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An equivalence in generalized almost-Jordan algebras

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Abstract

In this paper we work with the variety of commutative algebras satisfying the identity $\beta((x^2y)x - ((yx)x)x) + \gamma(x^3y - ((yx)x)x) = 0$, *where* β, γ *are scalars. They are called generalized almost-Jordan algebras. We prove that this variety is equivalent to the variety of commutative algebras satisfying* $(3\beta + \gamma)(G_y(x, z, t) - G_x(y, z, t)) + (\beta + 3\gamma)(J(x, z, t)y - J(y, z, t)x) = 0$, *for all* $x, y, z, t \in A$, *where* $J(x, y, z) = (xy)z + (yz)x + (zx)y$ *and* $G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y)$. *Moreover, we prove that if* A *is a commutative algebra, then* $J(x, z, t)y = J(y, z, t)x$, *for all* $x, y, z, t \in A$, *if and only if* A *is a generalized almost-Jordan algebra for* $\beta = 1$ *and* $\gamma = -3$, *that is,* A *satisfies the identity* $(x^2y)x + 2\big((yx)x\big)x - 3x^3y = 0$ *and we study this identity. We also prove that if* A *is a commutative algebra, then* $G_y(x, z, t) = G_x(y, z, t)$, *for all* $x, y, z, t \in A$, *if and only if* A *is an almost-Jordan or a Lie Triple algebra.*

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1. Introduction

In this work, F is a field of char$F \neq 2$ and A be a commutative not necessarily associative algebra over F.

The algebra A is called Jordan algebra if satisfies $(y^2, x, y) = 0$, for all $y, x \in A$. For properties of these algebras see [10]. It is know, see Osborn [7], that a Jordan algebra satisfies the identity

$$3(x^2y)x - 2((yx)x)x - x^3y = 0. \tag{1.1}$$

Algebras satisfying identity (1.1), called *Lie Triple algebras or almost-Jordan algebras* have been studied by Hentzel, Peresi, Osborn, Peterson and Sidorov [5, 7, 8, 9, 11].

Identity (1.1) was generalized in 1988 by Carini, Hentzel and Piaccentini-Cattaneo, see [3]. After that, Arenas and Labra call them generalized almost-Jordan algebras, see [1].

We say that A is *a generalized almost-Jordan algebra* if it satisfies:

$$\beta\Big((x^2y)x - ((yx)x)x\Big) + \gamma\Big(x^3y - ((yx)x)x\Big) = 0, \tag{1.2}$$

for all $x, y \in A$, where $\beta, \gamma \in F$ and $(\beta, \gamma) \neq (0, 0)$.

In the study of degree four identities not implied by conmutativity, Osborn [8] classified those that were implied by the fact of possessing a unit element. Carini, Hentzel and Piacentini-Cattaneo [3] extended this work by dropping the restriction on the existence of the unit element. The identity defining a generalized almost-Jordan algebra with $\beta, \gamma \in F$ appears as one of these identities.

We have:

$$(x^2, y, x) = (x^2y)x - x^2(yx), (x^2, x, y) = x^3y - x^2(yx), (yx, x, x) = ((yx)x)x - (yx)x^2,$$

so

$$(x^2, y, x) - (yx, x, x) = (x^2y)x - ((yx)x)x, (x^2, x, y) - (yx, x, x) = x^3y - ((yx)x)x$$

and

$$0 = \beta\Big((x^2y)x - ((yx)x)x\Big) + \gamma\Big(x^3y - ((yx)x)x\Big) =$$
$$\beta\Big((x^2, y, x) - (yx, x, x)\Big) + \gamma\Big((x^2, x, y) - (yx, x, x)\Big)$$

