Proyecciones Journal of Mathematics
Vol. 28, N^o 3, pp. 271–283, December 2009.
Universidad Católica del Norte
Antofagasta - Chile
DOI: 10.4067/S0716-09172009000300007

M-FUZZIFYING BASES *

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Received : October 2009. Accepted : October 2009

Abstract

In this paper, we continue the study of M-fuzzifying matroids. We define the notion of an M-fuzzifying base and discuss some properties of the dual matroids of basic M-fuzzifying matroids.

Keywords : *M-fuzzifying bases; M-fuzzifying matroids; Dual matroids*

Mathematics Subject Classification (2000) : *05B35, 52B40*

*The project is supported by the National Natural Science Foundation of China (10971242)

1. Introduction

In [12, 13], when M is a complete lattice, Shi defined an M-fuzzifying matroid to be the pair $(E, \mathcal{I})$, where $\mathcal{I}$ is a map from 2^E to M satisfying three axioms. Thus each subset of E can be regarded as an independent set to some degree. Moreover, Shi defined the M-fuzzifying rank of a set A as an M-fuzzy natural number $R(A) : \mathbf{N} \to M$. M-fuzzifying matroids and M-fuzzifying rank functions are one-to-one corresponding. This paper treats the notion of M-fuzzifying bases. In subsequent papers we will deals with M-fuzzifying circuits and other fuzzy concepts related to M-fuzzifying matroids.

2. Preliminaries

We will use the following notation in establishing the results of this paper.

If E is a finite set, $\mathcal{A} \subseteq 2^E$, define

$$Com(\mathcal{A}) = \{A \subseteq E : E - A \in \mathcal{A}\},$$

$$Low(\mathcal{A}) = \{A \subseteq E : \exists B \in \mathcal{A}, A \subseteq B\},$$

$$Max(\mathcal{A}) = \{A \in \mathcal{A} : \forall B \in \mathcal{A}, \text{if } A \subseteq B, \text{then } A = B\}.$$

Throughout this paper, M always denotes a completely distributive lattice and M^E is the set of all M-fuzzy sets on E. The smallest element and the largest element in M are denoted by $\perp$ and $\top$, respectively. We often do not distinguish a crisp subset A of E and its characteristic function χ_A.

An element a in M is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in M is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ [2]. The set of non-unit prime elements in M is denoted by $P(M)$. The set of non-zero co-prime elements in M is denoted by $J(M)$.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [1]. $\{a \in M : a \prec b\}$ is called the greatest minimal family of b in the sense of [14], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap J(M)$. Moreover, for $b \in M$, we define $\alpha(b) = \{a \in M : a \prec^{op} b\}$ and $\alpha^*(b) = \alpha(b) \cap P(M)$. In a completely distributive lattice M, α is an $\bigwedge$-$\bigcup$ map, β is a union-preserving map, and there exist $\alpha(b)$ and $\beta(b)$ for each $b \in M$ such that $b = \bigvee \beta(b) = \bigwedge \alpha(b)$ (see [14]). Note that $\beta(\perp) = \emptyset$ and $\alpha(\top) = \emptyset$.

For any $A \in M^E$ and any $a \in M$, we define

$$A_{[a]} = \{x \in E : A(x) \geq a\}, \quad A^{(a)} = \{x \in E : A(x) \nleq a\},$$
$$A_{(a)} = \{x \in E : a \in \beta(A(x))\}, \quad A^{[a]} = \{x \in E : a \notin \alpha(A(x))\}.$$

Some properties of these cut sets can be found in [4, 6, 8, 9, 10, 11].

In [5, 7, 15], a crisp matroid and a crisp base for a crisp matroid are usually defined as follows:

Definition 2.1. *Let E be a finite set. $\mathcal{I} \subseteq 2^E$ is called a system of matroid independent sets on E if it satisfies*

(I1) $\emptyset \in \mathcal{I}$;

(I2) *$A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$;*

(I3) *For any $A, B \in \mathcal{I}$ which satisfy $|A| < |B|$, there exists $e \in B - A$ such that $A \cup \{e\} \in \mathcal{I}$, where $|A|, |B|$ denote the cardinality of A, B.*
The set of all systems of matroid independent sets on E is denoted by $\mathbf{I}(E)$ and $(E, \mathcal{I})$ is called a crisp matroid.

Definition 2.2. *Let E be a finite set. A subset $\mathcal{B} \subseteq 2^E$ is called a crisp base on E if it satisfies*

(B1) *$\mathcal{B} \neq \emptyset$;*

(B2) *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$ for some $y \in B_2 - B_1$.*
The set of all crisp bases on E is denoted by $\mathbf{B}(E)$.

