mathbf{R}$ is a differentiable function on K° and f' is integrable on $[ma, ma+\eta(b,a,m)]$. Then, for each $x \in [ma, ma+\eta(b,a,m)]$, we have that*

$$\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)}-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du-\int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]$$
$$=\frac{\eta(x,a,m)^2}{\eta(b,a,m)}\int_0^1(t-1)f'(ma+t\eta(x,a,m))dt$$
$$+\frac{\eta(x,b,m)^2}{\eta(b,a,m)}\int_0^1(1-t)f'(mb+t\eta(x,b,m))dt. \tag{3.1}$$

Proof. Denote

$$I=\frac{\eta(x,a,m)^2}{\eta(b,a,m)}\int_0^1(t-1)f'(ma+t\eta(x,a,m))dt$$
$$+\frac{\eta(x,b,m)^2}{\eta(b,a,m)}\int_0^1(1-t)f'(mb+t\eta(x,b,m))dt.$$

Integrating by parts, we get

$$I=\frac{\eta(x,a,m)^2}{\eta(b,a,m)}\left[(t-1)\frac{f(ma+t\eta(x,a,m))}{\eta(x,a,m)}\Big|_0^1-\int_0^1\frac{f(ma+t\eta(x,a,m))}{\eta(x,a,m)}dt\right]$$
$$+\frac{\eta(x,b,m)^2}{\eta(b,a,m)}\left[(1-t)\frac{f(mb+t\eta(x,b,m))}{\eta(x,b,m)}\Big|_0^1+\int_0^1\frac{f(mb+t\eta(x,b,m))}{\eta(x,b,m)}dt\right]$$
$$=\frac{\eta(x,a,m)^2}{\eta(b,a,m)}\left[\frac{f(ma)}{\eta(x,a,m)}-\frac{1}{\eta(x,a,m)^2}\int_{ma}^{ma+\eta(x,a,m)} f(u)du\right]$$
$$+\frac{\eta(x,b,m)^2}{\eta(b,a,m)}\left[-\frac{f(mb)}{\eta(x,b,m)}+\frac{1}{\eta(x,b,m)^2}\int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]$$
$$=\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)}$$
$$-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du-\int_{mb}^{mb+\eta(x,b,m)} f(u)du\right].\ \square$$

Remark 3.2. *Clearly, if we choose $m=1$ and $\eta(x,y,m)=x-my$ in Lemma 3.1, we get Lemma (see [9], Lemma 1).*

Using the Lemma 3.1 the following results can be obtained.

Theorem 3.3. *Let $A\subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta: A\times A\times(0,1]\longrightarrow \mathbf{R}_0$ for some fixed $m\in(0,1]$ and let $a,b\in A$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f: A\longrightarrow \mathbf{R}$ is a differentiable*

function on A°. If $|f'|$ is a MT_m-preinvex function on $[ma, ma+\eta(b,a,m)]$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma+\eta(b,a,m)]$, we have that

$$\left|\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)}\right.$$
$$\left. -\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$
$$\leq \frac{Mm\pi}{4|\eta(b,a,m)|}\Big[\eta(x,a,m)^2+\eta(x,b,m)^2\Big]. \tag{3.2}$$

Proof. Using Lemma 3.1, Definition 2.6, MT_m-preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma+\eta(b,a,m)]$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)} - \frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 |t-1||f'(ma+t\eta(x,a,m))|dt$$
$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 |1-t||f'(mb+t\eta(x,b,m))|dt$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 (1-t)\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|\right]dt$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 (1-t)\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|\right]dt$$

$$\leq \frac{Mm}{2}\frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 (1-t)\left[\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}}\right]dt$$
$$+\frac{Mm}{2}\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 (1-t)\left[\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}}\right]dt$$
$$= \frac{Mm\pi}{4|\eta(b,a,m)|}\Big[\eta(x,a,m)^2+\eta(x,b,m)^2\Big].$$

□

Remark 3.4. *In Theorem 3.3, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ then we get Theorem (see [10], Theorem 2.2).*

The corresponding version for power of the absolute value of the first derivative is incorporated in the following results.