Therefore, in terms of associators a generalized almost-Jordan algebra satisfies,

$$\beta(x^2, y, x) + \gamma(x^2, x, y) = (\beta + \gamma)(yx, x, x) \tag{1.3}$$

If $\beta = 3$ and $\gamma = -1$, we obtain an almost-Jordan algebra, that is, A satisfies

$$3(x^2, y, x) = (x^2, x, y) + 2(yx, x, x).$$

Generalized almost-Jordan algebras A have been studied in [3] where the authors proved that for almost all the algebras, simplicity implies associativity, in [1], where the authors proved that these algebras always have a trace form in terms of the trace of right multiplication operators. They also prove that if A is finite-dimensional and solvable, then it is nilpotent. In [2] the author found the Wedderburn decomposition of A assuming that for every ideal I of A either I has a non zero idempotent or $I \subset R, R$ the solvable radical of A and the quotient A/R is separable, in [4] the authors give a characterization of representations and irreducibles modules of these algebras, and in [6] where, assuming that A also satisfies $((xx)x)x = 0$ the authors proved the existence of an ideal I of A such that $AI = IA = 0$ and the quotient algebra A/I is power-associative.

In this paper we prove the equivalence between generalized almost-Jordan algebras, and commutative algebras satisfying the identity $(3\beta + \gamma)(G_y(x, z, t) - G_x(y, z, t)) + (\beta + 3\gamma)(J(x, z, t)y - J(y, z, t)x) = 0$, for all $x, y, z, t \in A$, where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ and $G_x(y, z, t) = (yz, x, t) + (yt, x, z) + (zt, x, y)$, Theorem 3.2. We prove that a Jordan algebra satisfies the identity $G_x(y, z, t) = 0$ for all $x, y, z, t \in A$. Conversely if $\mathrm{char} F \neq 3$, then every commutative algebra $G_x(y, z, t) = 0$ for all $x, y, z, t \in A$ is a Jordan algebra, Proposition 3.1. Moreover, we prove that if A is a commutative algebra, then $J(x, z, t)y = J(y, z, t)x$ for all $x, y, z, t \in A$, if and only if A is a generalized almost-Jordan algebra for $\beta = 1$ and $\gamma = -3$, that is, A satisfies the identity $(x^2y)x + 2\Big((yx)x\Big)x - 3x^3y = 0$, Proposition 3.4. We also prove that if A is a commutative algebra, then $G_y(x, z, t) = G_x(y, z, t)$, for all $x, y, z, t \in A$, if and only if A is an almost-Jordan algebra, Proposition 3.5. Finally, we give same new identities, Theorem 3.13 and Proposition 3.15 for commutative algebras satisfying the identity $(x^2y)x + 2\Big((yx)x\Big)x - 3x^3y = 0$.

2. Preliminaries

In this section we found relationships among generalized almost-Jordan algebras and alternative algebras, Jordan algebras, baric algebras or b-algebras.

Proposition 2.1. *Let A be a commutative right alternative algebra. Then A is a generalized almost-Jordan algebra, for $\beta = \gamma = 1$.*

Proof: Since A is a right alternative algebra, then A is an alternative algebra and $(x, y, z) = -(x, z, y)$, so by (1.2) we have, $(x^2, y, x)+(x^2, x, y) = (x^2, y, x) - (x^2, y, x) = 0 = 2(yx, x, x)$. □

If A is a F-algebra, then we will define a new algebra $A' = Fe \oplus A$, as vector space, and the multiplication given by:

$$(\alpha e + u)(\beta e + v) = \alpha\beta e + uv,$$

where e is an idempotent, $\alpha, \beta \in F$ and $u, v \in A$.

Proposition 2.2. *Let A be a generalized almost-Jordan algebra. Then A' is a generalized almost-Jordan algebra and $\omega \colon A' \to F$, given by $\omega(\alpha e+u) = \alpha$, is a nonzero homomorphism of algebras.*

Proof: Let $\alpha, \beta \in F, u, v \in A, x = \alpha e + u$ and $y = \beta e + v$. Since $ez = 0$ and $e^2 = e$ for all $z \in A$, then $(a, b, c) = 0$, if $a, b, c \in A \cup \{e\}$, and at least one of them is equal to e, so

$$x^2 = \alpha^2 e + u^2, yx = \alpha\beta e + vu,$$
$$(x^2, y, x) = (\alpha^2 e + u^2, \beta e + v, \alpha e + u) = \alpha^3\beta(e, e, e) + (u^2, v, u) = (u^2, v, u)$$
$$(x^2, x, y) = (u^2, u, v) \text{ and } (yx, x, x) = (vu, u, u).$$

therefore

$\beta(x^2, y, x) + \gamma(x^2, x, y) = \beta(u^2, v, u) + \gamma(u^2, u, v) = (\beta + \gamma)(vu, u, u) = (\beta + \gamma)(yx, x, x)$. □