Theorem 2.3. *Let E be a finite set. A subset $\emptyset \neq \mathcal{B} \subseteq 2^E$ is a crisp base on E if and only if it satisfies*

(B2)$'$ *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then $(B_2 - \{y\}) \cup \{x\} \in \mathcal{B}$ for some $y \in B_2 - B_1$.*

Theorem 2.4. (1) *If $\mathcal{I} \in \mathbf{I}(E)$, then $Max(\mathcal{I}) \in \mathbf{B}(E)$;*

(2) *If $\mathcal{B} \in \mathbf{B}(E)$, then $Low(\mathcal{B}) \in \mathbf{I}(E)$;*

(3) *$Max \circ Low(\mathcal{B}) = \mathcal{B}$ ($\forall \mathcal{B} \in \mathbf{B}(E)$); $Low \circ Max(\mathcal{I}) = \mathcal{I}$ ($\forall \mathcal{I} \in \mathbf{I}(E)$).*

Definition 2.5. *Let $(E, \mathcal{I})$ be a crisp matroid. Define $(\mathcal{B}_{\mathcal{I}})^* = Com(Max(\mathcal{I}))$ and $\mathcal{I}^* = Low((\mathcal{B}_{\mathcal{I}})^*)$, then $(\mathcal{B}_{\mathcal{I}})^*$ is a crisp base on E and $(E, \mathcal{I}^*)$ is a crisp matroid. $(E, \mathcal{I}^*)$ is called the dual matroid of $(E, \mathcal{I})$.*

Theorem 2.6. *Let $(E, \mathcal{I})$ be a crisp matroid, for every $A \subseteq E$, then*

(1) *$R_{\mathcal{I}^*}(A) = |A| - R_{\mathcal{I}}(E) + R_{\mathcal{I}}(E - A)$;*

(2) *$R_{\mathcal{I}}(A) + R_{\mathcal{I}}(E - A) - R_{\mathcal{I}}(E) = R_{\mathcal{I}}(A) + R_{\mathcal{I}^*}(A) - |A| = R_{\mathcal{I}^*}(A) + R_{\mathcal{I}^*}(E - A) - R_{\mathcal{I}^*}(E)$.*
Where $R_{\mathcal{I}}$ and $R_{\mathcal{I}^*}$ denote the rank functions for $(E, \mathcal{I})$ and $(E, \mathcal{I}^*)$.

In [12, 13], Shi defined an M-fuzzifying matroid as follows:

Definition 2.7. *Let E be a finite set. If a map $\mathcal{I} : 2^E \to M$ satisfies the following conditions:*
(FI1) $\mathcal{I}(\emptyset) = \top$;
(FI2) *For any $A, B \in 2^E$, $A \subseteq B \Rightarrow \mathcal{I}(A) \geq \mathcal{I}(B)$;*
(FI3) *If $A, B \in 2^E$ and $|A| < |B|$, then $\bigvee_{e \in B-A} \mathcal{I}(A \cup \{e\}) \geq \mathcal{I}(A) \wedge \mathcal{I}(B)$.*
Then the pair $(E, \mathcal{I})$ is called an M-fuzzifying matroid. $\mathcal{I}$ is called a fuzzy family of independent sets on E. For $A \in 2^E$, $\mathcal{I}(A)$ can be regarded as the degree of the set A to be an independent set. A $[0, 1]$-fuzzifying matroid is also called a fuzzifying matroid for short.

Theorem 2.8. *Let E be a finite set and $\mathcal{I} : 2^E \to M$ be a map. Then $(E, \mathcal{I})$ is an M-fuzzifying matroid if and only if for each $a \in J(M)$, $(E, \mathcal{I}_{[a]})$ is a crisp matroid.*

Definition 2.9. *Let $\mathbf{N}$ denote the set of all natural numbers. An M-fuzzy natural number is an antitone map $\lambda : \mathbf{N} \to M$ satisfying*

$$\lambda(0) = \top, \bigwedge_{n \in \mathbf{N}} \lambda(n) = \bot.$$

The set of all M-fuzzy natural numbers is denoted by $\mathbf{N}(M)$.