Theorem 3.5. *Let $A \subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbf{R}_0$ for some fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma, ma + \eta(b, a, m)]$, $q > 1$, $p^{-1} + q^{-1} = 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$, we have that*

$$\left| \frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)} \right.$$

$$\left. - \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \right|$$

$$\leq \frac{M}{(p+1)^{1/p}} \left(\frac{m\pi}{2} \right)^{\frac{1}{q}} \left[\frac{\eta(x,a,m)^2 + \eta(x,b,m)^2}{|\eta(b,a,m)|} \right]. \tag{3.3}$$

Proof. Suppose that $q > 1$. Using Lemma 3.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

$$\left| \frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)} \right.$$

$$\left. - \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \right|$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \int_0^1 |t-1||f'(ma + t\eta(x,a,m))|dt$$

$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \int_0^1 |1-t||f'(mb + t\eta(x,b,m))|dt$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)^p dt\right)^{\frac{1}{p}}\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2+\eta(x,b,m)^2}{|\eta(b,a,m)|}\right].$$

□

Remark 3.6. *In Theorem 3.5, if we choose $m=1$ and $\eta(x,y,m)=x-my$ then we get Theorem (see [10], Theorem 2.4).*

Theorem 3.7. *Let $A\subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A\times A\times(0,1]\longrightarrow \mathbf{R}_0$ for some fixed $m\in(0,1]$ and let $a,b\in A$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f:A\longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma,ma+\eta(b,a,m)]$, $q\geq 1$ and $|f'(x)|\leq M$, then for each $x\in[ma,ma+\eta(b,a,m)]$, we have that*

$$\left|\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)}\right.$$

$$\left.-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du-\int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$

$$\leq M\left(\frac{1}{2}\right)^{1+\frac{1}{q}}(m\pi)^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2+\eta(x,b,m)^2}{|\eta(b,a,m)|}\right]. \tag{3.4}$$

Proof. Using Lemma 3.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma+\eta(b,a,m)]$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)f(ma)-\eta(x,b,m)f(mb)}{\eta(b,a,m)}\right.$$

$$\left.-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 |t-1||f'(ma+t\eta(x,a,m))|dt + \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}$$

$$\int_0^1 |1-t||f'(mb+t\eta(x,b,m))|dt$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)|f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}$$

$$\left[\int_0^1 (1-t)\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 (1-t)dt\right)^{1-\frac{1}{q}}$$

$$\left[\int_0^1 (1-t)\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq M\left(\frac{1}{2}\right)^{1+\frac{1}{q}}(m\pi)^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2+\eta(x,b,m)^2}{|\eta(b,a,m)|}\right].$$

□

Remark 3.8. *In Theorem 3.7, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ then we get Theorem (see [10], Theorem 2.6). Also, in Theorem 3.7, if we choose $q = 1$, we get Theorem 3.3.*

Theorem 3.9. *Let $A \subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbf{R}_0$ for some fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma, ma+\eta(b, a, m)]$, $q \geq 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$, we have that*

$$\left|\frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)}\right.$$
$$\left.-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$
$$\leq M\left[\frac{m\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{2\Gamma(q+1)}\right]^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2 + \eta(x,b,m)^2}{|\eta(b,a,m)|}\right]. \tag{3.5}$$

Proof. Using Lemma 3.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)f(ma) - \eta(x,b,m)f(mb)}{\eta(b,a,m)}\right.$$
$$\left.-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du\right]\right|$$
$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 |t-1||f'(ma+t\eta(x,a,m))|dt$$
$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 |1-t||f'(mb+t\eta(x,b,m))|dt$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^q|f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^q|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left[\int_0^1(1-t)^q\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}}\left[\int_0^1(1-t)^q\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq M\left[\frac{m\Gamma\left(\frac{1}{2}\right)\Gamma\left(q+\frac{1}{2}\right)}{2\Gamma(q+1)}\right]^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2+\eta(x,b,m)^2}{|\eta(b,a,m)|}\right]. \quad \square$$

Remark 3.10. *In Theorem 3.9, if we choose $m=1$ and $\eta(x,y,m)=x-my$ then we get Theorem (see [10], Theorem 2.8).*

4. Hermite-Hadamard type fractional integral inequalities for MT_m-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for MT_m-preinvex functions via fractional integrals, we need the following Lemma:

Lemma 4.1. *Let $K \subseteq \mathbf{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0,1]$ and let $a, b \in K$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : K \longrightarrow \mathbf{R}$ is a differentiable function on K° and f' is integrable on $[ma, ma+\eta(b,a,m)]$. Then, for each $x \in [ma, ma+\eta(b,a,m)]$ and $\alpha > 0$, we have that*

$$\frac{\eta(x,a,m)^\alpha f(ma)-\eta(x,b,m)^\alpha f(mb)}{\eta(b,a,m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^\alpha_{(ma+\eta(x,a,m))-}f(ma)-J^\alpha_{(mb+\eta(x,b,m))-}f(mb)\right]$$

$$= \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (t^\alpha - 1) f'(ma + t\eta(x,a,m))dt$$

$$+ \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (1 - t^\alpha) f'(mb + t\eta(x,b,m))dt, \tag{4.1}$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$ *is the Euler Gamma function.*

Proof. Denote

$$I = \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (t^\alpha - 1) f'(ma + t\eta(x,a,m))dt$$

$$+ \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \int_0^1 (1 - t^\alpha) f'(mb + t\eta(x,b,m))dt.$$