Definition 2.3. *Let A be a F-algebra. If $\omega \colon A \to F$ is a nonzero algebra homomorphism, then the ordered pair (A, ω) is called a* baric algebra *or* b-algebra*. When a b-algebra (A, ω) is a generalized almost-Jordan algebra, then we call it generalized almost-Jordan b-algebra.*

Corollary 2.4. *Let A be a generalized almost-Jordan algebra. Then (A', ω) is a generalized almost-Jordan b-algebra.*

If A is a F-algebra, then we will define a new algebra $A^{\#} = F \oplus A$, as vector space, and the multiplication given by:

$$(\alpha + u)(\beta + v) = \alpha\beta + \alpha v + \beta u + uv,$$

where $\alpha, \beta \in F$ and $u, v \in A$, $A^{\#}$ has unit element $1 + 0 = 1$.

Theorem 2.5. *Let A be a generalized almost-Jordan algebra. Then $A^{\#}$ is a generalized almost-Jordan algebra if and only if $\beta + 3\gamma = 0$ or A is an alternative algebra.*

Proof: Let $\alpha, \beta \in F, u, v \in A, x = \alpha + u$ and $y = \beta + v$. We note that $(1, a, b) = (a, 1, b) = (a, b, 1) = 0 = (a, b, a)$, for all $a, b \in A^{\#}$, so

$$x^2 = \alpha^2 + 2\alpha u + u^2, yx = \alpha\beta + \alpha v + \beta u + vu,$$
$$(x^2, y, x) = (\alpha^2 + 2\alpha u + u^2, \beta + v, \alpha + u) = 2\alpha(u, v, u) + (u^2, v, u) = (u^2, v, u),$$
$$(x^2, x, y) = (\alpha^2 + 2\alpha u + u^2, \alpha + u, \beta + v) = 2\alpha(u, u, v) + (u^2, u, v),$$
$$(yx, x, x) = (\alpha\beta + \alpha v + \beta u + vu, \alpha + u, \alpha + u) = \alpha(v, u, u) + (vu, u, u),$$
$$(u, u, v) = u^2 v - u(uv) = -\Big((vu)u - vu^2\Big) = -(v, u, u).$$

therefore

$\beta(x^2, y, x) + \gamma(x^2, x, y) - (\beta + \gamma)(yx, x, x) = \beta(u^2, v, u) + 2\alpha\gamma(u, u, v) + \gamma(u^2, u, v) - \alpha(\beta + \gamma)(v, u, u) - (\beta + \gamma)(vu, u, u) = 2\alpha\gamma(u, u, v) - \alpha(\beta + \gamma)(v, u, u) = -2\alpha\gamma(v, u, u) - \alpha(\beta + \gamma)(v, u, u) = -\alpha(3\gamma + \beta)(v, u, u)$. Since α is arbitrary, then the Theorem follows. □

Corollary 2.6. *Let A be an almost-Jordan algebra. Then $A^{\#}$ is an almost-Jordan algebra.*

Corollary 2.7. *If A is an almost-Jordan algebra and $\omega: A^{\#} \to F$ is given by $\omega(\alpha + u) = \alpha$. Then $(A^{\#}, \omega)$ is an almost-Jordan b-algebra.*

Example 2.8. *Let F be a field of characteristic not 2 and A be a commutative F-algebra of basis $\{s, t\}$ with the multiplication:*

	s	t
s	$s + t$	$\frac{1}{2}t$
t	$\frac{1}{2}t$	0

This algebra is an almost-Jordan algebra, but is not a Jordan algebra, see [7]. Moreover, it is a b-algebra and the only idempotent is zero.

In fact, let $\omega: A \to F$ given by $\omega(as + bt) = a$, where $a, b \in F$, then

$$\omega((as+bt)(a's+b't)) = \omega\left(aa's + \tfrac{1}{2}(2aa' + ab' + a'b)t\right) = aa' =$$
$$\omega(as+bt)\omega(a's+b't), \text{ for all } a, a', b, b' \in F,$$

so, this algebra is a b-algebra.