Definition 2.10. *For any $m \in \mathbf{N}$, define $\underline{m} \in \mathbf{N}(M)$ such that*

$$\underline{m} = \begin{cases} \top, & \text{if} \quad t \leq m, \\ \bot, & \text{if} \quad t \geq m + 1. \end{cases}$$

Definition 2.11. *For any $\lambda, \mu \in \mathbf{N}(M)$, define the addition $\lambda + \mu$ of λ and μ as follows: for any $n \in \mathbf{N}$,*

$$(\lambda + \mu)(n) = \bigvee_{k+l=n} (\lambda(k) \wedge \mu(l)).$$

Theorem 2.12. *For any $\lambda, \mu \in \mathbf{N}(M)$ and any $a \in J(M)$, it follows that*

$$(\lambda + \mu)_{[a]} = \lambda_{[a]} + \mu_{[a]}.$$

Definition 2.13. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid. The map $R_{\mathcal{I}} : 2^E \to \mathbf{N}(M)$ defined by*

$$R_{\mathcal{I}}(A)(n) = \bigvee \{\mathcal{I}(B) : B \subseteq A, |B| \geq n\}$$

is called the M-fuzzifying rank function for $(E, \mathcal{I})$. If $A \in 2^E$, then $R_{\mathcal{I}}(A)$ is called the M-fuzzifying rank of A.

Theorem 2.14. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid and $R_{\mathcal{I}}$ be the M-fuzzifying rank function for $(E, \mathcal{I})$. For each $a \in J(M)$, let $R_{\mathcal{I}_{[a]}}$ denote the rank function for $(E, \mathcal{I}_{[a]})$. Then $R_{\mathcal{I}_{[a]}}(A) = R_{\mathcal{I}}(A)_{[a]}$ for each $A \in 2^E$.*

3. M-fuzzifying bases

The obvious M-fuzzifying analog of a crisp base is the following:

Definition 3.1. *Let E be a finite set. A map $\mathcal{B} : 2^E \to M$ is called an M-fuzzifying base on E if it satisfies*

(FB1) $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$;

(FB2) $\forall B_1, B_2 \in 2^E$, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$.

A $[0,1]$-fuzzifying base is also called a fuzzifying base for short.

Example 3.2. *Let $E = \{x, y\}$. Define $\mathcal{B} : 2^E \to [0,1]$ by*

$$\mathcal{B}(A) = \begin{cases} 0, & A \in \{\emptyset, \{x, y\}\}; \\ 1, & A = \{x\}; \\ \frac{1}{2}, & A = \{y\}. \end{cases}$$

Obviously, $\mathcal{B}$ is a fuzzifying base on E.

Theorem 3.3. *Let E be a finite set and $\mathcal{B} : 2^E \to M$ be a map. Then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is an M-fuzzifying base on E;*

(2) *For each $a \in J(M), \mathcal{B}_{[a]}$ is a crisp base on E;*

(3) *For each $a \in P(M), \mathcal{B}^{(a)}$ is a crisp base on E.*

Proof. (1) $\Rightarrow$ (2). For each $a \in J(M)$. By (FB1), $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top \geq a$, thus $\mathcal{B}(B) \geq a$ for some $B \in 2^E$, hence $\mathcal{B}_{[a]} \neq \emptyset$, i.e $\mathcal{B}_{[a]}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B}_{[a]}$ and $x \in B_1 - B_2$, then $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \geq a$ by (FB2). As $a \in J(M)$, $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{[a]}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}_{[a]}$ satisfies (B2).

(2) $\Rightarrow$ (1). By (2), for each $a \in J(M)$, $\mathcal{B}(B) \geq a$ for some $B \in 2^E$, thus $\bigvee_{B \in 2^E} \mathcal{B}(B) \geq a$, hence $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$, i.e. (FB1) holds. Let $B_1, B_2 \in 2^E$, $x \in B_1 - B_2$, and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \neq \bot$. For every $a \in J(M)$ and

$a \le \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$, then $B_1, B_2 \in \mathcal{B}_{[a]}$, thus there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{[a]}$, hence $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \ge a$. Therefore, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \ge \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$.

(1) $\Rightarrow$ (3). For each $a \in P(M)$. By (FB1), $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top \not\le a$, thus $\mathcal{B}(B) \not\le a$ for some $B \in 2^E$, hence $\mathcal{B}^{(a)} \ne \emptyset$, i.e $\mathcal{B}^{(a)}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B}^{(a)}$ and $x \in B_1 - B_2$, by (FB2) and $a \in P(M)$, we have $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \ge \mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \not\le a$, hence $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{(a)}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}^{(a)}$ satisfies (B2).