Integrating by parts, we get

$$I = \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \left[(t^\alpha - 1) \frac{f(ma + t\eta(x,a,m))}{\eta(x,a,m)} \Bigg|_0^1 \right.$$

$$\left. -\alpha \int_0^1 \frac{t^{\alpha-1} f(ma + t\eta(x,a,m))}{\eta(x,a,m)} dt \right]$$

$$+ \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \left[(1 - t^\alpha) \frac{f(mb + t\eta(x,b,m))}{\eta(x,b,m)} \Bigg|_0^1 \right.$$

$$\left. +\alpha \int_0^1 \frac{t^{\alpha-1} f(mb + t\eta(x,b,m))}{\eta(x,b,m)} dt \right]$$

$$= \frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)} \left[\frac{f(ma)}{\eta(x,a,m)} - \frac{\Gamma(\alpha+1)}{\eta(x,a,m)^{\alpha+1}} J^\alpha_{(ma+\eta(x,a,m))-} f(ma) \right]$$

$$+ \frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)} \left[- \frac{f(mb)}{\eta(x,b,m)} + \frac{\Gamma(\alpha+1)}{\eta(x,b,m)^{\alpha+1}} J^\alpha_{(mb+\eta(x,b,m))-} f(mb) \right]$$

$$= \frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb)\right].$$

□

Remark 4.2. *Clearly, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ in Lemma 4.1, we get Lemma (see [10], Lemma 3.1).*

By using Lemma 4.1, one can extend to the following results.

Theorem 4.3. *Let $A \subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbf{R}_0$ for some fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|$ is a MT_m-preinvex function on $[ma, ma+\eta(b, a, m)]$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$ and $\alpha > 0$, we have that*

$$\left|\frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb)\right]\right|$$

$$\leq \frac{Mm}{2}\left[\frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\pi - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]. \tag{4.2}$$

Proof. Using Lemma 4.1, Definition 2.6, MT_m-preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, $\alpha > 0$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)}\right.$$

$$-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma)-J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]\Bigg|$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 |t^{\alpha}-1||f'(ma+t\eta(x,a,m))|dt$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 |1-t^{\alpha}||f'(mb+t\eta(x,b,m))|dt$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 (1-t^{\alpha})\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|\right]dt$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 (1-t^{\alpha})\left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|\right]dt$$

$$\leq \frac{Mm}{2}\frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 (1-t^{\alpha})\left[\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right]dt$$

$$\times+\frac{Mm}{2}\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 (1-t^{\alpha})\left[\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right]dt$$

$$=\frac{Mm}{2}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\pi-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right].$$

□

Remark 4.4. *In Theorem 4.3, if we choose $m=1$ and $\eta(x,y,m)=x-my$ then we get Theorem (see [10], Theorem 3.2). Also, in Theorem 4.3, if we choose $\alpha=1$, we get the inequality in Theorem 3.3.*

Theorem 4.5. *Let $A\subseteq\mathbf{R}_0$ be an open m-invex subset with respect to $\eta: A\times A\times(0,1]\longrightarrow\mathbf{R}_0$ for some fixed $m\in(0,1]$ and let $a,b\in A$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f:A\longrightarrow\mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma,ma+\eta(b,a,m)]$,*

$q > 1$, $p^{-1}+q^{-1} = 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma+\eta(b,a,m)]$ and $\alpha > 0$, we have that

$$\left|\frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb)\right]\right|$$

$$\leq M\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}.$$

(4.3)

Proof. Suppose that $q > 1$. Using Lemma 4.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma+\eta(b,a,m)]$, $\alpha > 0$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)^{\alpha} f(ma) - \eta(x,b,m)^{\alpha} f(mb)}{\eta(b,a,m)}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb)\right]\right|$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 |t^{\alpha}-1||f'(ma+t\eta(x,a,m))|dt$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 |1-t^{\alpha}||f'(mb+t\eta(x,b,m))|dt$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 (1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\int_0^1|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}$$

$$\left[\int_0^1\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1(1-t^{\alpha})^p dt\right)^{\frac{1}{p}}$$

$$\left[\int_0^1\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq M\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\frac{\Gamma(p+1)\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma\left(p+1+\frac{1}{\alpha}\right)}\right]^{\frac{1}{p}}. \quad \square$$

Remark 4.6. *In Theorem 4.5, if we choose $m=1$ and $\eta(x,y,m)=x-my$ then we get Theorem (see [10], Theorem 3.5). Also, in Theorem 4.5, if we choose $\alpha=1$, we get the inequality in Theorem 3.5.*