If $e = as + bt \in A$, such that $e^2 = e$, then

$$as + bt = a^2 s + \frac{1}{2}(2a^2 + 2ab)t = a^2 s + (a^2 + ab)t,$$

so $a = a^2$ and $b = a^2 + ab$, therefore, $a = b = 0$, then $e = 0$ is the only idempotent of A.

Example 2.9. *Let F be a field and A be a commutative F-algebra of basis $\{x_1, x_2, x_3, x_4\}$ with the multiplication:*

	x_1	x_2	x_3	x_4
x_1	x_2	x_3	0	0
x_2	x_3	x_3	0	x_3
x_3	0	0	0	0
x_4	0	x_3	0	$x_2 + x_3$

In [1] the authors prove that this algebra is a generalized almost-Jordan algebra for all $\beta, \gamma \in F$, because $(x^2 y)x = ((yx)x)x = x^3 y = 0$ for all $x, y \in A$. Since $(x_1, x_1, x_2) = x_1^2 x_2 - x_1(x_1 x_2) = x_3$, then A is not alternative algebra.

We will to prove that this algebra is not a b-algebra.

Let $\omega\colon A \to F$ be an algebra homomorphism, since $x_3^2 = 0, x_2^2 = x_3, x_1^2 = x_2$ and $x_4^2 = x_2 + x_3$, then $\omega(x_3) = \omega(x_2) = \omega(x_1) = \omega(x_4) = 0$, so A is not a b-algebra.

3. Main Results

Let A be a generalized almost-Jordan algebra.

Linearising (1.3) we have,

$$\begin{aligned}\beta\Big((x^2, y, z) + 2(xz, y, x)\Big) + \gamma\Big((x^2, z, y) + 2(xz, x, y)\Big) &= \\ = (\beta + \gamma)\Big((yx, x, z) + (yx, z, x) + (yz, x, x)\Big)&\end{aligned} \tag{3.1}$$

$$\begin{aligned}
(3.2)\qquad & 2\beta\Big((tx,y,z)+(xz,y,t)+(tz,y,x)\Big)+\\
&+2\gamma\Big((tx,z,y)+(xz,t,y)+(tz,x,y)\Big)=\\
&=(\beta+\gamma)\Big((yx,t,z)+(yx,z,t)+\\
&+(yt,x,z)+(yt,z,x)+(yz,x,t)+(yz,t,x)\Big)
\end{aligned}$$

Let $G_x\colon A\times A\times A\to A$ given by

$$G_x(y,z,t)=(yz,x,t)+(yt,x,z)+(zt,x,y)$$

It is easy to see that, G_x is 3-lineal function and symmetric in every two variables. Moreover, the complete linearization of the (x^2,y,x) is $2G_x(y,z,t)$. If A is a Jordan algebra, then $G_x(y,z,t)=0$, for all $x,y,z,t\in A$. Conversely we have.

Proposition 3.1. *Let A be a commutative algebra over a field of characteristic not 3, such that*

$$G_x(y,z,t)=0,$$

for all $x,y,z,t\in A$. Then A is a Jordan algebra.

Proof: Setting $z=t=y$ in $G_x(y,z,t)=(yz,x,t)+(yt,x,z)+(zt,x,y)=0$, we get $(y^2,x,y)+(y^2,x,y)+(y^2,x,y)=0$, so $3(y^2,x,y)=0$, then A is a Jordan algebra. □

Theorem 3.2. *A is a generalized almost-Jordan algebra, if and only if A is a commutative algebra satisfying*

$$(3\beta+\gamma)\Big(G_y(x,z,t)-G_x(y,z,t)\Big)+(\beta+3\gamma)\Big(J(x,z,t)y-J(y,z,t)x\Big)=0,$$

for all $x,y,z,t\in A$, where $J(a,b,c)=(ab)c+(bc)a+(ca)b$.