(3) $\Rightarrow$ (1). By (3), for each $a \in P(M)$, $B \in \mathcal{B}^{(a)}$ for some $B \in 2^E$, i.e. $\mathcal{B}(B) \not\le a$ for some $B \in 2^E$, thus $\bigvee_{B \in 2^E} \mathcal{B}(B) \not\le a$, hence $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$. Let $B_1, B_2 \in 2^E$, $x \in B_1 - B_2$, and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \ne \bot$. For every $a \in P(M)$ and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \not\le a$, then $\mathcal{B}(B_1) \not\le a$ and $\mathcal{B}(B_2) \not\le a$, i.e. $B_1, B_2 \in \mathcal{B}^{(a)}$, thus there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{(a)}$, hence $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \not\le a$. Therefore, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \ge \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$. $\square$

By Theorem 2.3 and Theorem 3.3, we can obtain the following theorem.

Theorem 3.4. *Let E be a finite set and $\mathcal{B} : 2^E \to M$ be a map. Then $\mathcal{B}$ is an M-fuzzifying base on E if and only if it satisfies the following conditions:*

(FB1) $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$;

(FB2)$'$ $\forall B_1, B_2 \in 2^E$, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_2 - \{y\}) \cup \{x\}) \ge \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$.

Theorem 3.5. *Let E be a finite set and $\mathcal{B} : 2^E \to M$ be a map. If $\alpha(a \vee b) = \alpha(a) \cap \alpha(b)$ for any $a, b \in M$, then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is an M-fuzzifying base on E;*

(2) *For each $a \in M \backslash \{\top\}, \mathcal{B}^{[a]}$ is a crisp base on E;*

(3) *For each $a \in P(M), \mathcal{B}^{[a]}$ is a crisp base on E.*

Proof. (1) $\Rightarrow$ (2). For each $a \in M \backslash \{\top\}$. By (FB1), $\bigvee_{B \in 2^E} \mathcal{B}(B) = \top$, hence $a \notin \alpha(\top) = \alpha\left(\bigvee_{B \in 2^E} \mathcal{B}(B)\right) = \bigcap_{B \in 2^E} \alpha(\mathcal{B}(B))$, thus $a \notin \alpha(\mathcal{B}(B))$ for some $B \in 2^E$, i. e. $B \in \mathcal{B}^{[a]}$. This implies $\mathcal{B}^{[a]}$ satisfies (B1). Let

$B_1, B_2 \in \mathcal{B}^{[a]}$ and $x \in B_1 - B_2$, then $a \notin \alpha(\mathcal{B}(B_1)) \cup \alpha(\mathcal{B}(B_2)) = \alpha(\mathcal{B}(B_1) \wedge \mathcal{B}(B_1))$ and $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$ by (FB2), hence $a \notin \alpha\left(\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\})\right) = \bigcap_{y \in B_2 - B_1} \alpha(\mathcal{B}((B_1 - \{x\}) \cup \{y\}))$, thus $a \notin \alpha((\mathcal{B}(B_1 - \{x\}) \cup \{y\}))$ for some $y \in B_2 - B_1$, i.e. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{[a]}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}^{[a]}$ satisfies (B2).

(2) $\Rightarrow$ (1). For each $a \in M \backslash \{\top\}$, by $\mathcal{B}^{[a]}$ satisfies (B1), then $a \notin \alpha(\mathcal{B}(B))$ for some $B \in 2^E$, thus $a \notin \bigcap_{B \in 2^E} \alpha(\mathcal{B}(B)) = \alpha\left(\bigvee_{B \in 2^E} \mathcal{B}(B)\right)$, hence $\bigvee_{B \in 2^E} \mathcal{B}(B) = \bigwedge \alpha\left(\bigvee_{B \in 2^E} \mathcal{B}(B)\right) = \top$, i.e. (FB1) holds. Let $B_1, B_2 \in 2^E$, $x \in B_1 - B_2$, and $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \neq \bot$. For every $a \in M \backslash \{\top\}$ and $a \notin \alpha(\mathcal{B}(B_1) \wedge \mathcal{B}(B_2)) = \alpha(\mathcal{B}(B_1)) \cup \alpha(\mathcal{B}(B_2))$, then $a \notin \alpha(\mathcal{B}(B_1))$ and $a \notin \alpha(\mathcal{B}(B_2))$, i.e. $B_1, B_2 \in \mathcal{B}^{[a]}$, thus there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}^{[a]}$, hence $a \notin \alpha(\mathcal{B}((B_1 - \{x\}) \cup \{y\}))$, $a \notin \alpha\left(\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\})\right)$. Therefore, $\bigwedge_{x \in B_1 - B_2} \bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$.