Theorem 4.7. *Let $A\subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta: A\times A\times(0,1]\longrightarrow \mathbf{R}_0$ for some fixed $m\in(0,1]$ and let $a,b\in A$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f:A\longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma,ma+\eta(b,a,m)]$, $q\geq 1$ and $|f'(x)|\leq M$, then for each $x\in[ma,ma+\eta(b,a,m)]$ and $\alpha>0$, we have that*

$$\left|\frac{\eta(x,a,m)^{\alpha}f(ma)-\eta(x,b,m)^{\alpha}f(mb)}{\eta(b,a,m)}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma)-J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]\right|$$

$$\leq M\left(\frac{m}{2}\right)^{\frac{1}{q}}\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\pi-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]. \tag{4.4}$$

Proof. Using Lemma 4.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)|\leq M$ for each $x\in[ma,ma+\eta(b,a,m)]$, $\alpha>0$, and taking the modulus, we have

$$\left|\frac{\eta(x,a,m)^{\alpha}f(ma)-\eta(x,b,m)^{\alpha}f(mb)}{\eta(b,a,m)}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma)-J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]\right|$$

$$\leq\frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1|t^{\alpha}-1||f'(ma+t\eta(x,a,m))|dt$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1|1-t^{\alpha}||f'(mb+t\eta(x,b,m))|dt$$

$$\leq\frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})|f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t^{\alpha})|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}$$

$$\times\left(\int_0^1(1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left[\int_0^1(1-t^{\alpha})\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}$$

$$\times\left(\int_0^1(1-t^\alpha)dt\right)^{1-\frac{1}{q}}\left[\int_0^1(1-t^\alpha)\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq M\left(\tfrac{m}{2}\right)^{\frac{1}{q}}\left(\tfrac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\pi-\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]. \qquad \square$$

Remark 4.8. *In Theorem 4.7, if we choose $m=1$ and $\eta(x,y,m)=x-my$ then we get Theorem (see [10], Theorem 3.8). Also, in Theorem 4.7, if we choose $\alpha=1$, we get Theorem 3.7.*

5. Ostrowski type classical integral inequalities for MT_m-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for MT_m-preinvex functions via classical integrals, we need the following Lemma:

Lemma 5.1. *Let $K\subseteq\mathbf{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbf{R}$ for some fixed $m\in(0,1]$ and let $a,b\in K$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f:K\longrightarrow\mathbf{R}$ is a differentiable function on K° and f' is integrable on $[ma,ma+\eta(b,a,m)]$. Then, for each $x\in[ma,ma+\eta(b,a,m)]$, we have that*

$$\frac{\eta(x,a,m)f(ma+\eta(x,a,m))-\eta(x,b,m)f(mb+\eta(x,b,m))}{\eta(b,a,m)}$$

$$-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)}f(u)du-\int_{mb}^{mb+\eta(x,b,m)}f(u)du\right]$$

$$=\frac{\eta(x,a,m)^2}{\eta(b,a,m)}\int_0^1 tf'(ma+t\eta(x,a,m))dt$$

$$-\frac{\eta(x,b,m)^2}{\eta(b,a,m)}\int_0^1 tf'(mb+t\eta(x,b,m))dt. \tag{5.1}$$

Proof. Denote

$$I = \frac{\eta(x,a,m)^2}{\eta(b,a,m)} \int_0^1 tf'(ma + t\eta(x,a,m))dt$$
$$- \frac{\eta(x,b,m)^2}{\eta(b,a,m)} \int_0^1 tf'(mb + t\eta(x,b,m))dt.$$

Integrating by parts, we get

$$I = \frac{\eta(x,a,m)^2}{\eta(b,a,m)} \left[t\frac{f(ma + t\eta(x,a,m))}{\eta(x,a,m)}\bigg|_0^1 - \int_0^1 \frac{f(ma + t\eta(x,a,m))}{\eta(x,a,m)}dt \right]$$

$$- \frac{\eta(x,b,m)^2}{\eta(b,a,m)} \left[t\frac{f(mb + t\eta(x,b,m))}{\eta(x,b,m)}\bigg|_0^1 - \int_0^1 \frac{f(mb + t\eta(x,b,m))}{\eta(x,b,m)}dt \right]$$

$$= \frac{\eta(x,a,m)^2}{\eta(b,a,m)} \left[\frac{f(ma + \eta(x,a,m))}{\eta(x,a,m)} - \frac{1}{\eta(x,a,m)^2} \int_{ma}^{ma+\eta(x,a,m)} f(u)du \right]$$

$$- \frac{\eta(x,b,m)^2}{\eta(b,a,m)} \left[\frac{f(mb + \eta(x,b,m))}{\eta(x,b,m)} - \frac{1}{\eta(x,b,m)^2} \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right]$$

$$= \frac{\eta(x,a,m)f(ma + \eta(x,a,m)) - \eta(x,b,m)f(mb + \eta(x,b,m))}{\eta(b,a,m)}$$

$$- \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right].$$

□

Remark 5.2. *Clearly, if we choose $m = 1$ and $\eta(x,y,m) = x - my$ in Lemma 5.1, we get Lemma (see [12], Lemma 1).*

Using the Lemma 5.1 the following results can be obtained.