Proof: By (3.2) we have,

$$2\beta G_y(x,z,t)+2\gamma\Big((tx,z,y)+(xz,t,y)+(tz,x,y)\Big)=(\beta+\gamma)\Big(G_t(x,y,z)+G_x(y,z,t)+G_z(x,y,t)\Big)-(\beta+\gamma)\Big((xz,t,y)+(tz,x,y)+(tx,z,y)\Big),\quad\text{so}$$

$$\begin{aligned}
(3.3)\qquad & 2\beta G_y(x,z,t)+(\beta+3\gamma)\Big((xz,t,y)+(tz,x,y)+(tx,z,y)\Big)=\\
&(\beta+\gamma)\Big(G_t(x,y,z)+G_x(y,z,t)+G_z(x,y,t)\Big).
\end{aligned}$$

In (3.3), replacing x by y and y by x, we have

$$\begin{gathered}2\beta G_x(y,z,t)+(\beta+3\gamma)\Big((yz,t,x)+(tz,y,x)+(ty,z,x)\Big)=\\(\beta+\gamma)\Big(G_t(x,y,z)+G_y(x,z,t)+G_z(x,y,t)\Big).\end{gathered}\tag{3.4}$$

By (3.3) and (3.4),

$$\begin{gathered}2\beta\Big(G_y(x,z,t)-G_x(y,z,t)\Big)+\\(\beta+3\gamma)\Big((xz,t,y)+(tz,x,y)+(tx,z,y)-\\-(yz,t,x)-(tz,y,x)-(ty,z,x)\Big)=\\(\beta+\gamma)\Big(G_x(y,z,t)-Hy(x,z,t)\Big),\quad \textit{so}\end{gathered}$$

$$\begin{gathered}(3\beta+\gamma)\Big(G_y(x,z,t)-G_x(y,z,t)\Big)+\\(\beta+3\gamma)\Big((xz,t,y)+(tz,x,y)+(tx,z,y)\\-(yz,t,x)-(tz,y,x)-(ty,z,x)\Big)=0,\quad \textit{but}\end{gathered}$$

$$\begin{gathered}(xz,t,y)+(tz,x,y)+(tx,z,y)-(yz,t,x)-(tz,y,x)-(ty,z,x)=\\((xz)t)y-(xz)(ty)+((tz)x)y-(tz)(xy)+((tx)z)y-(tx)(zy)-\\((yz)t)x+(yz)(tx)-((tz)y)x+(tz)(yx)-((ty)z)x+(ty)(zx)=\\\Big((xz)t+(tz)x+(tx)z\Big)y-\Big((yz)t+(tz)y+(ty)z\Big)x=J(x,z,t)y-J(y,z,t)x,\end{gathered}$$

where $J(a,b,c)=(ab)c+(bc)a+(ca)b$. Therefore,

$$(3\beta+\gamma)\Big(G_y(x,z,t)-G_x(y,z,t)\Big)+(\beta+3\gamma)\Big(J(x,z,t)y-J(y,z,t)x\Big)=0.\tag{3.5}$$

Conversely, setting $z=t=x$ in (3.5) we have

$$(3\beta+\gamma)\Big(G_y(x,x,x)-G_x(y,x,x)\Big)+(\beta+3\gamma)\Big(J(x,x,x)y-J(y,x,x)x\Big)=0.\quad(*)$$

Then, using the definition of G_x, G_y and the conmutativity of the algebra we obtain:

$$\begin{gathered}G_y(x,x,x) - G_x(y,x,x) = 3(x^2,y,x) - 2(yx,x,x) - (x^2,x,y) \\ = 3(x^2y)x - 3x^2(yx) - 2((yx)x)x + 2(yx)x^2 - x^3y + x^2(yx) \\ = 3(x^2y)x - 2((yx)x)x - x^3y.\end{gathered}$$

Moreover, $J(x,z,t)y - J(y,z,t)x = 3x^3y - 2((yx)x)x - (x^2y)x$.

