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About the solutions of linear control systems on Lie groups

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Abstract

In this paper we prove in details the completeness of the solutions of a linear control system on a connected Lie group. On the other hand, we summarize some results showing how to compute the solutions. Some examples are given.

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1. Introduction

In [1] Ayala and Tirao introduced the concept of linear control system Σ on a connected Lie group G as the family of ordinary differential equations given by

$$\Sigma : \qquad \dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) X^j(g(t)),$$

where $\mathcal{X}$ is a linear vector field, (see Definition 2.1), X^j are right-invariant vector fields for $j = 1, \ldots, m$ and $u = (u_1, \ldots, u_m)$ belongs to the class of admissible control functions $\mathcal{U} \subset L^\infty(\mathbf{R}, \mathbf{R}^m)$.

Linear control systems are important for at least two reasons. First, they are a natural generalization of the well known linear control system on the Euclidean space $G = \mathbf{R}^n$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in \mathbf{R}^{n\times n}, \ B \in \mathbf{R}^{n\times m} \ \text{ and } u = (u_1, \ldots, u_m) \in \mathcal{U}.$$

Besides that, [9] Jouan proved that Σ is relevant from theoretical and practical point of view, see also [2], [3], [4], [6], [7] and [8]. Actually, he shows that any general affine control system

$$\Sigma_M : \qquad \dot{x}(t) = X(x(t)) + \sum_{j=1}^{m} u_j(t) X^j(x(t)),$$

on a connected manifold M whose dynamics generate a finite dimensional Lie algebra, i.e.

$$\dim Span_{\mathcal{LA}} \left\{ X, Y^1, ..., Y^m \right\} < \infty.$$

is equivalent to a linear control system on a Lie group G or on a homogeneous space of G.

In this paper we prove in details the completeness of the Σ-solution $\phi(t, g, u)$ for any initial condition $g \in G$, control $u \in \mathcal{U}$ and $t \in \mathbf{R}$. Furthermore, we show a way to compute the flow of an arbitrary drift $\mathcal{X}$ and then the solution $\phi(t, g, u)$, which is specially suitable for nilpotent and simply connected Lie groups. In the same spirit we also recall a series solution-formula which appears in [1].

2. Linear vector fields

In [1] the authors introduced the notion of the **normalizer** of a Lie algebra $\mathbf{g}$ which is by definition the space

$$\eta := norm_{\mathcal{X}}(G)(\mathbf{g}) := \{F \in \mathcal{X}(\mathcal{G});\ \text{for all } \mathcal{Y} \in \mathbf{g},\ \ [\mathcal{F}, \mathcal{Y}] \in \mathbf{g}\}$$

where $\mathcal{X}(\mathcal{G})$ stands for the set of all smooth vector fields on G.

Let us denote by $e \in G$ the identity element of G.

Definition: 2.1. *A vector field $\mathcal{X}$ on G is said to be* **linear** *if it belongs to η and $\mathcal{X}(e) = 0$.*

The following result (Theorem 1 of [7]) gives equivalent conditions for a vector field on G to be linear.

Theorem: 2.2. *Let $\mathcal{X}$ be a vector field on a connected Lie group G. The following conditions are equivalent:*

1. *$\mathcal{X}$ is linear*
2. *$\mathcal{X}$ is an infinitesimal automorphism*
3. *$\mathcal{X}$ satisfies*

$$\mathcal{X}(gh) = (dL_g)_h\mathcal{X}(h) + (dR_h)_g\mathcal{X}(g), \quad \text{for all} \quad g, h \in G. \tag{2.1}$$

If we denote by $(\varphi_t)_{t\in\mathbf{R}}$ the flow associated to the linear vector field $\mathcal{X}$, by definition an infinitesimal automorphism is a vector field such that its flow

$$\{\varphi_t : t \in \mathbf{R}\} \subset Aut(G)$$

is a subgroup of the Lie group of the automorphism of G. As usual, L_g and R_g denote the left and right translations on G and dL_g, dR_g their derivatives.