Theorem 5.3. *Let $A \subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbf{R}_0$ for some fixed $m \in (0,1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : A \longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|$ is a MT_m-preinvex function on $[ma, ma + \eta(b,a,m)]$ and $|f'(x)| \le M$, then for each $x \in [ma, ma + \eta(b,a,m)]$, we have that*

$$\left| \frac{\eta(x,a,m)f(ma+\eta(x,a,m)) - \eta(x,b,m)f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. -\frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \right|$$

$$\le \frac{Mm\pi}{4|\eta(b,a,m)|} \Big[\eta(x,a,m)^2 + \eta(x,b,m)^2 \Big]. \tag{5.2}$$

Proof. Using Lemma 5.1, Definition 2.6, MT_m-preinvexity of $|f'|$, the fact that $|f'(x)| \le M$ for each $x \in [ma, ma + \eta(b,a,m)]$, and taking the modulus, we have

$$\left| \frac{\eta(x,a,m)f(ma+\eta(x,a,m)) - \eta(x,b,m)f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. -\frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \right|$$

$$\le \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \int_0^1 t|f'(ma+t\eta(x,a,m))|dt$$

$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \int_0^1 t|f'(mb+t\eta(x,b,m))|dt$$

$$\le \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \int_0^1 t \left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)| \right] dt$$

$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \int_0^1 t \left[\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)| \right] dt$$

$$\leq \frac{Mm}{2}\frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\int_0^1 t\left[\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right]dt$$

$$+\frac{Mm}{2}\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\int_0^1 t\left[\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right]dt$$

$$=\frac{Mm\pi}{4|\eta(b,a,m)|}\left[\eta(x,a,m)^2+\eta(x,b,m)^2\right].$$

□

Remark 5.4. *In Theorem 5.3, if we choose $m=1$ and $\eta(x,y,m)=x-my$ then we get Theorem (see [12], Theorem 2).*

The corresponding version for power of the absolute value of the first derivative is incorporated in the following results.

Theorem 5.5. *Let $A\subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta: A\times A\times(0,1]\longrightarrow \mathbf{R}_0$ for some fixed $m\in(0,1]$ and let $a,b\in A$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f:A\longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma,ma+\eta(b,a,m)]$, $q>1$, $p^{-1}+q^{-1}=1$ and $|f'(x)|\leq M$, then for each $x\in[ma,ma+\eta(b,a,m)]$, we have that*

$$\left|\frac{\eta(x,a,m)f(ma+\eta(x,a,m))-\eta(x,b,m)f(mb+\eta(x,b,m))}{\eta(b,a,m)}\right.$$

$$\left.-\frac{1}{\eta(b,a,m)}\left[\int_{ma}^{ma+\eta(x,a,m)}f(u)du-\int_{mb}^{mb+\eta(x,b,m)}f(u)du\right]\right|$$

$$\leq \frac{M}{(p+1)^{1/p}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{\eta(x,a,m)^2+\eta(x,b,m)^2}{|\eta(b,a,m)|}\right]. \tag{5.3}$$

Proof. Suppose that $q > 1$. Using Lemma 5.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

$$\left| \frac{\eta(x,a,m)f(ma+\eta(x,a,m)) - \eta(x,b,m)f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. - \frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \right|$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \int_0^1 t|f'(ma + t\eta(x,a,m))|dt$$

$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \int_0^1 t|f'(mb + t\eta(x,b,m))|dt$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(ma + t\eta(x,a,m))|^q dt \right)^{\frac{1}{q}}$$

$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(mb + t\eta(x,b,m))|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right) dt \right]^{\frac{1}{q}}$$

$$+ \frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right) dt \right]^{\frac{1}{q}}$$

$$\leq \frac{M}{(p+1)^{1/p}} \left(\frac{m\pi}{2} \right)^{\frac{1}{q}} \left[\frac{\eta(x,a,m)^2 + \eta(x,b,m)^2}{|\eta(b,a,m)|} \right].$$

□

Remark 5.6. *In Theorem 5.5, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ then we get Theorem (see [12], Theorem 3).*

Theorem 5.7. *Let $A \subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbf{R}_0$ for some fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma, ma+\eta(b, a, m)]$, $q \geq 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma + \eta(b, a, m)]$, we have that*

$$\left| \frac{\eta(x,a,m)f(ma+\eta(x,a,m)) - \eta(x,b,m)f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. -\frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \right|$$

$$\leq M \left(\frac{1}{2}\right)^{1+\frac{1}{q}} (m\pi)^{\frac{1}{q}} \left[\frac{\eta(x,a,m)^2 + \eta(x,b,m)^2}{|\eta(b,a,m)|} \right]. \tag{5.4}$$