Replacing these values in (*) we get

$$(3\beta+\gamma)\Big(3(x^2y)x-2((yx)x)x-x^3y\Big)+(\beta+3\gamma)\Big(3x^3y-2((yx)x)x-(x^2y)x\Big) = 0$$

Reordering these terms we obtain

$$8\gamma x^3y - (8\beta+8\gamma)((yx)x)x + 8\beta(x^2y)x = 0.$$

Since characteristic of the field is different of 2 we get

$$\gamma x^3y - (\beta+\gamma)((yx)x)x + \beta(x^2y)x = 0,$$

and by identity (2), A is a generalized almost-Jordan algebra. □

In [7], Osborn introduced two mappings,

$$\begin{gathered}H(y;x,z,t) = \Big(y(xz)\Big)t + \Big(y(zt)\Big)x + \Big(y(tx)\Big)z \text{ and} \\ K(y,x,z,t) = (xy)(zt) + (yz)(xt) + (yt)(xz), \text{ so}\end{gathered}$$

$$\begin{gathered}G_y(x,z,t) = (xz,y,t) + (xt,y,z) + (zt,y,x) = \Big((xz)y\Big)t + \Big((xt)y\Big)z + \\ \Big((zt)y\Big)x - (xz)(yt) - (xt)(yz) - (zt)(yx) = H(y;x,z,t) - K(y,x,z,t),\end{gathered}$$

but $K(x,y,z,t) = (yx)(zt) + (xz)(yt) + (xt)(yz) = K(y,x,z,t)$, then

$$G_y(x,z,t) - G_x(y,z,t) = H(x;y,z,t) - H(y;x,z,t), \tag{3.6}$$

for all $x,y,z,t \in A$.

Corollary 3.3. *If A satisfies the identity $(x^2)^2 = x^4$ for all $x \in A$ and $\beta + 3\gamma \neq 0$, then $J(x,z,t)y = J(y,z,t)x$.*

Proof: By [7], we have $H(y;x,z,t) = H(x;y,z,t)$ for all $x,y,z,t \in A$ and by Theorem 3.2, $J(x,z,t)y = J(y,z,t)x$. □

Proposition 3.4. *Let A be a commutative algebra. Then the following identities are equivalent:*

1. $(x^2y)x + 2((yx)x)x - 3x^3y = 0$,
2. $J(x, z, t)y = J(y, z, t)x$.

Proof: Since A satisfies the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$, then A is a generalized almost-Jordan algebra for $\beta = 1, \gamma = -3$ and $\beta + 3\gamma \neq 0$, so by (3.5), $J(x, z, t)y = J(y, z, t)x$.

Conversely, setting $z = t = x$ in $J(x, z, t)y = J(y, z, t)x$, we get $J(x, x, x)y = J(y, x, x)x$, that is $3x^3y = ((yx)x)x + (x^2y)x + ((yx)x)x$, so A satisfies the identity $(x^2y)x + 2((yx)x)x - 3x^3y = 0$. □

Proposition 3.5. *Let A be a commutative algebra. Then A is an almost-Jordan algebra if and only if*

$$G_y(x, z, t) = G_x(y, z, t),$$

for all $x, y, z, t \in A$.

Proof: Since A is an almost-Jordan algebra, $\beta + 3\gamma = 0$ so $3\beta + \gamma \neq 0$, and by (3.5), $\Big(G_y(x, z, t) - G_x(y, z, t)\Big) = 0$, so $G_y(x, z, t) = G_x(y, z, t)$.

Conversely, if A satisfies the identity, $G_y(x, z, t) = G_x(y, z, t)$, then developing the associators we have

$$[(yz)x - (xz)y]t + [(yt)x - (xt)y]z + ((zt)x)y - ((zt)y)x = 0.$$

Since $(y, z, x) = (yz)x - y(zx)$ and $(y, t, x) = (yt)x - y(tx)$, we get

$$(y, z, x)t + (y, t, x)z + ((zt)x)y - ((zt)y)x = 0.$$

Replacing $(zt, x, y) = ((zt)x)y - (zt)(xy)$ and $(zt, y, x) = ((zt)y)x - (zt)(xy)$ in the above expression we obtain

$$(y, z, x)t + (y, t, x)z + (zt, x, y) - (zt, y, x) = 0. \tag{3.7}$$

Since A is a commutative algebra, then $(a, b, c) = -(c, b, a)$ and (3.7) becames

$$(x, z, y)t + (x, t, y)z + (y, x, zt) - (x, y, zt) = 0. \tag{3.8}$$

Setting $z = t = x$ in (3.8), we obtain

$$2(x, x, y)x + (y, x, x^2) - (x, y, x^2) = 0.$$

Developing the associators and using the commutativity we get

$$3(x^2y)x - 2(x(xy))x - yx^3 = 0.$$

By identity (1.1), A is an almost-Jordan algebra. □

By identity (3.6), we have

Corollary 3.6. *Let A be a commutative algebra. Then A is an almost-Jordan algebra if and only if*

$$H(y;x,z,t) = H(x;y,z,t),$$

for all $x, y, z, t \in A$.