Remark: 2.3. *Item 2. of Theorem 2.2 shows that $\mathcal{X}$ is complete. In fact, since $\mathcal{X}(e) = 0$ for $t \in \mathbf{R}$ it follows that φ_t is well defined in a neighborhood V_t of $e \in G$. Since G is connected, V_t generates G. So, for any $g \in G$ there exist $g_1, \ldots, g_n \in V_t$ such that $g = g_1 \cdots g_n$ implying that*

$$\varphi_t(g) = \varphi_t(g_1 \cdots g_n) = \varphi_t(g_1) \cdots \varphi_t(g_n)$$

is well defined and therefore $\mathcal{X}$ is complete.

Next we show that associated with any linear vector field $\mathcal{X}$ there exists a $\mathbf{g}$-derivation that is related with the flow of $\mathcal{X}$. Recall that $D : \mathbf{g} \to \mathbf{g}$ is a derivation if

$$D[X, Y] = [DX, Y] + [X, DY], \text{ for any } X, Y \in \mathbf{g}.$$

Of course $\varphi_t(e) = e$ for all $t \in \mathbf{R}$, so if $Y \in \mathbf{g}$ we obtain

$$(2.2)\quad [\mathcal{X}, Y](e) = \left(\frac{d}{dt}\right)_{t=0} (d\varphi_{-t})_{\varphi_t(e)} Y(\varphi_t(e)) = \left(\frac{d}{dt}\right)_{t=0} (d\varphi_{-t})_e Y(e).$$

The vector field Y is right invariant, therefore at any point $g \in G$

$$[\mathcal{X}, Y](g) = \left(\frac{d}{dt}\right)_{t=0} (d\varphi_{-t})_{\varphi_t(g)} Y(\varphi_t(g)) = \left(\frac{d}{dt}\right)_{t=0} (d\varphi_{-t})_{\varphi_t(g)} (dR_{\varphi_t(g)})_e Y(e)$$

$$= \left(\frac{d}{dt}\right)_{t=0} (dR_g)_e (d\varphi_{-t})_e Y(e) = (dR_g)_e [\mathcal{X}, Y](e).$$

In fact for any $t \in \mathbf{R}$, φ_t is an automorphism of G therefore

$$\varphi_{-t} \circ R_{\varphi_t(g)} = R_g \circ \varphi_{-t}.$$

Thus, for a given linear vector field $\mathcal{X}$, one can associate the derivation $\mathcal{D}$ of $\mathbf{g}$ defined as

$$\mathcal{D}Y = -[\mathcal{X}, Y](e), \text{ for all } Y \in \mathbf{g}.$$

The minus sign in the above formula comes from the fact $[Ax, b] = -Ab$ in $\mathbf{R}^d$. It is also used in order to avoid a minus sign in the equality stated in Proposition 2 of [7].

Remark: 2.4. *For all $t \in \mathbf{R}$, $(d\varphi_t)_e = e^{t\mathcal{D}}$. In particular, from the commutative diagram*

$$\begin{array}{ccc} \mathbf{g} & \xrightarrow{(d\varphi_t)_e} & \mathbf{g} \\ \exp \downarrow & & \downarrow \exp \\ G & \xrightarrow{\varphi_t} & G \end{array}$$

it follows that

$$\varphi_t(\exp Y) = \exp(d\varphi_t)_e Y = \exp(e^{t\mathcal{D}} Y), \text{ for all } t \in \mathbf{R}, Y \in \mathbf{g}.$$

Remark: 2.5. *If G is a connected and simply connected nilpotent Lie group the exponential map is a diffeomorphism. In particular, given a derivation it is possible to explicitly compute the drift $\mathcal{X}$ through the formula above via the logarithm map, as follows. Let* $\log(g) = Y$, *then*

$$\varphi_t(g) = \exp(e^{t\mathcal{D}} \log(g)), \text{ for all } t \in \mathbf{R},\ g \in G.$$