Proof. Using Lemma 5.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, and taking the modulus, we have

$$\left| \frac{\eta(x,a,m)f(ma+\eta(x,a,m)) - \eta(x,b,m)f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. -\frac{1}{\eta(b,a,m)} \left[\int_{ma}^{ma+\eta(x,a,m)} f(u)du - \int_{mb}^{mb+\eta(x,b,m)} f(u)du \right] \right|$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|} \int_0^1 t|f'(ma + t\eta(x,a,m))|dt$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|} \int_0^1 t|f'(mb + t\eta(x,b,m))|dt$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\eta(x,a,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left[\int_0^1 t\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{\eta(x,b,m)^2}{|\eta(b,a,m)|}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left[\int_0^1 t\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq M\left(\tfrac{1}{2}\right)^{1+\frac{1}{q}}(m\pi)^{\frac{1}{q}}\left[\tfrac{\eta(x,a,m)^2+\eta(x,b,m)^2}{|\eta(b,a,m)|}\right]. \quad \square$$

Remark 5.8. *In Theorem 5.7, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ then we get Theorem (see [12], Theorem 4). Also, in Theorem 5.7, if we choose $q = 1$, we get Theorem 5.3.*

6. Ostrowski type fractional integral inequalities for MT_m-preinvex functions

In this section, in order to prove our main results regarding some generalizations of Ostrowski type inequalities for MT_m-preinvex functions via fractional integrals, we need the following Lemma:

Lemma 6.1. *Let $K \subseteq \mathbf{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbf{R}$ for some fixed $m \in (0, 1]$ and let $a, b \in K$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : K \longrightarrow \mathbf{R}$ is a differentiable function on K° and f' is integrable on $[ma, ma+\eta(b, a, m)]$. Then, for each $x \in [ma, ma + \eta(b, a, m)]$ and $\alpha > 0$, we have that*

$$\frac{\eta(x,a,m)^\alpha f(ma+\eta(x,a,m)) - \eta(x,b,m)^\alpha f(mb+\eta(x,b,m))}{\eta(b,a,m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma)-J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]$$

$$=\frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)}\int_0^1 t^{\alpha}f'(ma+t\eta(x,a,m))dt$$

$$-\frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)}\int_0^1 t^{\alpha}f'(mb+t\eta(x,b,m))dt, \tag{6.1}$$

where $\Gamma(\alpha)=\int_0^{+\infty}e^{-u}u^{\alpha-1}du$ *is the Euler Gamma function.*

Proof. Denote

$$I=\frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)}\int_0^1 t^{\alpha}f'(ma+t\eta(x,a,m))dt$$

$$-\frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)}\int_0^1 t^{\alpha}f'(mb+t\eta(x,b,m))dt.$$

Integrating by parts, we get

$$I=\frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)}\left[t^{\alpha}\frac{f(ma+t\eta(x,a,m))}{\eta(x,a,m)}\bigg|_0^1-\alpha\int_0^1\frac{t^{\alpha-1}f(ma+t\eta(x,a,m))}{\eta(x,a,m)}dt\right]$$

$$-\frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)}\left[t^{\alpha}\frac{f(mb+t\eta(x,b,m))}{\eta(x,b,m)}\bigg|_0^1-\alpha\int_0^1\frac{t^{\alpha-1}f(mb+t\eta(x,b,m))}{\eta(x,b,m)}dt\right]$$

$$=\frac{\eta(x,a,m)^{\alpha+1}}{\eta(b,a,m)}\left[\frac{f(ma+\eta(x,a,m))}{\eta(x,a,m)}-\frac{\Gamma(\alpha+1)}{\eta(x,a,m)^{\alpha+1}}J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma)\right]$$

$$-\frac{\eta(x,b,m)^{\alpha+1}}{\eta(b,a,m)}\left[\frac{f(mb+\eta(x,b,m))}{\eta(x,b,m)}-\frac{\Gamma(\alpha+1)}{\eta(x,b,m)^{\alpha+1}}J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]$$

$$= \frac{\eta(x,a,m)^{\alpha} f(ma+\eta(x,a,m)) - \eta(x,b,m)^{\alpha} f(mb+\eta(x,b,m))}{\eta(b,a,m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb)\right].$$

□

Remark 6.2. *Clearly, if we choose $m = 1$ and $\eta(x,y,m) = x - my$ in Lemma 6.1, we get Lemma (see [11], Lemma 1).*

By using Lemma 6.1, one can extend to the following results.