Analogously, we can obtain (1) $\Leftrightarrow$ (3). $\square$

Theorem 3.6. *Let E be a finite set and $\mathcal{B} : 2^E \to M$ be a map. If $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for any $a, b \in M$, then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is an M-fuzzifying base on E;*

(2) *For each $a \in \beta(\top), \mathcal{B}_{(a)}$ is a crisp base on E.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathcal{B}$ is an M-fuzzifying base on E. Then for any $a \in \beta(\top)$, $a \in \beta\left(\bigvee_{B \in 2^E} \mathcal{B}(B)\right) = \bigcup_{B \in 2^E} \beta(\mathcal{B}(B))$ by (FB1), thus $a \in \beta(\mathcal{B}(B))$ for some $B \in 2^E$, hence $B \in \mathcal{B}_{(a)}$, which means that $\mathcal{B}_{(a)}$ satisfies (B1). Let $B_1, B_2 \in \mathcal{B}_{(a)}$ and $x \in B_1 - B_2$, then $a \in \beta(\mathcal{B}(B_1)) \cap \beta(\mathcal{B}(B_1)) = \beta(\mathcal{B}(B_1) \wedge \mathcal{B}(B_2))$, and $\bigvee_{y \in B_2 - B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$ by (FB2), hence $a \in \beta(\mathcal{B}((B_1 - \{x\}) \cup \{y\}))$ for some $y \in B_2 - B_1$, i.e. $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{(a)}$ for some $y \in B_2 - B_1$. This means that $\mathcal{B}_{(a)}$ satisfies (B2).

$(2) \Rightarrow (1)$. By (B1), for each $a \in \beta(\top)$, $B \in \mathcal{B}_{(a)}$ for some $B \in 2^E$, thus $a \in \beta(\mathcal{B}(B)) \subseteq \beta\left(\bigvee_{B\in 2^E} \mathcal{B}(B)\right)$, hence $\bigvee_{B\in 2^E} \mathcal{B}(B) = \bigvee \beta\left(\bigvee_{B\in 2^E} \mathcal{B}(B)\right) \geq \bigvee_{a\in\beta(\top)} a = \top$, i.e (FB1) holds. Let $B_1, B_2 \in 2^E$, $\mathcal{B}(B_1) \wedge \mathcal{B}(B_2) \neq \perp$, $x \in B_1 - B_2$, and $a \in \beta(\mathcal{B}(B_1) \wedge \mathcal{B}(B_2))$. Then $a \in \beta(\mathcal{B}(B_1))$ and $a \in \beta(\mathcal{B}(B_2))$, i.e. $B_1, B_2 \in \mathcal{B}_{(a)}$. Hence there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}_{(a)}$ by $\mathcal{B}_{(a)}$ is a crisp base on E. This shows that $a \in \beta(\mathcal{B}((B_1 - \{x\}) \cup \{y\})) \subseteq \beta\left(\bigvee_{y\in B_2-B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\})\right)$. Therefore, $\bigvee_{y\in B_2-B_1} \mathcal{B}((B_1 - \{x\}) \cup \{y\}) \geq \mathcal{B}(B_1) \wedge \mathcal{B}(B_2)$. This means that (FB2) holds. $\square$

Theorem 3.7. *Let E be a finite set, Then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is a fuzzifying base on E;*
(2) *For each $a \in (0,1], \mathcal{B}_{[a]}$ is a crisp base on E;*
(3) *For each $a \in [0,1), \mathcal{B}_{(a)}$ is a crisp base on E.*

4. The relation between M-fuzzifying bases and M-fuzzifying matroids

Definition 4.1. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid on E, if it satisfies $Max(\mathcal{I}_{[b]}) \subseteq Max(\mathcal{I}_{[a]})$ for every $a, b \in J(M)$ and $a \leq b$, then it is called a basic M-fuzzifying matroid.*

Remark 4.2. *Some M-fuzzifying matroids are not basic M-fuzzifying matroids. For example, let $E = \{x, y, z\}$ and $M = [0,1]$. Define $\mathcal{I} : 2^E \to M$ by*

$$\mathcal{I}(A) = \begin{cases} 1, & A \in \{\emptyset, \{x\}, \{y\}\}; \\ \frac{1}{2}, & A \in \{\{x, y\}\}; \\ \frac{1}{3}, & A \in \{\{z\}, \{x, z\}\}; \\ 0, & A \in \{\{y, z\}, E\}. \end{cases}$$