Remark: 2.6. *In [1] the authors proved that for connected and simply connected Lie groups the normalizer is isomorphic to the semidirect product* $\mathbf{g} \otimes_s \partial\mathbf{g}$ *between* $\mathbf{g}$ *and the Lie algebra of derivations* $\partial\mathbf{g}$ *of* $\mathbf{g}$. *Therefore, any linear vector field defines a derivation, but the converse is true just for simply connected Lie groups.*

Remark: 2.7. *For a general Lie algebra* $\mathbf{g}$ *pick an inner derivation* $D \in \partial\mathbf{g}$ *which means* $D = [\cdot, Y]$, *where* $Y \in \mathbf{g}$. *In this situation D defines a linear vector field* $\mathcal{X} = \mathcal{X}^D$ *and it is easy to determine*

$$\mathcal{X} = \left(\frac{d}{dt}\right)_{t=0} \varphi_t$$

through the computation of its flow as follows

$$\varphi_t(g) = \exp(tY) g \exp(-tY), \text{ for all } t \in \mathbf{R}.$$

3. Completeness of the solutions

Consider a linear control system on a Lie group G introduced in [1] as

$$\Sigma : \qquad \dot{g}(t) = \mathcal{X}(g(t)) + \sum_{j=1}^{m} u_j(t) X^j(g(t)),$$

where the drift $\mathcal{X}$ is a linear vector field, X^j are right-invariant vector fields, $u \in \mathcal{U} \subset L^\infty(\mathbf{R}, \Omega \subset \mathbf{R}^m)$ is the class of admissible controls, with $\Omega \subset \mathbf{R}^m$ a convex subset satisfying $0 \in int\Omega$.

Since all the vector fields involved are analytical, for each control function $u \in \mathcal{U}$ and each initial value $g \in G$ there exists a unique solution $\phi(t, g, u)$ defined on an open interval containing $t = 0$ and satisfying $\phi(0, g, u) = g$. Note that in general $\phi(t, g, u)$ is just a solution in the sense of Carathéodory, i.e., a locally absolutely continuous curve satisfying the corresponding differential equation almost everywhere. In its domain we know that any solution of Σ satisfies the **cocycle property**

$$\phi(t+s, g, u) = \phi(t, \phi(s, g, u), \Theta_s u)$$

where the map Θ_t is the **shift flow** on $\mathcal{U}$ defined by

$$(\Theta_t u)(s) := u(t+s).$$

In the sequel, instead of $\phi(t, g, u)$ we usually write $\phi_{t,u}(g)$. Note that smoothness of the vector fields $\mathcal{X}$, $X^1, \ldots, X^m$ implies the smoothness of $\phi_{t,u}$. Moreover, for a fixed t and u the map $g \in G \mapsto \phi_{t,u}(g) \in G$ is a diffeomorphism whose inverse is given by $g \in G \mapsto \phi_{-t,\Theta_t u}(g) \in G$.

The next result shows that in order to compute solutions of Σ starting from an arbitrary initial condition it is enough to compute the corresponding solution at the identity element.

Proposition: 3.1. *For a given $u \in \mathcal{U}$, $t \in \mathbf{R}$, let us denote by $\phi_{t,u} := \phi_{t,u}(e)$ the solution of Σ starting at the origin $e \in G$. Then, the solution of Σ starting at $g \in G$ satisfies*

$$\phi(t, g, u) = \phi_{t,u} \cdot \varphi_t(g) = L_{\phi_{t,u}}(\varphi_t(g)).$$

Proof: Let us consider the curve $\alpha(t)$ given by

$$\alpha(t) = \phi_{t,u} \cdot \varphi_t(g).$$

Therefore, $\alpha(0) = g$ and

$$\dot{\alpha}(t) = (dL_{\phi_{t,u}})_{\varphi_t(g)} \tfrac{d}{dt}\varphi_t(g) + (dR_{\varphi_t(g)})_{\phi_{t,u}} \tfrac{d}{dt}\phi_{t,u}$$