Theorem 6.3. *Let $A \subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow \mathbf{R}_0$ for some fixed $m \in (0,1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : A \longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|$ is a MT_m-preinvex function on $[ma, ma+\eta(b,a,m)]$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma+\eta(b,a,m)]$ and $\alpha > 0$, we have that*

$$\left| \frac{\eta(x,a,m)^{\alpha} f(ma+\eta(x,a,m)) - \eta(x,b,m)^{\alpha} f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. -\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb)\right] \right|$$

$$\leq \frac{Mm}{2}\left[\frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]\left[\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]. \tag{6.2}$$

Proof. Using Lemma 6.1, Definition 2.6, MT_m-preinvexity of $|f'|$, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b,a,m)]$, $\alpha > 0$, and taking the modulus, we have

$$\left| \frac{\eta(x,a,m)^{\alpha} f(ma + \eta(x,a,m)) - \eta(x,b,m)^{\alpha} f(mb + \eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{\eta(b,a,m)} \left[J^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) - J^{\alpha}_{(mb+\eta(x,b,m))-} f(mb) \right] \right|$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} |f'(ma + t\eta(x,a,m))| dt$$

$$+ \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} |f'(mb + t\eta(x,b,m))| dt$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(a)| \right] dt$$

$$+ \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} \left[\frac{m\sqrt{t}}{2\sqrt{1-t}} |f'(x)| + \frac{m\sqrt{1-t}}{2\sqrt{t}} |f'(b)| \right] dt$$

$$\leq \frac{Mm}{2} \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} \left[\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}} \right] dt$$

$$+ \frac{Mm}{2} \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} \left[\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}} \right] dt$$

$$= \frac{Mm}{2} \left[\frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \right] \left[\frac{\Gamma\left(\alpha + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)} \right].$$

□

Remark 6.4. *In Theorem 6.3, if we choose $m = 1$ and $\eta(x, y, m) = x - my$ then we get Theorem (see [11], Theorem 5). Also, in Theorem 6.3, if we choose $\alpha = 1$, we get the inequality in Theorem 5.3.*

Theorem 6.5. *Let $A \subseteq \mathbf{R}_0$ be an open m-invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbf{R}_0$ for some fixed $m \in (0, 1]$ and let $a, b \in A$, $a < b$ with $ma < ma + \eta(b, a, m)$. Assume that $f : A \longrightarrow \mathbf{R}$ is a differentiable function on A°. If $|f'|^q$ is a MT_m-preinvex function on $[ma, ma+\eta(b, a, m)]$, $q > 1$, $p^{-1}+q^{-1} = 1$ and $|f'(x)| \leq M$, then for each $x \in [ma, ma+\eta(b, a, m)]$ and $\alpha > 0$, we have that*

$$\left| \frac{\eta(x,a,m)^\alpha f(ma+\eta(x,a,m)) - \eta(x,b,m)^\alpha f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{\eta(b,a,m)} \left[J^\alpha_{(ma+\eta(x,a,m))-} f(ma) - J^\alpha_{(mb+\eta(x,b,m))-} f(mb) \right] \right|$$

$$\leq \frac{M}{(1+p\alpha)^{1/p}} \left(\frac{m\pi}{2}\right)^{\frac{1}{q}} \left[\frac{|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \right]. \tag{6.3}$$

Proof. Suppose that $q > 1$. Using Lemma 6.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, Hölder inequality, the fact that $|f'(x)| \leq M$ for each $x \in [ma, ma + \eta(b, a, m)]$, $\alpha > 0$, and taking the modulus, we have

$$\left| \frac{\eta(x,a,m)^\alpha f(ma+\eta(x,a,m)) - \eta(x,b,m)^\alpha f(mb+\eta(x,b,m))}{\eta(b,a,m)} \right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{\eta(b,a,m)} \left[J^\alpha_{(ma+\eta(x,a,m))-} f(ma) - J^\alpha_{(mb+\eta(x,b,m))-} f(mb) \right] \right|$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^\alpha |f'(ma + t\eta(x,a,m))| dt$$

$$+ \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^\alpha |f'(mb + t\eta(x,b,m))| dt$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left[\int_0^1 \left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq \frac{M}{(1+p\alpha)^{1/p}}\left(\frac{m\pi}{2}\right)^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right].$$

□

Remark 6.6. *In Theorem 6.5, if we choose* $m = 1$ *and* $\eta(x, y, m) = x - my$ *then we get Theorem (see [11], Theorem 6). Also, in Theorem 6.5, if we choose* $\alpha = 1$*, we get the inequality in Theorem 5.5.*

Theorem 6.7. *Let* $A \subseteq \mathbf{R}_0$ *be an open* m*-invex subset with respect to* $\eta : A \times A \times (0, 1] \longrightarrow \mathbf{R}_0$ *for some fixed* $m \in (0, 1]$ *and let* $a, b \in A$, $a < b$ *with* $ma < ma + \eta(b, a, m)$*. Assume that* $f : A \longrightarrow \mathbf{R}$ *is a differentiable function on* A°*. If* $|f'|^q$ *is a* MT_m*-preinvex function on* $[ma, ma+\eta(b, a, m)]$, $q \geq 1$ *and* $|f'(x)| \leq M$*, then for each* $x \in [ma, ma + \eta(b, a, m)]$ *and* $\alpha > 0$*, we have that*

$$\left|\frac{\eta(x,a,m)^\alpha f(ma+\eta(x,a,m))-\eta(x,b,m)^\alpha f(mb+\eta(x,b,m))}{\eta(b,a,m)}\right.$$

$$-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma)-J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]\Bigg|$$