Then

$$\mathcal{I}_{[r]} = \begin{cases} \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}\}, & r \in (0, \frac{1}{3}]; \\ \{\emptyset, \{x\}, \{y\}, \{x, y\}\}, & r \in (\frac{1}{3}, \frac{1}{2}]; \\ \{\emptyset, \{x\}, \{y\}\}, & r \in (\frac{1}{2}, 1]. \end{cases}$$

Hence $(E,\mathcal{I})$ is an M-fuzzifying matroid, but it is not a basic M-fuzzifying matroid since $Max(\mathcal{I}_{[1]}) \nsubseteq Max(\mathcal{I}_{[\frac{1}{2}]})$.

Theorem 4.3. *Let $\mathcal{B}$ be an M-fuzzifying base on E. Define a map $\mathcal{I}_{\mathcal{B}} : 2^E \to M$ by*

$$\mathcal{I}_{\mathcal{B}}(A) = \bigvee_{A\subseteq B} \mathcal{B}(B).$$

Then $\mathcal{I}_{\mathcal{B}}$ is a basic M-fuzzifying matroid on E.

Proof. For every $a \in J(M)$, $A \in (\mathcal{I}_{\mathcal{B}})_{[a]} \Leftrightarrow \bigvee_{A\subseteq B} \mathcal{B}(B) \geq a \Leftrightarrow A \subseteq B$ and $\mathcal{B}(B) \geq a$ for some $B \in 2^E \Leftrightarrow A \subseteq B$ and $B \in \mathcal{B}_{[a]}$ for some $B \in 2^E \Leftrightarrow A \in Low(\mathcal{B}_{[a]})$. Therefore, $(\mathcal{I}_{\mathcal{B}})_{[a]} = Low(\mathcal{B}_{[a]})$ and $Max((\mathcal{I}_{\mathcal{B}})_{[a]}) = \mathcal{B}_{[a]}$. By Theorem 3.3, $(E,(\mathcal{I}_{\mathcal{B}})_{[a]})$ is a crisp matroid for every $a \in J(M)$. Therefore, $\mathcal{I}_{\mathcal{B}}$ is a basic M-fuzzifying matroid on E. □

When M is a boolean algebra, we have Theorems 4.4-4.6.

Theorem 4.4. *Let $(E,\mathcal{I})$ be an M-fuzzifying matroid. Define a map $\mathcal{B}_{\mathcal{I}} : 2^E \to M$ by*

$$\mathcal{B}_{\mathcal{I}}(B) = \bigwedge_{B\subset A} (\mathcal{I}(A))' \wedge \mathcal{I}(B).$$

Then $\mathcal{B}_{\mathcal{I}}$ is an M-fuzzifying base.

Proof. $\forall a \in J(M)$, $B \in (\mathcal{B}_{\mathcal{I}})_{[a]} \Leftrightarrow (\mathcal{B}_{\mathcal{I}})(B) \geq a \Leftrightarrow (\mathcal{I}(A))' \wedge \mathcal{I}(B) \geq a$ for every $A \supset B \Leftrightarrow \mathcal{I}(B) \geq a$ and $(\mathcal{I}(A))' \geq a$ for every $A \supset B \Leftrightarrow \mathcal{I}(B) \geq a$ and $\mathcal{I}(A) \ngeq a$ for every $A \supset B \Leftrightarrow B \in \mathcal{I}_{[a]}$ and $A \notin \mathcal{I}_{[a]}$ for every $A \supset B \Leftrightarrow B \in Max(\mathcal{I}_{[a]})$. Hence $(\mathcal{B}_{\mathcal{I}})_{[a]} = Max(\mathcal{I}_{[a]})$ $(\forall a \in J(M))$, and $\mathcal{B}_{\mathcal{I}}(B) = \bigvee\{a \in J(M) : B \in Max(\mathcal{I}_{[a]})\}$. By Theorem 3.3, $\mathcal{B}_{\mathcal{I}}$ is an M-fuzzifying base. □

Theorem 4.5. (1) *For an M-fuzzifying matroid $(E,\mathcal{I})$, it follows that $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$;*

(2) *For an M-fuzzifying base $\mathcal{B}$, it follows that $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}} = \mathcal{B}$.*

Proof. (1) For every $a \in J(M)$, by Theorem 4.3 and Theorem 4.4, $(\mathcal{I}_{\mathcal{B}_{\mathcal{I}}})_{[a]} = Low((\mathcal{B}_{\mathcal{I}})_{[a]}) = Low(Max(\mathcal{I}_{[a]})) = \mathcal{I}_{[a]}$.