$$= (\mathrm{dL}_{\phi_{t,u}})_{\varphi_t(g)} \mathcal{X}(\varphi_t(g)) + (dR_{\varphi_t(g)})_{\phi_{t,u}} \left\{ \mathcal{X}(\phi_{t,u}) + \textstyle\sum_{j=1}^m u_j(t) X^j(\phi_{t,u}) \right\}$$

$$= \left\{ (dL_{\phi_{t,u}})_{\varphi_t(g)} \mathcal{X}(\varphi_t(g)) + (dR_{\varphi_t(g)})_{\phi_{t,u}} \mathcal{X}(\phi_{t,u}) \right\} + \textstyle\sum_{j=1}^m u_j(t) X^j(\alpha(t)).$$

By item 2. of Theorem 2.2

$$(dL_{\phi_{t,u}})_{\varphi_t(g)} \mathcal{X}(\varphi_t(g)) + (dR_{\varphi_t(g)})_{\phi_{t,u}} \mathcal{X}(\phi_{t,u}) = \mathcal{X}(\phi_{t,u}\varphi_t(g)) = \mathcal{X}(\alpha(t)).$$

Consequently

$$\dot{\alpha}(t) = \mathcal{X}(\alpha(t)) + \sum_{j=1}^{m} u_j(t) X^j(\alpha(t)).$$

By the uniqueness of the solution, we have the desired conclusion. $\square$

Theorem: 3.2. *For each $u \in \mathcal{U}$ and $g \in G$ the corresponding solution $\phi_{t,u}(g)$ of Σ is defined in the whole real line.*

Proof: Since the solution of Σ starting at g and control u is given by $\phi_{t,u}(g) = \phi_{t,u}\varphi_t(g)$ we only have to show that the solution starting at the identity element $e \in G$ is defined for any $t \in \mathbf{R}$.

Consider $u \in \mathcal{U}$ and let $\alpha(t)$ defined on $(-\tau', \tau)$ be the maximal solution of Σ associated with u satisfying $\alpha(0) = e$. Let $\beta(t)$ be also a solution associated with u satisfying $\beta(\tau) = e$ and defined on $(\tau - \delta, \tau + \delta)$. Consider the curve

$$\gamma(t) := \begin{cases} \alpha(t) & t \in (-\tau', \tau - \frac{1}{2}\delta) \\ \beta(t)\varphi_{t-\left(\tau - \frac{1}{2}\delta\right)}(g^{-1}h) & [\tau - \frac{1}{2}\delta, \tau + \delta) \end{cases}$$

where $g = \alpha\left(\tau - \frac{1}{2}\delta\right)$ and $h = \beta\left(\tau - \frac{1}{2}\delta\right)$. It is straightforward to check that γ is well defined and continuous. Moreover, for all $t \in (-\tau', \tau - \frac{1}{2}\delta)$, $\gamma(t)$ it is a solution of Σ. If we denote by $\eta(t) := \varphi_{t-\left(\tau-\frac{1}{2}\delta\right)}(g^{-1}h)$ we have, for all $t \in [\tau - \frac{1}{2}\delta, \tau + \delta)$

$$\dot{\gamma}(t) = \frac{d}{dt}\beta(t)\eta(t) = (dL_{\beta(t)})_{\eta(t)}\dot{\eta}(t) + (dR_{\eta(t)})_{\beta(t)}\dot{\beta}(t).$$

However,

$$\dot{\eta}(t) = \mathcal{X}(\eta(t)) \quad \text{and} \quad \dot{\beta}(t) = \mathcal{X}(\beta(t)) + \sum_{j=1}^{m} X^j(\beta(t))$$

and so

$$\eta(t) = (dL_{\beta(t)})_{\eta(t)}\mathcal{X}(\eta(t)) + (dR_{\eta(t)})_{\beta(t)}\left(\mathcal{X}(\beta(t)) + \sum_{j=1}^{m} X^j(\beta(t))\right)$$

$$= (dL_{\beta(t)})_{\eta(t)}\mathcal{X}(\eta(t)) + (dR_{\eta(t)})_{\beta(t)}\mathcal{X}(\beta(t)) + \sum_{j=1}^{m} X^j\left(\beta(t)\eta(t)\right)$$

