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COMPACT OPERATORS

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Abstract

We give a characterization in terms of the transpose operator for a continuous linear operator between locally convex spaces to map bounded sets into relatively weakly compact [relatively compact, pre-compact] sets. Our results give a known characterization for compact operators between Banach spaces.

In a recent paper in this journal Khurana ([Ku]) gave a characterization of continuous linear operators between locally convex spaces which map bounded sets into relatively weakly compact sets and also established a necessary condition which such operators must satisfy. In this note we show that an alteration of this necessary condition can be used to characterize such operators. We also establish an abstract setup which gives characterizations of operators which map bounded sets into relatively weakly compact [relatively compact, precompact] sets.

Our notation is fairly standard and follows for example [Sw], [Ko1], [Wi]. Throughout let E, F be Hausdorff locally convex spaces and $T : E \rightarrow F$ be a weakly continuous linear operator with transpose (adjoint) $T' : F' \rightarrow E'$. There are two natural extensions of the notion of weakly compact [compact, precompact] operators between normed spaces. A continuous linear operator between normed spaces is defined to be weakly compact [compact, precompact] if the operator carries the unit ball of the domain space onto a relatively weakly compact [relatively compact, precompact] subset of the range space. Since the unit ball of a normed space is both a neighborhood of 0 and a bounded set, one could define an operator between locally convex spaces to be weakly compact [relatively precompact] if either it maps a neighborhood of 0 or bounded sets into relatively weakly compact [relatively compact, precompact] subsets. The former definition is adopted in [Ko2],[Sch],[Tr] while Khurana ([Ku]) considers operators satisfying the latter condition and this is the condition which we consider.

Let $\mathcal{F}$ be a family of $\sigma(F, F')$ bounded subsets of F such that each member of $\mathcal{F}$ is absolutely convex, $\mathcal{F}$ is closed under finite unions and $\cup \mathcal{F} = F$. Let $\tau_{\mathcal{F}}$ be the polar topology on F' of uniform convergence on the members of $\mathcal{F}$ ([Sw]17,[Ko1]21.7,[Wi]8.5).

Definition 1. *The operator T is an $\mathcal{F}$ -operator if whenever $B \subset E$ is bounded, there exists $W \in \mathcal{F}$ such that $TB \subset \overline{W}$.*

Example 2. $\mathcal{F}$ consists of all absolutely convex, relatively compact subsets of F . Then $\mathcal{F}$ -operators would correspond to weakly compact operators in the sense that bounded sets are mapped into relatively weakly compact sets.

Example 3. $\mathcal{F}$ consists of all absolutely convex, relatively compact subsets of E . Then $\mathcal{F}$ -operators would correspond to compact operators in the sense that bounded sets are mapped into relatively compact sets.

Example 4. $\mathcal{F}$ consists of all precompact subsets of E . Then $\mathcal{F}$ -operators would correspond to precompact operators in the sense that bounded sets are mapped into precompact sets.

We give a characterization of $\mathcal{F}$ -operators. If $B \subset E$, we denote the polar of B in E' by $B^0 = \{x' \in E' : |x'(x)| \leq 1 \text{ for all } x \in B\}$, and if $A \subset E'$, we denote the polar of A in E by $A_0 = \{x \in E : |x'(x)| \leq 1 \text{ for all } x' \in A\}$. The strong topology of E' is denoted by $\beta(E', E)$ ([Sw]19, [Wi]8.5).

Theorem 5. T is an $\mathcal{F}$ -operator $\Leftrightarrow T' : F' \rightarrow E'$ is $\tau_{\mathcal{F}} - \beta(E', E)$ continuous.

Proof : $\Leftarrow$: Let $B \subset E$ be bounded so B^0 is a basic $\beta(E', E)$ neighborhood of 0. There exists a $\tau_{\mathcal{F}}$ neighborhood of 0, W^0 , with $W \in \mathcal{F}$ such that $(T')^{-1}(B^0) = (TB)^0 \supset W^0$. The Bipolar Theorem ([Sw]15.5, [Wi]8.3.8) implies that $TB \subset (TB)_0^0 \subset W_0^0 = \overline{W}$ so T is an $\mathcal{F}$ -operator.

$\Rightarrow$: Let B^0 be a basic $\beta(E', E)$ neighborhood of 0 with $B \subset E$ bounded and absolutely convex. There exists $W \in \mathcal{F}$ such that $TB \subset \overline{W}$. Therefore, $(TB)^0 \supset W^0 = (\overline{W})^0$ is a basic $\tau_{\mathcal{F}}$ neighborhood of 0 and $(T')^{-1}(B^0) = (TB)^0 \supset W^0$ implies T' is $\tau_{\mathcal{F}} - \beta(E', E)$ continuous.