By [7], we have

Proposition 3.7. *If A satisfies the identity $(x^2)^2 = x^4$ for all $x \in A$, then A is an almost-Jordan algebra.*

Remark 3.8. *The converse of the Proposition 3.7 is note true. Let A be the algebra of Example 2.8, so $s^4 = s + \frac{7}{4}t$ and $(s^2)^2 = s + 2t \neq s^4$.*

An algebra A is called *power-associative algebra* if for all $x \in A$, the subalgebra $A(x)$ of A generated by x is associative algebra.

Corollary 3.9. *If A is a commutative power-associative algebra, then A is an almost-Jordan algebra.*

By Corollaries 3.3 and 3.9, we have

Proposition 3.10. *If A is a commutative power-associative algebra and $\beta+3\gamma \neq 0$, then A is an almost-Jordan algebra and $J(x,z,t)y = J(y,z,t)x$.*

Corollary 3.11. *If A is a commutative power-associative algebra and $\beta + 3\gamma \neq 0$, then the following identities hold,*

1. $3(x^2y)x - 2\Big((yx)x)\Big)x - x^3y = 0,$
2. $(x^2y)x + 2\Big((yx)x\Big)x - 3x^3y = 0.$

for all $x, y \in A$.

Remark 3.12. *The converse of the Corollary 3.11 is note true. Let A be the algebra of Example 2.9, so A satisfies both identities, but A is not power-associative algebra, because $x_1^4 = 0$ and $(x_1^2)^2 = x_3$.*

Let A be a commutative algebra which satisfies the identities of Corollary 3.11, then $(x^2y)x = ((yx)x)x = x^3y$ for all $x, y \in A$. In this case the converse is true.

Next, let A be a commutative non necessarily power-associative algebra, so identities (1) and (2) of the above Corollary are not equivalent. Since identity (1), a Lie triple or almost-Jordan algebra has been largely studied we will study an algebra A satisfying

$$(x^2y)x + 2\Big((yx)x\Big)x - 3x^3y = 0, \tag{3.9}$$

for all $x, y \in A$, which is identity (2) of Corollary 3.11.

It is known (see [1]) that every finite dimensional solvable algebra satisfiyng (3.9) is nilpotent. If R is radical of A and A/R is solvable, then A has Wedderburn decomposition, (see [2]). Moreover, if A has an idempotent element, then $A = A_0 \oplus A_1 \oplus A_{-\frac{3}{2}}$, where $A_i = \{x \in A \mid ex = ix\}, i = 0, 1, -\frac{3}{2}$, is the Peirce decomposition of A. The subspaces A_i satisfies the relations, (see [2]):

$$A_0^2 \subseteq A_0,\ A_1^2 \subseteq A_1,\ A_0A_1 = \{0\} = A_{-\frac{3}{2}}A_0 = A_{-\frac{3}{2}}^2,\ A_{-\frac{3}{2}}A_1 \subseteq A_{-\frac{3}{2}}.$$

In this work we give same new identities.

Substituting $y = x^k$ in (3.9), we get $(x^2x^k)x + 2x^{k+3} - 3x^3x^k = 0$, so

$$2x^{k+3} = 3x^3x^k - (x^2x^k)x, \quad k \geq 2 \tag{3.10}$$

The identity (3.9) is equivalent to

$$(x^2, y, x) + 2(yx, x, x) - 3(x^2, x, y) = 0, \tag{3.11}$$

for all $x, y \in A$.