(2) For every $a \in J(M)$, by Theorem 4.3 and Theorem 4.4, $(\mathcal{B}_{\mathcal{I}_{\mathcal{B}}})_{[a]} = Max((\mathcal{I}_{\mathcal{B}})_{[a]}) = Max(Low(\mathcal{B}_{[a]})) = \mathcal{B}_{[a]}$. □

By Theorems 4.3-4.5, we can obtain the following two results:

Theorem 4.6. *There is a one-to-one correspondence between M-fuzzifying matroids and M-fuzzifying bases. That is, an M-fuzzifying matroid can be completely characterized by an M-fuzzifying base.*

Remark 4.7. *When M is a boolean algebra, an M-fuzzifying matroid is equivalent to a basic M-fuzzifying matroid.*

When $M = [0,1]$, we have Theorems 4.8-4.10.

Theorem 4.8. *Let $(E, \mathcal{I})$ be a basic fuzzifying matroid. Define a map $\mathcal{B}_{\mathcal{I}} : 2^E \to [0,1]$ by*

$$\mathcal{B}_{\mathcal{I}}(B) = \bigvee\{a \in (0,1] : B \in Max(\mathcal{I}_{[a]})\}.$$

Then $\mathcal{B}_{\mathcal{I}}$ is a fuzzifying base on E.

Proof. For each $a \in (0,1]$, obviously, $Max(\mathcal{I}_{[a]}) \subseteq (\mathcal{B}_{\mathcal{I}})_{[a]}$. $B \in (\mathcal{B}_{\mathcal{I}})_{[a]} \Leftrightarrow \bigvee\{a \in (0,1] : B \in Max(\mathcal{I}_{[a]})\} \geq a \Leftrightarrow a \leq b$ for some $b \in (0,1]$ and $B \in Max(\mathcal{I}_{[b]})$. As $(E, \mathcal{I})$ is a basic fuzzifying matroid, $B \in Max(\mathcal{I}_{[b]}) \subseteq Max(\mathcal{I}_{[a]})$. This means that $(\mathcal{B}_{\mathcal{I}})_{[a]} \subseteq Max(\mathcal{I}_{[a]})$. Therefore, $(\mathcal{B}_{\mathcal{I}})_{[a]} = Max(\mathcal{I}_{[a]})$. By Theorem 3.3, $\mathcal{B}_{\mathcal{I}}$ is a fuzzifying base on E. □

Theorem 4.9. (1) *For a basic fuzzifying matroid $(E, \mathcal{I})$, it follows that $\mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$;*

(2) *For a fuzzifying base $\mathcal{B}$, it follows that $\mathcal{B}_{\mathcal{I}_{\mathcal{B}}} = \mathcal{B}$.*

Proof. (1) For every $a \in (0,1]$, by Theorem 4.3 and Theorem 4.8, $(\mathcal{I}_{\mathcal{B}_{\mathcal{I}}})_{[a]} = Low((\mathcal{B}_{\mathcal{I}})_{[a]}) = Low(Max(\mathcal{I}_{[a]})) = \mathcal{I}_{[a]}$.

(2) For every $a \in (0,1]$, by Theorem 4.3 and Theorem 4.8, $(\mathcal{B}_{\mathcal{I}_{\mathcal{B}}})_{[a]} = Max((\mathcal{I}_{\mathcal{B}})_{[a]}) = Max(Low(\mathcal{B}_{[a]})) = \mathcal{I}_{[a]}$. □

By Theorem 4.3, Theorem 4.8 and Theorem 4.9, we can obtain the following result:

Theorem 4.10. *There is a one-to-one correspondence between basic fuzzifying matroids and fuzzifying bases. That is, a basic fuzzifying matroid can be completely characterized by a fuzzifying base.*

5. The dual matroids of basic M-fuzzifying matroids

In this section, M is a boolean algebra or $M = [0, 1]$.