$$= \mathcal{X}(\gamma(t)) + \sum_{j=1}^{m} X^j\left(\gamma(t)\right)$$

showing that $\gamma(t)$ is a solution of Σ defined on $(-\tau', \tau+\delta)$ associated with $u \in \mathcal{U}$ and satisfying $\gamma(0) = e$ which is a contradiction with the maximality of $\alpha(t)$. It turns out that α must be defined in $(-\tau', \infty)$. Analogously it is possible to show the same for negative times. Thus, the solutions of Σ starting at $e \in G$ are defined in the whole real line. $\square$

4. Solution and series

In this section we recall a result that appears in [1] which shows how to compute the Σ solutions when you know the flow of the drift.

Theorem: 4.1. *Let us consider a constant admissible control, $u \in \mathbf{R}^m$. Therefore, the vector field $\mathcal{X} + \sum_{j=1}^{m}$ has the solution given by*

$$\phi(t, g, u) = \varphi_t(x)\ \exp\left(\sum_{j=1}^{\infty}(-1)^{n+1} t^n d_n(X^u, D)\right) \tag{4.1}$$

where $X^u = \sum_{j=1}^{m} u_j X^j \in \mathbf{g}$ and for each $n \geq 1$,

$$d_n : \mathbf{g} \otimes_s \partial\mathbf{g} \longrightarrow \mathbf{g}$$

is a homogeneous polynomial map of degree n.

In particular, some of the first terms of d_n are obtained by recursive formula as follows:

$$\begin{aligned}
d_1(Y^u, D) &= Y^u \\
d_2(Y^u, D) &= \frac{1}{2} D(Y^u) \\
d_3(Y^u, D) &= \frac{1}{12}[Y^u, D(Y^u)] + \frac{1}{6} D^2(Y^u) \\
d_4(Y^u, D) &= \frac{1}{24}[Y^u, D^2(Y^u)] + \frac{1}{24} D^3(Y^u), \text{ etc.}
\end{aligned}$$

5. Examples

In order to build examples of linear control systems on a Lie group G it is worth to compute first the Lie algebra $\partial\mathbf{g}$ of $\mathbf{g}$, see [5]. Of course, the dimension of $\partial\mathbf{g}$ varies from Abelian to semisimple Lie groups. In fact, in the Euclidean case any linear transformation $D : \mathbf{R}^d \to \mathbf{R}^d$ is trivially a derivation. However, in a semisimple Lie group every derivation is inner. Thus, the dimension of $\partial\mathbf{g}$ varies from d^2 to d.

In this section we show some examples

Example: 5.1. *Let $G = E(2)$ the Lie group of the Euclidean motions of the plane*

$$G=\left\{ g = \begin{pmatrix} 1 & 0 & 0 \\ x & a & b \\ y & -b & \alpha \end{pmatrix} : (x,y) \in \mathbf{R}\ ,\ a^2+b^2=1 \right\}.$$

Any point (x,y) in $\mathbf{R}^2$ is both translated and rotated by the action of elements in G. The Lie algebra $\mathbf{g}$ of G is given by

$$g = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} : a,b,c \in \mathbf{R} \right\}.$$

Let us consider the basis

$$\mathbf{g}=\text{Span}\left\{ Y^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ Y^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ Y^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

and the inner derivations D_1 and D_3 determined by Y^1 and Y^3 respectively. We obtain linear vector fields $\mathcal{X}^1 = \mathcal{X}^{D_1}$ and $\mathcal{X}^3 = \mathcal{X}^{D_3}$ as follows

$$\begin{aligned} \mathcal{X}^1(g) &= \left(\frac{d}{dt}\right)_{t=0} \exp(tY^1)g\exp(-tY^1) \\ &= \left(\frac{d}{dt}\right)_{t=0} \begin{pmatrix} 1 & 0 & 0 \\ x+t-at & a & b \\ y+bt & -b & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1-a & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}\mathcal{X}^3(g) &= \left(\frac{d}{dt}\right)_{t=0} \exp(tY^3)g\exp(-tY^3)\\ &= \left(\frac{d}{dt}\right)_{t=0} \begin{pmatrix} 1 & 0 & 0\\ x\cos t + y\sin t & a & b\\ -x\sin t + y\cos t & -b & a\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0\\ y & 0 & 0\\ -x & 0 & 0\end{pmatrix}\end{aligned}$$