We give a restatement of the theorem for later reference.

Corollary 6. T is an $\mathcal{F}$ -operator $\Leftrightarrow T'$ carries nets in F' which converge uniformly on members of $\mathcal{F}$ into nets in E' which converge uniformly on bounded subsets of E .

Remark 7. The special case of Theorem 5 from Example 2 (weakly compact operators) corresponds to condition (i) of Theorem 1 in [Ku] where the sufficiency of the condition is established; Theorem 5 shows that the condition is also necessary. Even in the case of normed spaces this characterization seems to be new. Theorem 5 also gives a characterization of compact and precompact operators from Examples 3 and 4.

We give some specializations of Theorem 5 and Corollary 6 for precompact [compact] operators.

Corollary 8. *Assume F barrelled. T is precompact $\Rightarrow$ (#) T' carries nets in F' which are $\sigma(F', F)$ bounded and $\sigma(F', F)$ convergent to 0 to nets in E' which converge uniformly on bounded subsets of E .*

Proof : Suppose $\{y'_\delta\}$ is a $\sigma(F', F)$ bounded net which is $\sigma(F', F)$ convergent to 0. Since F is barrelled, $\{y'_\delta\}$ is equicontinuous so $y'_\delta(y) \rightarrow 0$ uniformly for y belonging to precompact subsets of F ([Sw]23.6,24.12,[Ko1]21.6.2). From Corollary 6, $T'y'_\delta(x) \rightarrow 0$ uniformly for x belonging to bounded subsets of E .

Remark 9. *For Banach spaces, Corollary 8 is one part of a characterization of compact operators ([DS]VI.5.6,[Sw]28.1.4); we address the other part later in Corollary 11.*

The barrelledness assumption in Corollary 8 is important.

Example 10. *Let c_{00} be the sequence space of all scalar sequences which are eventually 0 with the sup-norm; the dual of c_{00} is then l^1 . Define $T : c_{00} \rightarrow c_{00}$ by $T\{t_j\} = \{t_j/j\}$. Then T is precompact ([Sw]10.1.15-the proof). The transpose of T , $T' : l^1 \rightarrow l^1$, is given by $T'\{s_j\} = \{s_j/j\}$. The sequence $\{i^2 e^i\}$, where e^i is the sequence with 1 in the i^{th} coordinate and 0 in the other coordinates, is $\sigma(l^1, c_{00})$ bounded and $\sigma(l^1, c_{00})$ convergent to 0, but $\{T'(i^2 e^i)\} = \{i e^i\}$ is not $\beta(l^1, c_{00}) = \|\cdot\|_1$ convergent to 0.*

Corollary 11. *Assume F barrelled. Then condition (#) of Corollary 8 $\Rightarrow T' : (F', \sigma(F', F)) \rightarrow (E', \beta(E', E))$ is compact.*

Proof : Suppose $B \subset F'$ is $\sigma(F', F)$ bounded. To show $T'B$ is $\beta(E', E)$ relatively compact let $\{T'y'_\delta\}$ be a net in E' with $y'_\delta \in B$. Since F is barrelled and $B \subset E'$ is $\sigma(F', F)$ bounded, B is $\sigma(F', F)$ relatively compact ([Sw]24.6,[Ko1]21.4.4). Therefore, there exists a subnet $\{y'_\delta\}$ which is $\sigma(F', F)$ convergent to some $y' \in F'$. Condition (#) implies $T'y'_\delta(x) \rightarrow T'y'(x)$ uniformly for x belonging to bounded subsets of E or $T'y'_\delta \rightarrow T'y'$ in $\beta(E', E)$. Therefore, $T'B$ is relatively $\beta(E', E)$ compact.

Remark 12. If F is barrelled, $\sigma(F', F)$ bounded subsets of F' are $\beta(E', E)$ bounded ([Sw]24.6) so the conclusion of Corollary 11 can also be stated that $T' : (F', \beta(F', F)) \rightarrow (E', \beta(E', E))$ is compact. If E and F are Banach spaces, then Corollaries 8 and 11 imply that if $T : E \rightarrow F$ is compact, then condition (#) is satisfied and condition (#) implies $T' : F' \rightarrow E'$ is compact and then Schauder's Theorem ([Sw]28.11, [Ko2]42.1.7) implies that T is compact. Hence, in the case of Banach spaces T is compact iff (#) is satisfied. This is the characterization of compact operators given in [DS]VI.5.6, [Sw]28.1.4.

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