$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left(\frac{m}{2}\right)^{\frac{1}{q}}\left[\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right]. \tag{6.4}$$

Proof. Using Lemma 6.1, Definition 2.6, MT_m-preinvexity of $|f'|^q$, the well-known power mean inequality, the fact that $|f'(x)|\leq M$ for each $x\in[ma,ma+\eta(b,a,m)]$, $\alpha>0$, and taking the modulus, we have

$$\Bigg|\frac{\eta(x,a,m)^{\alpha}f(ma+\eta(x,a,m))-\eta(x,b,m)^{\alpha}f(mb+\eta(x,b,m))}{\eta(b,a,m)}$$

$$-\frac{\Gamma(\alpha+1)}{\eta(b,a,m)}\left[J^{\alpha}_{(ma+\eta(x,a,m))-}f(ma)-J^{\alpha}_{(mb+\eta(x,b,m))-}f(mb)\right]\Bigg|$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 t^{\alpha}|f'(ma+t\eta(x,a,m))|dt$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\int_0^1 t^{\alpha}|f'(mb+t\eta(x,b,m))|dt$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(ma+t\eta(x,a,m))|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}|f'(mb+t\eta(x,b,m))|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}$$

$$\left[\int_0^1 t^{\alpha}\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(a)|^q\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}$$

$$\left[\int_0^1 t^{\alpha}\left(\frac{m\sqrt{t}}{2\sqrt{1-t}}|f'(x)|^q+\frac{m\sqrt{1-t}}{2\sqrt{t}}|f'(b)|^q\right)dt\right]^{\frac{1}{q}}$$

$$\leq \frac{M}{(1+\alpha)^{1-\frac{1}{q}}}\left(\frac{m}{2}\right)^{\frac{1}{q}}\left[\frac{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\alpha+1)}\right]^{\frac{1}{q}}\left[\frac{|\eta(x,a,m)|^{\alpha+1}+|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\right].$$

□

Remark 6.8. *In Theorem 6.7, if we choose* $m=1$ *and* $\eta(x,y,m)=x-my$ *then we get Theorem (see [11], Theorem 7). Also, in Theorem 6.7, if we choose* $\alpha=1$, *we get Theorem 5.7.*

References

[1] T. S. Du, J. G. Liao, Y. J. Li Properties and integral inequalities of Hadamard-Simpson type for the generalized (s,m)-preinvex functions *J. Nonlinear Sci. Appl.*, **9**, pp. 3112-3126, (2016).

[2] S. S. Dragomir, J. Pečarić, L. E. Persson Some inequalities of Hadamard type *Soochow J. Math.*, **21**, pp. 335-341, (1995).

[3] T. Antczak Mean value in invexity analysis *Nonlinear Anal.*, **60**, pp. 1473-1484, (2005).

[4] X. M. Yang, X. Q. Yang, K. L. Teo Generalized invexity and generalized invariant monotonicity *J. Optim. Theory Appl.*, **117**, pp. 607-625, (2003).

[5] R. Pini Invexity and generalized convexity *Optimization.*, **22**, pp. 513-525, (1991).

[6] D. D. Stancu, G. Coman, P. Blaga Analiză numerică şi teoria aproximării *Cluj-Napoca: Presa Universitară Clujeană.*, Vol. **II**, (2002).

[7] W. Liu New integral inequalities involving beta function via P-convexity *Miskolc Math Notes.*, **15**, 2, pp. 585-591, (2014).

[8] M. E. Özdemir, E. Set, M. Alomari Integral inequalities via several kinds of convexity *Creat. Math. Inform.*, **20**, 1, pp. 62-73, (2011).

[9] H. Kavurmaci, M. Avci, M. E. Özdemir New inequalities of Hermite-Hadamard type for convex functions with applications *arXiv:1006.1593v1 [math. CA]*, pp. 1-10, (2010).

[10] W. Liu, W. Wen, J. Park Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals *J. Nonlinear Sci. Appl.*, **9**, pp. 766-777, (2016).

[11] W. Liu, W. Wen, J. Park Ostrowski type fractional integral inequalities for MT-convex functions *Miskolc Mathematical Notes.*, **16**, 1, pp. 249-256, (2015).

[12] M. Tunç Ostrowski type inequalities for functions whose derivatives are MT-convex *J. Comput. Anal. Appl.*, **17**, 4, pp. 691-696, (2014).

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