Example: 5.2. *Let* $\mathbf{g} = \mathbf{R}X^1 + \mathbf{R}X^2 + \mathbf{R}X^3$ *the Lie algebra of the connected and simply connected Heisenberg Lie group* G *with the following generators*

$$X^1 = \frac{\partial}{\partial x_1},\ X^2 = x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \text{ and } X^3 = \frac{\partial}{\partial x_3}$$

The only one non-vanishing Lie bracket is $[X^3, X^2] = X^1$. The group G is diffeomorphic to $\mathbf{R}^3$ with the non-Abelian group operation $* : G \to G$ given by

$$(x_1, x_2, x_3) * (y_1, y_2, y_3) = (x_1 + y_1 + x_3y_2, x_2 + y_2, x_3 + y_3), \text{and}$$
$$(x_1, x_2, x_3)^{-1} = (-x_1 + x_2x_3, -x_2, -x_3).$$

On the other hand, the exponential and logarithm maps are given by

$$\exp(a_1X^1 + a_2X^2 + a_3X^3) = (a_1 + \frac{1}{2}a_2a_3, a_2, a_3)$$

and

$$\log(x_1, x_2, x_3) = (x_1 - \frac{1}{2}x_2x_3)X^1 + x_2X^2 + x_3X^3.$$

Let us consider the linear control system Σ given by

$$\dot{g} = \mathcal{X}(g) + uX^2(g),\ u \in \mathbf{R},$$

where $g = (x_1, x_2, x_3) \in G$ and the infinitesimal automorphism $\mathcal{X}$ is determined by

$$\mathcal{X}_t(x_1, x_2, x_3) = (x_1 + x_2t + \frac{1}{2}x_2^2t, x_2,\ tx_2 + x_3).$$

In fact, it can be checked that $\mathcal{X}_t \in Aut(G)$ for every real number t. In coordinates, the system Σ reads

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 + \frac{1}{2}x_2^2 + ux_3 \\ \dot{x}_2 = u \\ \dot{x}_3 = x_2 \end{cases}$$

According to our previous results, the solution exists and is given by the series-solution as

$$\phi_{t,u}(g) = \varphi_t(g) \exp\left(\sum_{j=1}^{\infty}(-1)^{n+1}t^n d_n(uY^2, D)\right).$$

The derivation D associated to $\mathcal{X}$ is the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Since D is nilpotent with nilpotency degree 2, it follows that d_n is zero for $n \geq 4$. The non-null terms of the series are listed below:

$$\begin{aligned} d_1 &= uY^2, \\ d_2 &= \frac{1}{2}u(Y^1 + Y^3), \\ d_3 &= -\frac{1}{12}u^2Y^1 \text{ and} \\ d_4 &= d_5 = \cdots = 0. \end{aligned}$$

In such a case, we get a finite series

$$\zeta(t) = td_1 - t^2d_2 + t^3d_3$$

so that

$$\exp\zeta(t) = \exp\left(\left(-\frac{t^3}{12}u^2 - \frac{t^2}{2}u\right)Y^1 + utY^2 - \frac{t^2}{2}uY^3\right).$$

By the exponential rule, the solution $\phi_{t,u}(g)$ with initial condition g and constant control u reads

$\phi(t, g, u) =$
$\left(x_1 + \left(x_2 + \frac{1}{2}x_2^2 + ux_3\right)t + \left(ux_2 - \frac{u}{2}\right)t^2 - \frac{t^3}{3}u^2, x_2 + ut, tx_2 + x_3 - \frac{t^2}{2}u\right).$

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