Definition 5.1. *Let $(E, \mathcal{I})$ be a basic M-fuzzifying matroid. Define a map $(\mathcal{B}_{\mathcal{I}})^* : 2^E \to M$ by $(\mathcal{B}_{\mathcal{I}})^*(B) = \mathcal{B}_{\mathcal{I}}(E - B)$ and let $\mathcal{I}^* = \mathcal{I}_{(\mathcal{B}_{\mathcal{I}})^*}$, then $(\mathcal{B}_{\mathcal{I}})^*$ is an M-fuzzifying base on E and $(E, \mathcal{I}^*)$ is a basic M-fuzzifying matroid. $(E, \mathcal{I}^*)$ is called the dual matroid of $(E, \mathcal{I})$.*

Proof. By the definition of $(\mathcal{B}_{\mathcal{I}})^*$, $(\mathcal{B}_{\mathcal{I}})^*_{[a]} = Com((\mathcal{B}_{\mathcal{I}})_{[a]})$ for every $a \in J(M)$. Therefore, for every $a \in J(M)$, $(\mathcal{B}_{\mathcal{I}})^*_{[a]}$ is a crisp base on E. By Theorem 3.3, $(\mathcal{B}_{\mathcal{I}})^*$ is an M-fuzzifying base on E. By Theorem 4.3, $(E, \mathcal{I}^*)$ is a basic M-fuzzifying matroid. $\square$

Theorem 5.2. *Let $(E, \mathcal{I})$ be a basic M-fuzzifying matroid, then*

(1) $(\mathcal{I}^*)^* = \mathcal{I}$;

(2) $Com((\mathcal{B}_{\mathcal{I}})_{[a]}) = (\mathcal{B}_{\mathcal{I}^*})_{[a]}$ *for every* $a \in J(M)$;

(3) $\mathcal{I}^*_{[a]} = (\mathcal{I}_{[a]})^*$ *for every* $a \in J(M)$.

Proof. (1) By the definition of $\mathcal{I}^*$, Theorem 4.5 and Theorem 4.9, $(\mathcal{I}^*)^* = (\mathcal{I}_{(\mathcal{B}_{\mathcal{I}})^*})^* = \mathcal{I}_{(\mathcal{B}_{\mathcal{I}_{(\mathcal{B}_{\mathcal{I}})^*}})^*} = \mathcal{I}_{((\mathcal{B}_{\mathcal{I}})^*)^*} = \mathcal{I}_{\mathcal{B}_{\mathcal{I}}} = \mathcal{I}$.

(2) By Theorem 4.5, Theorem 4.9 and the definition of $(\mathcal{B}_{\mathcal{I}})^*$, for every $a \in J(M)$, $(\mathcal{B}_{\mathcal{I}^*})_{[a]} = (\mathcal{B}_{\mathcal{I}})^*_{[a]} = Com((\mathcal{B}_{\mathcal{I}})_{[a]})$.

(3) For every $a \in J(M)$, $A \in \mathcal{I}^*_{[a]} \Leftrightarrow A \in Low((\mathcal{B}_{\mathcal{I}})^*_{[a]}) \Leftrightarrow A \in Low(Com((\mathcal{B}_{\mathcal{I}})_{[a]})) \Leftrightarrow A \in Low(Com(Max(\mathcal{I}_{[a]})) \Leftrightarrow A \in (\mathcal{I}_{[a]})^*$. $\square$

Theorem 5.3. *Let $(E, \mathcal{I})$ be a basic M-fuzzifying matroid, for every $A \subseteq E$, then*

(1) $R_{\mathcal{I}^*}(A) + R_{\mathcal{I}}(E) = \underline{|A|} + R_{\mathcal{I}}(E - A)$;

(2) $R_{\mathcal{I}}(A) + R_{\mathcal{I}^*}(E) = \underline{|A|} + R_{\mathcal{I}^*}(E - A)$.

Proof. (1) $\forall a \in J(M)$, $(R_{\mathcal{I}^*}(A) + R_{\mathcal{I}}(E))_{[a]} = (R_{\mathcal{I}^*}(A))_{[a]} + (R_{\mathcal{I}}(E))_{[a]} = R_{\mathcal{I}^*_{[a]}}(A) + R_{\mathcal{I}_{[a]}}(E) = R_{(\mathcal{I}_{[a]})^*}(A) + R_{\mathcal{I}_{[a]}}(E) = |A| + R_{\mathcal{I}_{[a]}}(E - A) = (\underline{|A|})_{[a]} + R_{\mathcal{I}}(E - A))_{[a]} = (\underline{|A|} + R_{\mathcal{I}}(E - A))_{[a]}$.

(2) By (1) and Theorem 5.2(1), $R_{\mathcal{I}}(A) + R_{\mathcal{I}^*}(E) = R_{(\mathcal{I}^*)^*}(A) + R_{\mathcal{I}^*}(E) = \underline{|A|} + R_{\mathcal{I}^*}(E - A)$. $\square$

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