Linearising (3.11), we have

$$\begin{aligned} 2(xz, y, x) + (x^2, y, z) + 2(yz, x, x) + 2(yx, z, x) + 2(yx, x, z)- \\ -6(xz, x, y) - 3(x^2, z, y) = 0 \end{aligned} \tag{3.12}$$

Interchanging y and z, we have

$$\begin{aligned} 2(xy, z, x) + (x^2, z, y) + 2(yz, x, x) + 2(zx, y, x) + 2(zx, x, y)- \\ -6(xy, x, z) - 3(x^2, y, z) = 0 \end{aligned}$$

Subtracting both identities and canceling out by the factor 4, we obtain

$$(x^2, y, z) - (x^2, z, y) + 2(yx, x, z) - 2(zx, x, y) = 0, \tag{3.13}$$

so $(x^2, y, z)+2(yx, x, z) = (x^2, z, y)+2(zx, x, y)$, and substituting in identity (3.11), we get

$$(xz, y, x) + (yz, x, x) + (yx, z, x) - 2(xz, x, y) - (x^2, z, y) = 0 \tag{3.14}$$

Theorem 3.13. *Let A be an algebra which satisfies identity (3.9). Then A satisfies the following identities for $i, j \geq 2, i \neq j$:*

1. $(x^2, x^i, x^j) = 2\Big(x^{j+2}x^i - x^{i+2}x^j\Big)$,
2. $2x^{j+2}x^i = (x^{j+1}x^i)x + (x^{i+1}x^j)x + ((x^ix^j)x)x - (x^2x^j)x^i$.

Proof: Setting $y = x^i,\ z = x^j$ and then $y = x^j,\ z = x^i$ in identity (3.14), we have,

$$(x^{j+1}, x^i, x) + (x^ix^j, x, x) + (x^{i+1}, x^j, x) - 2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) = 0,$$
$$(x^{i+1}, x^j, x) + (x^jx^i, x, x) + (x^{j+1}, x^i, x) - 2(x^{i+1}, x, x^j) - (x^2, x^i, x^j) = 0.$$

Subtracting both identities we obtain

$$-2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) + 2(x^{i+1}, x, x^j) + (x^2, x^i, x^j) = 0,$$

Developing the associators, we obtain

$$-2x^{j+2}x^i + 2x^{i+2}x^j + (x^2x^i)x^j - x^2(x^ix^j) = 0,$$

This is identity (1).

To get identity (2) we use the commutativity and we will develop the associator in the identity: $(x^{j+1}, x^i, x) + (x^ix^j, x, x) + (x^{i+1}, x^j, x) - 2(x^{j+1}, x, x^i) - (x^2, x^j, x^i) = 0$. □

Remark 3.14. *Setting $y = z = x^i$ in identity (3.14), we have*

$$2x^{i+2}x^i = 2(x^{i+1}x^i)x + ((x^i)^2x)x - (x^i)^2x^2 - (x^2x^i)x^i + x^2(x^i)^2.$$

Proposition 3.15. *Let A be an algebra which satisfies identity (3.9). Then A satisfies the following identities for $k \geq 1$:*

1. $2x^4x^k - 2x^{k+2}x^2 + (x^2)^2x^k - x^2(x^2x^k) = 0$,

2. $4x^{k+4} = 4(x^3x^k)x + 3x^3x^{k+1} - 2x^{k+2}x^2 - x^2(x^2x^k)$,

3. $4x^{k+4} = 4(x^3x^k)x + 3x^3x^{k+1} - 2x^4x^k - (x^2)^2x^k$.

Proof: Setting $i = 2, j = k$ in (1) of Theorem 3.13, we obtain identity (1).

Setting $i = 2, j = k$ in (2) of Theorem 3.13, we obtain

$$2x^{k+2}x^2 = (x^{k+1}x^2)x + (x^3x^k)x + ((x^2x^k)x)x - (x^2x^k)x^2,$$

Using the identity (3.10), we get

$$2x^{k+2}x^2 = (3x^{k+1}x^3 - 2x^{k+4}) + (x^3x^k)x + (3x^kx^3 - 2x^{k+3})x - (x^2x^k)x^2 = 3x^{k+1}x^3 - 4x^{k+4} + 4(x^3x^k)x - (x^2x^k)x^2,$$

which is identity (2).

Finally identity (3) follows from identities (1) and (2). □

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